An Attempt to the Unified Description of Elementary Particles*

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An attempt is made to give a unified and a more positive definition of elementary particles by introducing a substance of higher level than them. This substance is named "Urmaterie", and is assumed to be described by a non-local field proposed by Yukawa. Various states of the internal motion of Urmaterie are classified by eigenvalues of a complete set of mutually commutative operators with respect to internal coordinates, two of which can be taken as the spin and the mass operators. Each eigenstate thus classified is assumed to correspond to the elementary particle of the present theory.

In § 1, the present situation of the theory of elementary particles is analysed, and the phenomenological feature of the present theory and the necessity of introducing a substance of higher level to overcome it is stressed. In § 2, the spin and the mass operators are introduced. It is done according to the principle that the structure of elementary particles should determine uniquely the transformation properties of the wave functions under Lorentz-transformations. The equations of motion of Urmaterie are then derived from korrespondenzmassig considerations and from the principle of reciprocity. In § 3, the eigenvalue and the eigenfunction of the mass operator is given in the simplest case when the Urmaterie field is assumed as scalar. In § 4, the relation of this Urmaterie field to the local field is discussed, and it is shown that, in the case of the scalar as before, the equation of motion of Urmaterie reduces to that of Fierz for eigenstates of the spin and the mass operators. In § 5, a qualitative discussion on the interaction of Urmaterie is given. In § 6, an extension to the case of spinor is discussed. It is shown that the existence of a new structure constant other than the spin and the rest mass inevitably follows. It is shown that this is interpretable as expressing the essential difference of the heavy and light particles, and in its connection an attempt is made to deduce the conservation law of heavy particles. In § 7, the deductive perspective and the further outlook of the theory is briefly given.

§ 1. Introduction

The present theory of elementary particles, which has been formulated in a perfectly Lorentz covariant form, has succeeded to express beautifully one aspect of elementary particles, and to obtain numerous brilliant results. What is underlying the ground of the present theory is of course the concept of elementary particles, which has been defined through the course of its development as that characterized by structure constants such as the spin or the rest mass, and satisfying conservation laws such as of the energy or of the momentum by their mutual transformations. Needless to say that these definitions have been an excellent abstraction of one aspect of elementary particles, and have played an essential role in the development of the theory.

* This is a detailed account of our preliminary reports¹.

¹ This refers to the preliminary reports submitted earlier.
The recent progress of the theory of elementary particles, however, seems to have exposed its essential limitation that such definitions are no longer sufficient for the better understanding of elementary particles. For example, the present theory can not give answers to questions such as, "What kind of structure of elementary particles do the present structure constants express?" or, "What kind of intrinsic correlation of elementary particles does the present interaction scheme express?". In fact, in the present theory, the structure constants or the interaction Lagrangians are introduced phenomenologically merely as parameters or additive terms into the theory, and the assignment of the spin or the rest mass values to elementary particles or the introduction of the mutual interaction between them are done entirely ad hoc. In this sense, the present theory may be said to remain at the phenomenological stage* in the course of the development of the theory of elementary particles. A remarkable regularity between the rest masses of elementary particles as first pointed out by Nambu⁴, the universal Fermi interaction, or the divergence difficulties inherent in the quantum theory of the wave field from the day of its birth—a series of these facts suggest clearly such limitation of the present theory and also the necessity of more positively defining the concept of elementary particles.

Under this circumstance, it would be of great interest, as promoting a step in overcoming the limitation of the present theory, to introduce an internal structure to elementary particles, and thus try to grasp structure constants of elementary particles and the intrinsic correlation between them through its mediation. Of course there is no definite guiding principle in doing it. It would be reasonable, however, to suppose that the future theory will satisfy on one hand the requirement of relativity and on the other hand will reduce to the present theory in the approximation in which we disregard such internal freedom.

Along this line attempts have been done by many authors. We may quote among them the names of Heisenberg⁵, Bopp⁶, Wessel⁷, and Hönl and Papapetrou⁸.** Heisenberg's attempt is a very ambitious one, but it seems to us that his theory contains too speculative elements in its foundation and is difficult to try any further development. Bopp, Wessel, or Hönl and Papapetrou's attempts, although interesting in as much as they attack directly the study of the internal structure, also seems to lack firm principles in constructing their basic equations.

In previous paper on Yukawa's theory of non-local field¹⁰, one of the authors suggested to regard elementary particles as corresponding to various states of the internal motion of a kind of "Urmaterie", a substance of higher level than elementary particles,*

* This is after the nomenclature of Taketani¹², who pointed out that the development of the physical theory is made spirally through three stages of phenomenological, substantialistic and essentialistic, each of which corresponds to those of an sich, für sich, and an und für sich of Hegel. In the case of Newtonian mechanics, for example, they are Ticho Brahe's, Kepler's, and Newton's stage respectively. See also S. Sakata¹³, "The Theory of the Interaction of Elementary Particles".

** Recently, Pais¹⁴ proposed to introduce a internal structure in connection with r-spin, and pointed out that selection rules concerning with this new freedoms play important roles in explaining the longevity of V particles.
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and pointed out that the non-local field would be nothing but the one that would describe this Urmaterie. As emphasized by Yukawa, the non-local field is an elegant tool of introducing the internal freedom in a Lorentz covariant form, and the use of the non-local field is expected to give a new scope in investigating the internal structure of elementary particles. It is the purpose of this paper to discuss the development of this idea.

The outline of our idea is as follows; We introduce a new substance which is described by a non-local field $U(X_\mu, r_\mu)$. Of course $X_\mu$ can be identified to positional coordinates of elementary particles of the present theory, but $r_\mu$ cannot. This is assumed to describe the internal motion of Urmaterie. Various states of this internal motion will be classified by assigning eigenvalues of a complete set of mutually commutative operators with respect to internal coordinates, and two of them may be taken as the spin and the mass operators respectively. Each eigenstate thus classified may be assumed to correspond to the elementary particle of the present theory.

The actual development is done through the following steps;

i) To define the spin and the mass operators as constants of internal motion.

ii) To solve eigenvalue equations for the spin and the mass operators.

iii) To make clear the mathematical relation of the non-local Urmaterie field to the local field.

iv) To check the qualitative features of the interaction of Urmaterie.

(i) is done in § 2. What we introduce there as a principle of defining them is that the structure of elementary particles should determine uniquely the transformation properties of the wave functions under Lorentz-transformations. Using it, almost unique definitions of the spin and the mass operators are given. The former agrees with that of Fierz, and the latter is shown to reduce to an expectation value of the energy of the internal motion if a suitable condition is imposed. Equations of motion of Urmaterie are then derived from korrespondenzmassig considerations and from the principle of reciprocity.

(ii) is done in § 3. (iii) is done in § 4, and it is shown that in the case when $U(X_\mu, r_\mu)$ is assumed as scalar, the equation of motion of Urmaterie reduces to Fierz's equation. (iv) is done in § 5 using the S-matrix formalism given by Yukawa.

Finally, the concept of particle family of Fermi particles is studied in § 6 to show an advantage of our theory. It is shown that in our theory the concept of particle family follows very naturally. This is because when we extend our theory to include the spinor non-local field, which is necessary to get particles of half-integer spin, the spin and the mass operators no longer compose a complete set. Thus the introduction of a new structure constant becomes inevitable, and we can show that this new structure constant can be interpreted as expressing the essential difference of heavy and light particles.

* Recently, Yukawa published a similar attempt.

** It can be shown that the spin and the mass operators compose a complete set if $U(X_\mu, r_\mu)$ is taken as scalar, but this is not the case when it is spinor. (See § 2 and § 5)
§ 2, The introduction of the spin and the mass operators

As stated in the introduction, the most essential part of our discussion is to try a unified and a more positive definition of elementary particles by introducing the concepts of the spin and the rest mass. Thus, the first task is to introduce constants of internal motion which are responsible to structure constants such as the spin or the rest mass.

As for the spin, we have a fine analysis due to Fierz that the angular momentum of the internal motion of the non-local field just corresponds to the spin. Therefore, we may expect that the spin operator of our theory may be obtained by generalizing his angular momentum operator into an invariant form in such a way that it reduces to the ordinary one in the rest system of the external motion.

Contrary to the case of the spin, it is very difficult to define the mass operator. The clue to it, however, seems to be found in the concept of the wave function and of the space-time underlying the present theory.

The present theory of elementary particles is constructed by at the beginning ascribing a wave function of the specified transformation property to the elementary particle of the specified structure. This implies that it forms an integral part of the present theory to assume that the transformation property of the wave function is one of the most direct expression of the structure of the elementary particle and therefore is determined uniquely by the structure of the elementary particle.

Partly, this assumption is realized in the present theory as a relation between the spin and the transformation property of the wave function. Asserting this to the full, it seems to us very natural to assume that the rest mass, which is a very fundamental structure constant abreast with the spin, is also closely related to the transformation property of the wave function.

It seems instructive to study here the irreducible representation of the Lorentz group $D_{jj'}$. The wave function of the elementary particle is naturally assumed to be transformed according to it. $D_{jj'}$ being not irreducible under spatial rotations, however, the spin of the elementary particle whose wave function is transformed according to $D_{jj'}$ is not unique, and is given by

\[ S = j + j' - \lambda, \]
\[ \lambda = 0, 1, 2, \ldots, 2j' \quad (j \geq j'). \]  

(2.1)

This means that the transformation property of the wave function is not determined uniquely by assigning the value of spin only. The gap can be filled to some extent, since of $j$ and $j'$ what is physically significant is its combination $(j+j')$ only. Even

* In Dirac's theory of the generalized wave equation, special one of (2.1) corresponding to $\lambda=0$ is chosen out by imposing a subsidiary condition. The correspondence between the spin and the transformation property of the wave function is made unique by this. Although his theory is the materialization of the general viewpoint of the present theory, it seems to us that this restriction is not along the right course. An origin of the fact that the grasp of the rest mass in the present theory is very phenomenological seems to lie here.
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taking into account this situation, however, (2.1) is not sufficient to establish a unique correspondence.

Our assumption suggests here the existence of a new relation between \((j+j'), \lambda, s,\) and the rest mass \(m;\)

\[ f(j+j', \lambda, s, m) = 0. \]  

(2.2)

If this is the case, the transformation property of the wave function is determined uniquely by solving (2.1) and (2.2) when \(s\) and \(m\) is given.

Thus, it would be reasonable to make it the guiding principle of introducing the mass operator to try to find such constant of internal motion which, when mediated to external coordinates, its eigenvalue determines uniquely with that of the spin the transformation property of the wave function.

In the following, we restrict our attention to the simplest case, and assume that\(\text{Urmaterie}\) is described by a scalar non-local field \(U(X_{\mu}, r_{\mu}).\) Noticing the Fierz's remark that the internal angular momentum of the non-local field must corresponds to the spin, we introduce an infinitesimal rotational operator in four dimensional space;

\[ R_{\mu\nu} = (1/i) \left( r_{\mu} \partial / \partial r_{\nu} - r_{\nu} \partial / \partial r_{\mu} \right). \]  

(2.3)

Of course the square of \(R_{\mu\nu}\) contains spin part, since it includes spatial rotation as its special case. In general, however, it is not merely the spin and is given by a sum of the spin part and a quantity independent of it corresponding to rotations including time axis.

In order to separate these two components in a Lorentz-invariant way, we introduce \(v_{\lambda\mu\nu}\) and \(\Gamma_{\nu}\) defined by

\[ v_{\lambda\mu\nu} = P_{\mu} R_{\nu\lambda} + P_{\nu} R_{\lambda\mu} + P_{\lambda} R_{\mu\nu} \]  

(2.4)

\[ \Gamma_{\nu} = P_{\nu} R_{\mu\nu}, \]  

(2.5)

where \(P_{\mu}\) is the energy momentum four vector of the external motion

\[ P_{\mu} = (1/i) \cdot \partial / \partial X_{\mu}. \]  

(2.6)

Defining \(S^2\) and \(M^2\) by

\[ S^2 = (v_{\lambda\mu\nu} v_{\lambda\mu\nu}) / 6 P_{\mu}^2 \]  

(2.7)

\[ M^2 = - \Gamma_{\nu} \Gamma_{\nu} / P_{\mu}^2, \]  

(2.8)

we can show by a direct calculation that

\[ \left( 1/2 \right) R_{\mu\nu} R_{\mu\nu} = S^2 - M^2. \]  

(2.9)

\(P_{\mu}\) is commutative with \(S^2\) and \(M^2\). Therefore, expanding \(U(X_{\mu}, r_{\mu})\) into Fourier series

* \(x_{4}\) denotes \(ict.\) \(x_0 = x_4 / i\) is also used when to make hermitic character explicit is needed.
we consider in the following only one component with the wave number $k_\mu$. We further assume that $k_\mu^2 < 0$. Then we see, in the rest system of the external motion,

$$S^2 = S_1^2 + S_2^2 + S_3^2,$$

$$M^2 = \mu_1^2 + \mu_2^2 + \mu_3^2,$$

where $S$ and $\mu$ are two space vectors forming a skew symmetric tensor

$R_{\mu\nu} = \begin{pmatrix} 0, & S_3, & -S_2, & i\mu_1 \\ -S_3, & 0, & S_1, & i\mu_2 \\ S_2, & -S_1, & 0, & i\mu_3 \\ -\mu_1, & -i\mu_2, & -i\mu_3, & 0 \end{pmatrix},$

and each component of $S$ and $\mu$ satisfies the commutation relations

$$[S^\alpha, S^\beta] = iS^\gamma,$$

$$[\mu^\alpha, \mu^\beta] = -iS^\gamma,$$

$$[\mu^\alpha, S^\beta] = i\mu^\gamma \quad (i, j, k \text{ cyclic})$$

From (2.14), and from the fact that $S^2$ and $M^2$ behave as scalar under Lorentz-transformations, we can see at once that $S^2$ and $M^2$ are Lorentz-invariant separations of $R_{\mu\nu}$ into two independent components.

Thus, (2.9) can be regarded as a Lorentz-invariant separation of $R_{\mu\nu}$ into two independent components.

It is clear from (2.11) that $S^2$ is just the required generalization of Fierz's spin operator. Thus, in our theory, to the spin is given an intuitive image that it corresponds to the rotation of a rigid sphere. This is in marked contrast to that of the present theory, where it is given only as the number of independent components of the wave function in the rest system of the center of mass.

It would be natural, from the symmetry in (2.9), to conjecture that $M^2$ is also responsible to the structure of elementary particles just as $S^2$ was. This conjecture seems not misdirected, since we can show in fact that the transformation property of the wave function is determined uniquely by assigning the eigenvalue of $S^2$ and $M^2$. The proof is given in § 4.

It seems useful to investigate the relation of the eigenvalue of $M^2$ to dynamical variables of the internal motion to show clearer that $M^2$ should be taken as the mass operator. Of course, at the present, we know nothing of the law governing the internal motion. But the

* This assumption is self-consistent in the sense that the eigenvalue of the mass is given in fact in a positive definite form (see § 3). An essential revision seems to be necessary to include consistently the case of vanishing rest mass.

** The proof is given in Appendix IV. (b).
fact that the angular momentum of the internal motion corresponds just to the spin seems to suggest that, at least partly, we can expect that the law of the external motion may be applicable to that of the internal motion. We adopt this as a korrespondenzmässig suggestion, and specifically assume that the law of quantum mechanics can be applied to the internal motion in the sense that \( u(k_\mu, r_\mu) \) describes the state of the internal motion, and that \( \hbar / i \cdot \partial / \partial r_i \ (i = 1, 2, 3) \) and \( - \hbar c / i \cdot \partial / \partial r_0 \) correspond to the momentum and the energy operator of the internal motion respectively.

Thus, defining the normalization of \( u(k_\mu, r_\mu) \) by

\[
\int u^*(k_\mu, r_\mu) \delta(n_\mu r_\mu) u(k_\mu, r_\mu) \, (dr_\mu) = 1,
\]

(2.16)

the expectation value of various dynamical variables is assumed to be given by

\[
\langle F\left(r_i, \frac{\hbar}{i} \frac{\partial}{\partial r_i}, -\frac{\hbar c}{i} \frac{\partial}{\partial r_0}, \ldots\right) \rangle
= \int u^*(k_\mu r_\mu) \delta(n_\mu r_\mu) F\left(r_i, \frac{\hbar}{i} \frac{\partial}{\partial r_i}, -\frac{\hbar c}{i} \frac{\partial}{\partial r_0}, \ldots\right)
\times u(k_\mu r_\mu) \, (dr_\mu),
\]

(2.17)

where \( n_\mu \) is an arbitrary time-like vector.

Under these assumptions, we try to study further the meaning of \( M^2 \). In the rest system of the external motion, where \( k_\mu \) takes the form \((0, 0, 0, i\lambda c)\), \( M^2 \) is given by

\[
M^2 u(k_\mu, r_\mu) = m^2 u(k_\mu, r_\mu)
\]

(2.18)

\[
-\lambda^2 = \left(r_0 \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_0} \right)^2
\]

(2.19)

from which we have, after some calculations,

\[
m^2 = -\int u^*(k_\mu r_\mu) \delta(r_0) \left(r_0 \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_0} \right)^2 u(k_\mu r_\mu) \, (dr_\mu),
\]

(2.20)

where we have put \( n_\mu \) as \((0, 0, 0, i)\). Expanding the right hand side of (2.20),

\[
(r_0 \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_0})^2 = 2r_0 \frac{\partial}{\partial r_0} - 3r_0 \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_0}
\]

(2.21)

Of terms of (2.21), first three vanish owing to \( \delta(r_0) \) appearing in (2.20), and the fourth also vanishes if we assume Yukawa’s second equation for \( U(X_\mu, r_\mu) \);

\[
(r_\mu r_\mu - \lambda^2 U(X_\mu, r_\mu) = 0.
\]

(2.22)

Thus, under this assumption, we get

\[
m^2 = (\lambda / hc)^2 \left<E_\mu^2 \right>,
\]

(2.23)

where \( E_\mu^2 \) means the square of the energy of the internal motion;

* As for details see Appendix IV. (a).
\[ E_{\text{in}} = (\hbar c/\lambda)^2 \partial^2 / \partial r_0^2. \]

An analogous discussion can also be repeated for an operator defined by

\[ \mathbf{M} = \sqrt{\mathbf{M}^2}, \]

and the result is

\[ m = \lambda / \hbar c \cdot \langle \sqrt{E_{\text{in}}^2} \rangle. \]

The rest mass of elementary particles may be defined as such a constant which
(i) firstly is a scalar specifying the structure of elementary particles,
(ii) secondly expresses the pool of the energy of the system in the rest system of
center of mass, which means that it agrees there numerically with the fourth
component of the energy and momentum four vector,
and
(iii) thirdly determines the transformation property of the wave function abreast with
the spin.

Collecting above results, it seems perfectly reasonable to assume that \( \mathbf{M} \) is the mass
operator, and that \( m \) represents the rest mass of elementary particle measured in unit
\((\hbar / c \lambda)\). The invariance of \( m \) follows at once from that of \( \mathbf{M} \).

Moreover, we can show that \( r_\mu r_\mu, S^2 \) and \( \mathbf{M}^2 \) compose a complete set with respect
to internal coordinates.* This means that, so long as we assume Yukawa's non-local field
and his second equation, there is no alternative for the mass operator other than \( \mathbf{M} \). At
least we can say that the mass operator must be a function of it.

Thus, in our theory, \( S^2 \) and \( \mathbf{M}^2 \) give the spin and the mass spectrum of elementary
particles.

Introducing polar coordinates,

\[
\begin{align*}
    r_1 &= \lambda \cosh \xi \sin \theta \cos \varphi \\
    r_2 &= \lambda \cosh \xi \sin \theta \sin \varphi \\
    r_3 &= \lambda \cosh \xi \cos \theta \\
    r_0 &= \lambda \sinh \xi
\end{align*}
\]

\[ S^2 \]

is written in the form

\[ S^2 = -\left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right) \]

in the rest system of the external motion. Thus, eigenfunction of \( S^2 \) is the usual spherical
harmonics, and its eigenvalue is given by

\[ s(s+1), \quad s=0, 1, 2, \ldots \]

\[ \]

* The commutability of \( r_\mu r_\mu \) with \( S^2 \) and \( \mathbf{M}^2 \) is shown in Appendix IV. (c). Completeness of these
set follows from the fact that their simultaneous eigenfunction is not degenerated except for \( n \). (See (3.10)).
\( n \) is not scalar, and therefore can not be adopted as a structure constant. We can readily see that it corresponds
to the freedom of polarization.
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The eigenvalue equation of \( M^2 \) leads to a hypergeometric equation, and it will be shown in the next section that discrete eigenvalues are given by solving it under a suitable boundary condition.

The scalar non-local field \( U(X_\mu, r_\mu) \) provides us, therefore, with a unified description of Bose particles, and when expanded into simultaneous eigenfunctions of \( S^2 \) and \( M^2 \), each component describes elementary particles of definite spin and rest mass.

In view of the fact that the rest mass should be related to the external motion as being equal to the magnitude of the energy momentum four vector, it is legitimate to assume

\[
\left( \frac{\partial}{\partial x} - \left( \frac{m}{\lambda} \right)^2 \right) U(X_\mu, r_\mu; m, s) = 0, \tag{2.29}
\]

which must further be generalized to

\[
\left( \frac{\partial}{\partial x} - \frac{M^2}{\lambda^2} \right) U(X_\mu, r_\mu) = 0 \tag{2.30}
\]

for the general non-local field which is not the eigenfunction of \( S^2 \) and \( M^2 \).

(2.29) or (2.30) play an important role of connecting the internal motion to the external, and is nothing but the Yukawa's first equation. This stands to (2.22) just reciprocally adjoint. Although in Yukawa's theory (2.22) was introduced from a rather formal requirement of Born's reciprocity, it appears in our theory as an indispensable equation to secure that the eigenvalue of \( M \) is equal to an expectation value of the energy of the internal motion.

Merely to introduce the internal freedom, it would be sufficient to increase the number of variables, and to introduce the non-local field is not necessarily needed. The fact that the reciprocity appears in such a form, however, seems to suggest the necessity of introducing it when we go to describe the internal structure of elementary particles.

§ 3. The solution of eigenvalue problem

In this section we try to solve the eigenvalue equation for the mass spectrum

\[
M^2 u(k_\mu, r_\mu) = m^2 u(k_\mu, r_\mu). \tag{3.1}
\]

\( M^2 \) being scalar under Lorentz-transformations, it is convenient to solve it in the rest system of the external motion, where \( M^2 \) can be expressed as

\[
M^2 = \sum_{i=1,2,3} (r_i \partial / \partial r_i + r_i \partial / \partial r_i)^2.
\]

In terms of polar coordinates introduced in (2.29), this is simplified to

\[
M^2 = (1 + x^2) d^2 / dx^2 + 3 x d / dx + x^2 / (1 + x^2) \cdot \Omega, \tag{3.2}
\]

where \( x = \sinh \xi \) and \( -\Omega = \) is given by (2.27). Substituting (3.1) into (3.2), and replacing \( -\Omega \) by \( s(s+1) \), which means that we restrict our attention to the mass spectrum of elementary particles of spin \( s \), the equation to be solved becomes

\[
\left\{ \frac{d^2}{dx^2} + \frac{3x}{1+x^2} \frac{d}{dx} - \frac{x^2}{(1+x^2)^2} s(s+1) + \frac{m^2}{1+x^2} \right\} \phi(x) = 0 \tag{3.3}
\]

\((-\infty < x < \infty).\)
This is a hypergeometric differential equation, and performing a transformation

\[ y = 1 + x^2 \]

\[ \varphi = y^{-(1+s)/2} u(y) , \]

is readily written in the standard form;

\[ \left\{ \frac{d^2}{dy^2} + \gamma - \frac{(\alpha + \beta + 1)y}{y(1-y)} \frac{d}{dy} - \frac{\alpha \beta}{y(1-y)} \right\} u(y) = 0, \]

where

\[ \alpha = 1/2 \cdot \{-s + \sqrt{s(s+1)+1-m^2}\} \]

\[ \beta = 1/2 \cdot \{-s - \sqrt{s(s+1)+1-m^2}\} \]

and

\[ \gamma = 1/2 - s. \]

To get explicit solutions, it is necessary to impose boundary conditions. What boundary conditions should be imposed is a problem to be decided in relation with the law of internal motion, and at the present we can say nothing about it. Here we assume tentatively that they should be square integrable. Although this seems to be a natural consequence of extending quantum mechanics into the internal world, its precise meaning is not clear, and must seriously be re-examined at the next stage.

Under this boundary condition, the eigenfunction is given by (As for the details of the calculation, see Appendix 1),

\[ \varphi_{s,l} = (1 + x^2)^{-\frac{1}{2}(s-l+1)} \left( \frac{x^2}{1+x^2} \right)^{\varepsilon} G_{\frac{l-s}{2}} \left( s + \frac{1}{2}, \frac{1}{2} + 2\varepsilon, \frac{x^2}{1+x^2} \right), \]

with the corresponding eigenvalue

\[ m_{s,l} = s + 1 + 2sl - l^2, \]

\[ s \geq 1, \ l = 0, 1, 2, \cdots, s-1, \]

where \( G_{\frac{l-s}{2}} \) is Jacobi's polynome of \((l/2-\varepsilon)\)th order and \( \varepsilon \) stands for \( \varepsilon = \frac{1 + (-1)^{l+1}}{4} \)

\[ \begin{cases} 0 & \text{for even } l \\ 1/2 & \text{for odd } l \end{cases} \]

It should be noted that the difficulty of the infinite degeneracy as pointed out by Schrödinger \(^{35}\) and by Yukawa \(^{12}\) does not occur in our theory. This is because in our theory the eigenfunctions of \( M^2 \) is classified once more according to the eigenvalue of \( S^2 \). Group theoretically, these eigenfunctions transform according to an unitary representation

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\[ \text{\textcopyright Discrete eigenvalues appear below } s(s+1). \text{ Above } s(s+1) \text{ appear continuous ones instead. We neglected the latter since it is of less interest. As for details see Appendix I.} \]
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of the Lorentz-group. As shown by Bargman, the unitary representation of this group is specified by two parameters \( \xi \) and \( \eta \), which are eigenvalues of

\[
\xi = R_{01}R_{23} + R_{03}R_{12} + R_{02}R_{13} \\
\eta = R_{12}^2 + R_{23}^2 + R_{31}^2 - R_{01}^2 - R_{02}^2 - R_{03}^2
\]

respectively. Negative sign before \( \sum R_{ij}^2 \) is the characteristic of the pseudo-Euclid space, and is closely related to the above difficulty. That is, this makes it possible for infinitely large number of combinations of \( \sum R_{ij}^2 \) and \( \sum R_{ij} \) to give the same \( \eta \). Such degeneracy, however, can be removed at once if it is possible to assign eigenvalue of \( \sum R_{ij}^2 \) and \( \sum R_{ij} \) separately. In our case, this is made possible since the separation of (2.9) is Lorentz invariant, and each of \( S^0 \) and \( M^0 \) is scalar. Note that this is a consequence of introducing the external momentum which suffer the same Lorentz-transformation with \( r_{\mu} \) in the definition of \( S^0 \) and \( M^0 \).

(3.8) represent the rest mass of Bose particles measured in unit \( (h/c) \). In particular, Nambu’s mass unit \( 137 \times (\text{electron mass}) \) is obtained by taking \( \lambda \) to be of the order of the classical electron radius. Scalar \( S^0 \) discussed here, however, is an academic model adopted only for its simplicity as the starting point of the theory, and the detailed comparison of this result with experiment does not seem so meaningful.

The complete eigenfunction is given by a product of (3.7) and the corresponding spherical harmonics, and that in arbitrary system can be obtained at once by transforming it by a suitable Lorentz-transformation. Postponing again detailed calculations to Appendix III, the result is given by

\[
\Phi_{x, s, n}(k, r_{\mu}) = \frac{c(s, n)}{\{k^2 - (\xi \cdot \eta / k_{\mu} \cdot \eta / k_{\mu})\}^2} g_{s, \xi}(\frac{k_{\mu} r_{\mu}}{k_{\mu} r_{\mu}}) \times
\]

\[
\times \int_{-\infty}^{\infty} du (a_{\eta} r_{\mu} + i a_{\eta} r_{\mu} \cos u + i a_{\eta} r_{\mu} \sin u)^{a_{n, \eta}} u_{\leq |n|},
\]

where

\[
c(s, n) = \frac{(s - |n|)!}{2\pi i^{s}|s|!}
\]

\[
g_{s, \xi}(x) = (1 + x^2)^{-\frac{s - l + 1}{2}} (\frac{x^2}{1 + x^2})^s G_{s - l}(s + \frac{1}{2}, \frac{1}{2} + 2l, \frac{x^2}{1 + x^2}),
\]

and \( a_{\mu} \) are coefficients of Lorentz-transformation which transforms \( k_{\mu} \) into rest. The explicit form is given by

\[
a_{\mu} = \begin{pmatrix}
-1 + (k_1/K)^2, & k_1 k_2/K^2, & k_1 k_3/K^2, & -ik_1/k \\
k_1 k_2/K^2, & -1 + (k_2/K)^2, & k_2 k_3/K^2, & -ik_2/k \\
k_1 k_3/K^2, & k_2 k_3/K^2, & -1 + (k_3/K)^2, & -ik_3/k \\
i k_1/k, & i k_2/k, & i k_3/k, & -ik/k
\end{pmatrix}
\]
with

\[ K = \sqrt{\xi(k_0 - \kappa)}. \]

\((2s + 1)\) eigenfunctions belonging to the same eigenvalues of \(S^2\) and \(M^2\) span an irreducible subspace. The transformation property of \(\Phi_{s,m,n}(k_\mu, r_\mu)\) is determined essentially by that of spherical harmonics, namely

\[ \Phi'_{s,m,n}(k_\mu', r_\mu') = \sum_{n=0}^{2s+1} a_{n',n} \Phi_{s,m,n}(k_\mu, r_\mu), \]

where \(a_{n',n}\) are the transformation matrix of \(P_s^n(\cos \theta)\) by spatial rotations. The introduction of \(\kappa\) is merely to distinguish \((2s+1)\) independent components of this subspace. As will be easily anticipated, this expresses the freedom of polarization.

As is seen from \((3.8)\), the eigenvalue of the mass exists only when \(s \geq 1\). That is, particles of spin 0 cannot be given in our theory so long as we assume that \(\varphi\) is square integrable and \(U(X_{\mu}, r_\mu)\) is scalar. To obtain particles of spin 0, we must either modify the boundary condition or introduce vector or higher tensor Urmaterie. At the present, we cannot say which is the preferable one. As for the former, the provisional nature of our boundary condition should be stressed. When higher tensor Urmaterie is introduced, on the other hand, the spin of elementary particles becomes a combination of the "intrinsic" and the "orbital" angular momentum of the internal motion of Urmaterie. To take Urmaterie as scalar means to neglect intrinsic part entirely, and it is also very likely that such a simplification is not permissible.

§ 4. The relation between Urmaterie field and local field

The discussions of §2 enable us to define the irreducible Urmaterie field \(U(X_{\mu}, r_\mu; s, m)\) as a simultaneous eigenstate belonging to the specified eigenvalue of \(S^2\) and \(M^2\).

It was our fundamental assumption that \(U(X_{\mu}, r_\mu; s, m)\) describes an elementary particle of spin \(s\) and mass \(m\). In the local theory, however, such particle was described by generalized Dirac's equation or Fierz's equation. Thus, it would be important to investigate the relation between the equations of motion of the irreducible Urmaterie field and the corresponding local equations in clarifying the constitution of the Urmaterie field, or in clarifying the physical meaning of internal coordinates.

\(U(X_{\mu}, r_\mu; s, m)\) is obtained by expanding at first the internal wave function into eigenfunctions of \(S^2\) and \(M^2\)

\[ u(k_\mu, r_\mu) = \sum_{s,m,n} A(s, m; n) \Phi_{s,m,n}(k_\mu, r_\mu), \quad (4.1) \]

and then picking up from them a component belonging to a set of specified value of \(s\) and \(m\);

\[ U(X_{\mu}, r_\mu; s, m) = \{A(s, m; n) \Phi_{s,m,n}(k_\mu, r_\mu) e^{ikX}(dk). \quad (4.2) \]

To make the correspondence to the local theory clear, it is necessary to use a rearranged form of \((4.2)\). Using the explicit form given by \((3.10)\),
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\[ u(k_{\mu}, r_{\mu}) = \sum_{s, m, n} A(k_{\mu}; s, m, n) \]

\[ \times \frac{c(s, n)}{(r_{\mu} - k_{\mu})^{s+3/2}/(k_{\mu} k_{\mu})^{1/2}} e^{i(k_{\mu} r_{\mu})^{1/2}} \]

\[ \times \int \frac{du}{(a_{2n} r_{\mu} + i a_{1n} r_{\mu} \cos\theta + i a_{0n} r_{\mu} \sin\theta)^{s+1/2}}. \]

Picking up coefficients from this expression under the specified values of \( s \) and \( m \), we see that they transform under Lorentz-transformations as a symmetrical tensor of rank \( s \) contragradient to \( r_{\mu} r_{\nu} \cdots \), since \( u(k_{\mu}, r_{\mu}) \) is scalar. Denoting it as \( A_{\lambda, \mu, \nu, \cdots}(k_{\mu}; s, m) \) to make this character explicit, it is given by

\[ A_{\lambda, \mu, \nu, \cdots}(k_{\mu}; s, m) = \]

\[ \sum_{s, m, n} \int \frac{dnu}{(a_{2n} + ia_{1n} \cos\theta + ia_{0n} \sin\theta)^{s+1/2}}. \]

\( \alpha, \beta, \gamma \) and \( \delta \) denote the numbers of \( 1, 2, 3 \) and \( 4 \) respectively appearing in suffixes of \( A_{\lambda, \mu, \nu, \cdots}(k_{\mu}; s, m) \).

It is clear that \( A_{\lambda, \mu, \nu, \cdots}(k_{\mu}; s, m) \) must satisfy \( \{(s+1)(s+2)(s+3)/6 - (2s+1)\} \) subsidiary conditions. This is because \( (s+1)(s+2)(s+3)/6 \) components appear under \( A_{\lambda, \mu, \nu, \cdots}(k_{\mu}; s, m) \), whereas the number of independent component of \( A_{\lambda, \mu, \nu, \cdots}(k_{\mu}; s, m) \) is only \( (2s+1) \). The explicit form of these subsidiary conditions are given after some calculations, and the results are*

\[ A_{\lambda, \mu, \nu, \cdots}(k_{\mu}; s, m) = 0 \]

\[ k_{\lambda} A_{\lambda, \mu, \nu, \cdots}(k_{\mu}; s, m) = 0. \]  \hspace{1cm} (4.5)**

It is clear on the other hand that (2.29) is equivalent to the following equations for \( A_{\lambda, \mu, \nu, \cdots}(k_{\mu}; s, m) \);

\[ (k_{\mu}^2 - (m/\hbar)^2) A_{\lambda, \mu, \nu, \cdots}(k_{\mu}; s, m) = 0. \]  \hspace{1cm} (4.6)

Eqs. (4.5) and (4.6) are nothing but the equations of motion and subsidiary conditions for elementary particle of spin and mass \( m(\hbar/c) \) as given by Fierz. Therefore \( A_{\lambda, \mu, \nu, \cdots}(k_{\mu}; s, m) \) can be regarded as a local field for particle of spin \( s \) and mass \( m(\hbar/c) \).

* For details of the calculation, see Appendix III.

** As will be seen through the calculations of Appendix III, (4.5) is a direct consequence of the fact that the eigenfunction of \( S^2 \) is essentially spherical harmonics, and therefore is closely related to the rotation of a rigid sphere. In this point, too, the intuitive image given in the non-local field theory is very satisfactory.
The above result shows that the Urmaterie field is equal to the superposition of local fields with various spins and rest masses, the values of which are mediated by the internal motion described by eigenfunctions accompanying to each of them. Such constitution of the Urmaterie field suggests a way for its quantization, since the quantization is a procedure to reproduce particle aspect from that of the wave, and particles that appear in our observation seem to have definite spin and rest mass at least in the case of no interaction. Thus, the quantization of the Urmaterie field may be achieved by quantizing those parts of it which correspond to local fields. The eigenfunctions of the internal motion will, on the other hand, be responsible to the law of interaction between such local fields. A speculative discussion of it is given in § 6.

§ 5. Survey over the interaction of Urmaterie

Thus far we have dealt exclusively with the case of no interaction, and focussed our attention on the understanding of individual elementary particle.

We shall try to see in this section what a new scope is expected when we introduce the interaction of Urmaterie. In particular we want to see to what extent our theory is expected to succeed in elucidating the structure of the interaction of elementary particles, or in dissolving divergence difficulties. Unfortunately, satisfactory theory of the interaction of non-local fields has not been given, and therefore discussions in this section are restricted to very provisional ones. What is attempted here is to see the qualitative feature of these problems that can be seen without entering upon details of the formalism of treating the interaction of non-local fields.

The structure of the interaction of elementary particles is a problem recently proposed by Sakata\(^\text{21)\)}, who emphasized the importance of elucidating the qualitative difference as well as the intimate relationships between various kinds of interactions in nature. As the first step to attack this within the framework of the local theory, he suggested "the principle of renormalizability", and he and his collaborators classified the interactions into the first kind for which the renormalization procedure can be performed in a closed form, and the second for which this is not the case. Their work played an important role in clarifying the limit of the applicability of the present theory.

It is clear, however, that this work must go beyond such phenomenological stage, and in particular recent experiments seem to suggest strongly the necessity of it. A remarkable example is the establishment of the universal Fermi interaction. Although recent data show that the coupling constants of various direct Fermi interactions are nearly equal, in the current theory the introduction of the mutual interaction is done entirely ad hoc, and this remarkable fact can not help being regarded as accidental.

It will be shown that, by introducing the interaction in the form of the interaction of Urmaterie, a way of introducing various interactions in a unified way and of deducing such relationship is suggested in our theory.

As the formalism of treating the interactions of non-local fields we assume Yukawa's S-matrix\(^\text{22)\)}, and investigate the interaction of Urmaterie by assuming appropriate interaction...
Lagrangian density and therefrom constructing $S$-matrix according to him. Yukawa's $S$-matrix is given by

$$S = 1 + i\langle L \rangle + i^2\langle LD_L L \rangle + \cdots ,$$

(5.1)

the notations being the same with those used by him. As the interaction Lagrangian density we assume*,**

$$L = g\bar{\psi} O_{\mu \nu} \ldots \phi \cdot \bar{\psi} O_{\mu \nu} \ldots \phi ,$$

(5.2)

where $\psi$ is a spinor non-local field describing spinor Urmaterie, and $O_{\mu \nu} \ldots$ some Dirac matrices whose explicit form is, for example, one of Bethe's five covariants of $\beta$-interaction. As discussed in § 4, $\psi$ is a superposition of local fields with various spins and rest masses

$$(x^\mu | \psi | x'^\mu) = \sum_{s, m} \psi_{s, m} (X^\mu) \varphi_{s, m} (r^\mu) ,$$

(5.3)

where $\varphi_{s, m} (r^\mu)$ is the eigenfunction of the spin and the mass operators which compose the form factor of this interaction as will be seen below. Substituting (5.3) into (5.2), (5.2) is written in the form

$$L = g \sum_{s, m, l, \ldots, l', m', \ldots} \bar{\psi}_{s, m} (X^\mu) O_{\mu \nu} \ldots \psi_{s', m'} (X'^\mu) \times \bar{\psi}_{s', m'} (X'^\mu) O_{\nu \nu} \ldots \psi_{s'', m''} (X''^\mu) \times (\text{form factors}).$$

(5.4)

(5.4) represents various interactions of Fermi particles according to which eigenvalues of the spin and the rest mass Urmaterie takes. It gives, for example, the interaction of the nucleon and the electron, of the nucleon and the $\mu$ meson, or of the $\mu$ meson and the electron, ... according to the eigenvalue of $s$ and $m$. Here an important fact is that they all appear with the same coupling constant $g$. This explains at once the equality of coupling constants of various Fermi interactions, since form factors are expected to reduce approximately to $\delta$-functions in low energy region. (By low energy region we mean energy region below about $(\hbar c/\lambda)$ 1-Bev corresponding to $\lambda$ taken of the order of the compton wave length of the nucleon.)

Thus, the universal Fermi interaction follows as an immediate consequence of introducing the interaction in the form of the interaction of Urmaterie. We hope that our theory thus might serve as a first step to the substantialistic study of the structure of the interaction of elementary particles.

Next we go over to divergence difficulties. This has been one of the most serious difficulties of the quantum theory of the wave field, and many attempts have been done for it.

---

* Of course the interaction Lagrangian density must be so chosen that it satisfies required conservation laws. What kind of conservation laws should be satisfied can not be decided at the beginning. As the least requirement, however, it would be necessary that it should satisfy the conservation law of the energy and momentum, of the electric charge, and of the heavy particles. These requirements are satisfied by imposing invariance under translations and gauge transformations performed in a relation with electric and with mesic charge. As for mesic charge see also the discussion of § 6.

** Strictly speaking, (5.2) should not be called interaction Lagrangian density since all $\psi$ appearing there refer to the same Urmaterie. It should be regarded as a sort of self-stress.
Although remarkable progress has been made by the idea of renormalization, it seems a general feeling of recent years that the satisfactory solution of this problem would not be obtained within the framework of the local theory. The theory of non-local interaction was then proposed, and in particular Kristensen and Möller\textsuperscript{30} have shown that there is a hope of eliminating all divergences if a suitable form is taken for the form factor. The theory of the non-local interaction is thus promising in as much as this point is concerned, but it is clear that this theory has a serious limitation in that it does not give the principle of determining the form of form factors. If we could find such principle, therefore, it would mean a great advance.

It was shown by many authors\textsuperscript{10, 31, 32} that the interaction of non-local fields leads to non-local interactions with definite form of form factors. Of course these conclusions cannot be definite at the present when reliable theory of treating the interaction of non-local field is entirely lacking. But the analysis of §4 suggests that this is a very general feature of the interaction of non-local fields, and a way of overcoming the above limitation of the theory of the non-local interactions may be found from an approach of this line.

It must of course be stressed that discussions in this section should not be regarded more than an optimistic conjecture. But two problems quoted here are the most direct evidences of the limitation of the present theory, and it seems to us not meaningless that at least a clue of overcoming them is suggested by introducing the interaction in the form of the interaction of Urmaterie.

§6. The deduction of particle family

The conservation of heavy particles recently emphasized by Oneda\textsuperscript{15} is certainly one of the most important problems to be solved in the theory of the interaction of elementary particles.

For it theoretical explanations have been proposed by many authors under the guiding principle of restricting the type of the interactions by assuming the invariance of the theory against possible transformations\textsuperscript{24, 25, 26} such as the charge conjugation or the time reversal. Of course such attempts of seeking for selection rules within the framework of the current theory are orthodox ones.

But it seems to us that this is a problem to be understood in a relation with a more intrinsic structure of elementary particles. The fact that such disconnected families exist seems to suggest that all Fermi-particles are classified into two families by some unknown structure constants, and the selection rules concerning with this new degrees of freedom play important roles by their mutual transformations. From this view point, the conservation of heavy particles seems to be an evidence for the incompleteness of the definition of elementary particles in the present theory. We shall shown in this section that the existence of such new structure constant follows very naturally if we introduce Urmaterie which is described by a spinor non-local field.

Starting with a non-local spinor field \( \phi_{\rho}(X_\mu, r_\nu) \) \((\rho = 1, 2, 3, 4)\) which describes the spinor Urmaterie, we assume that the spin and the mass operators are defined in an
The transformation law of \( \phi_{\mu}(X_{\mu}, r_{\mu}) \) is of course given by
\[
\phi'_{\mu}(X'_{\mu}, r'_{\mu}) = \gamma_{\nu} \phi_{\nu}(X_{\mu}, r_{\mu})
\]
\[
X'_{\mu} = a_{\mu\nu} X_{\nu}, \quad r'_{\mu} = a_{\mu\nu} r_{\nu},
\]
and explicit form of \( \gamma \) is given by
\[
\gamma = \cos \theta/2 - \alpha_i \alpha_j \sin \theta/2
\]
\[
\bar{\gamma} = \cos \theta/2 + \alpha_i \alpha_j \sin \theta/2
\]
for the special rotation in \((x_i x_j)\) plane \((i, j = 1, 2, 3)\)
\[
x'_{i} = x_{i} \cos \theta + x_{j} \sin \theta
\]
\[
x'_{j} = -x_{i} \sin \theta + x_{j} \cos \theta
\]
and
\[
\gamma = \bar{\gamma} = \cos \theta/2 - \alpha_i \sinh \theta/2
\]
for the translation
\[
x'_{0} = x_{0} \cosh \theta - x_{i} \sinh \theta
\]
\[
x'_{i} = -x_{0} \sinh \theta + x_{i} \cosh \theta
\]
\[
\theta = \log \sqrt{c + \theta}
\]
Thus, we have
\[
R_{ij} = (L + 1/2 \cdot \sigma) \quad (i, j, k \text{ cyclic}),
\]
and
\[
R_{\mu \rho} = (\mu + 1/2 i \cdot \rho \cdot \sigma) \quad \text{cyclic},
\]
where \( L \) and \( \mu \) are given by \((2 \cdot 13)\), and \( \rho \) and \( \sigma \) are usual Dirac matrices. Definitions of \( S^2 \) and \( M^2 \) are made in an analogous way as that of the scalar, and separating \( R_{\mu \nu} R_{\mu \nu} \) into \( S^2 \) and \( M^2 \)
\[
1/2 \cdot R_{\mu \nu} R_{\mu \nu} = S^2 - M^2
\]
we can show at once the commutability of \( S^2 \) and \( M^2 \).**S^2 and \( M^2 \) are to be interpreted as the spin and the mass operators for this Urmaterie. In the rest system of the center of mass, \( S^2 \) reduces to

* Here we introduced \( L \) to distinguish the "orbital part" of the spin. This is what was written as \( S \) in § 2.

** The definition of \( R_{\mu \nu} \) is according to Yennie;\(^{27}\)
\[
(i/2) \varepsilon_{\mu \nu} R_{\mu \nu} \phi_{\mu}(X_{\mu}, r_{\mu}) = \phi'_{\mu}(X_{\mu}, r_{\mu}) - \phi_{\mu}(X_{\mu}, r_{\mu})
\]

*** The proof is given in Appendix IV. (b).
Thus, the spin is given in this case as the sum of the "orbital" and the "intrinsic" angular momentum of the internal motion of spinor Urmaterie.

As for the eigenvalue of the mass operator, we assume as before that it gives the rest mass of elementary particles measured in unit \((\hbar/c)\), where of course an equation analogous to (2.22) is assumed;

\[(r_\mu r_\mu - \vec{r}^2)\psi(X_\mu, r_\mu) = 0.\]

Writing \(M^2\) in terms of polar coordinate introduced in (2.26),

\[M^2 = - (1 + x^a) d^a/dx^a - 2x^a/dx^a - x^a \Omega + 3/4 + x_\mu (\sigma p),\]

where \(\rho_\mu\) are "momentum of the internal motion" conjugate to \(r_\mu\). With these spin and mass operators, and proceeding in an exactly analogous way as that of the scalar case, a unified description of particles of spin half integer is given.*

If \(S^2\) and \(M^2\) compose a complete set of mutually commuting operators with respect to internal coordinates, the specification of elementary particles with the spin and the rest mass is complete, and no other structure constant appears. If, however, another scalar operator that commutes with them exists, it means that the elementary particles possess a structure constant other than the spin and the rest mass.

In the case of the spinor, such an operator is provided by contracting \(R_{\mu \nu}\) with its dual tensor \(\bar{R}_{\mu \nu}\). Explicitly written,

\[R_{\mu \nu} \bar{R}_{\mu \nu} = (L + 1/2 \cdot \sigma) (\mu + 1/(2i) \cdot \rho_\mu \sigma),\]

which is simplified to

\[R_{\mu \nu} \bar{R}_{\mu \nu} = -i \rho_\mu/2 \cdot (i \rho_\mu (\mu \sigma) + (\bar{\sigma} \sigma) + 3/2),\]

since we can show by direct calculations that \((L, \mu)\) vanishes. It can be further shown that (see Appendix IV (d))

\[\begin{align*}
[S^2, R_{\mu \nu} \bar{R}_{\mu \nu}] &= 0 \\
[M^2, R_{\mu \nu} \bar{R}_{\mu \nu}] &= 0
\end{align*}\]

Thus, the states of the internal motion which are specified by the eigenvalues of \(S^2\) and \(M^2\) is always degenerated with respect \(R_{\mu \nu} \bar{R}_{\mu \nu}\). Some parts of \(R_{\mu \nu} \bar{R}_{\mu \nu}\) are not independent of \(S^2\) and \(M^2\), and omitting such irrelevant term, we are left with

\[\theta = \rho_\mu.\]

* It should be noted that \(R_{\phi}\) introduced in (6.7) is not hermitic, therefore the mass operator is also not hermitic in spinor case. This is because the representation of the Lorentz group by spinor is not unitary. A way of avoiding this difficulty is to use unitary trick, which consists in replacing \(x_\phi\) by \(ix_\phi\). In this case, however, the square integrability of eigenfunctions is violated. These difficulties are overcome only by introducing quantity like expandor recently introduced by Dirac[19]. Its actual study will be reported elsewhere. What is intended in this section is to give the simplest model needed to understand better the concept of particle family.
Although $\rho_1$ is pseudoscalar, it is easy to construct a scalar operator that inherits the essential feature of it, and we disregard this odd character in the following to simplify the discussion.

$\Theta$ has eigenvalues $\pm 1$. As will be seen from (6.10), $\Theta$ is closely related to the eigenvalue of $m$, and in classifying all spinor particles into two families according to this eigenvalue, the eigenvalue $\pm 1$ determine the minimum value of the rest mass appearing in these two families. Although eq. (6.10) has not yet been solved exactly, a preliminary estimation treating $\chi p (\sigma \mathbf{p})$ as a small perturbation yields

$$m_{\text{min}}^2 \approx \begin{cases} 5 & \text{for eigenvalue of } \Theta = 1 \\ 0 & \text{"", } \Theta = -1 \end{cases} \quad (6.15)$$

As was discussed before, $m$ represents the rest mass of elementary particles measured in unit ($\hbar/c\lambda$). Therefore, by taking $\lambda$ to be of the order of the Compton wave length of the nucleon, this mass separation becomes comparable to that of the nucleon and lepton families.

Thus, $\Theta$ is interpreted as expressing the intrinsic difference of these two families, and in our theory the classification of Fermi particles into heavy and light follows as an inevitable consequence of introducing spinor Urmatere. This means that in our theory the concept of particle family must be considered as very fundamental. We think that this result may be considered as a remarkable advantage of our theory.

In our theory, the difference of heavy and light particles is reduced to that of $\Theta$. Therefore, the conservation of $\Theta$ leads at once to the conservation of both families. To find a reason of assuming the conservation of $\Theta$, we introduce $\Theta$ defined by

$$\Theta = 1/2 \cdot (1 + \Theta) \quad (6.16)$$

$\Theta$ takes eigenvalues 1 and 0 for the nucleon and lepton family respectively, and just corresponds to $\lambda$ introduced by Oneda. Therefore, it would be natural to interpret it as mesic charge.

Thus being interpreted as mesic charge, it is natural to require its conservation. Formally, this is satisfied by assuming the invariance of the theory under the “gauge transformation” performed in a relation with it;

$$\phi \rightarrow \phi e^{i\Theta} \quad (6.17)$$

---

* A way is to double the components of $\psi_p (X_p, r_p)$ into eight, and to introduce an independent set of Pauli matrices $\sigma$ other than $\rho$ and $\phi$. It seems interesting to identify $\sigma$ to $\tau$ spin, and to try to elucidate a deeper relation between the electric and mesic charge, or between the conservation of the electric and mesic charge.

** (6.15) was obtained using unitary trick. In this case, unperturbed eigenfunctions and boundary conditions differ from those of § 3, and as remarked before, square integrability is violated. But here we do not touch these difficulties.

*** Thus, in our theory, to $\Theta$ is given two fold meanings; it classifies on the one hand the heavy and light particles, and on the other hand mediates the interaction of $\pi$-meson with Fermi particles. We are tempted to consider this fact as an answer to the question why only heavy particles interact strongly with $\pi$-meson.

**** The concept of mesic charge and its conservation was first introduced by Okayama and by Wigner.
where \( \alpha \) is an arbitrary constant. From this requirement follows the conservation of mesic current, which is nothing but the conservation of heavy particles.

§ 7. Summary of results and concluding remarks

It would be convenient to give first a brief summary in clarifying the logical construction of the theory we have developed thus far.

What is the most fundamental in our theory is a substance we named "Urmaterie"; which we assumed is described by a non-local field satisfying the equations of motion

\[
\left( \Box - \frac{M^2}{\lambda^2} \right) U(X, r_\mu) = 0, 
\]

and

\[
(r_\mu r_\nu - \lambda^2) U(X, r_\mu) = 0. 
\]

Various states of the internal motion of Urmaterie are classified by the eigenvalues of a complete set of mutually commuting scalar operators with respect to internal coordinates, two of which can be taken as the spin and the mass operators, and each eigenstate thus classified is assumed to correspond to the elementary particle of definite structure. What determines the structure of Urmaterie is the transformation property of the non-local field used in describing it, and the parameter \( \lambda \) appearing in (7.2). In this way a unified description of elementary particles is given, and the structure of Urmaterie determines the structure of individual elementary particle and the mutual correlation between them. That is, it determines the spectrum of the spin, of the rest mass, or of other structure constants, the structure of interactions, or the form of the form factors when interaction is introduced. For example, by taking \( U(X, r_\mu) \) as scalar, a unified description of Bose particles is given each of which is characterized by two structure constants, the spin and the rest mass, and which are subjected to the spin and the mass spectrum given by (2.28) and (3.8), and by taking it as spinor, that of Fermi-particles which are classified into two families according to mesic charge 1 and 0.

Thus, in our theory, elementary particles are regarded as phenomenal forms of Urmaterie. That the definition of elementary particles is made more positive by introducing Urmaterie is clear. For example, in our theory, the answer can be given at once to the question such as, "Is the specification of elementary particles by the spin and the rest mass complete?". This is never a self-evident thing, and in fact the discovery of particle families show its incompleteness at least in the case of Fermi particles. It may be a remarkable advantage of our theory that it can not only give answer to these questions, but also succeed in deducing just the required new structure constant.

In addition, the possibility of deducing the spectrum of various structure constants, the structure of the interactions, or of the form of form factors must also be stressed. Note that, as discussed in § 2, these are the most direct evidences of the limitation of the present theory.

Of course our theory is only a starting point for the unified theory and is far from satisfactory goal. The most serious limitation of our theory is the lack of law governing the internal motion. Keenly we feel that the attention of the future theory must be
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focussed on elucidating it. The use of Yukawa’s non-local field must also be criticized. Although his non-local field is a very elegant, probably the most elegant way of introducing internal coordinates known up to the present, there is no inevitable reason to use it at least at the present stage, and we can not rule out the possibility that the use of Yukawa’s non-local field impose a severe restriction to the whole of our theory.

In conclusion, it would be important to make clear the position of our theory in the course of the development of the theory of elementary particles.

The most essential point of our discussion was that the moment to the further development would only be found in more positively defining elementary particles. We regarded the present theory as "phenomenological", and it was as the first step to overcome this limitation that we introduced Urmaterie. In this sense, our theory may be regarded as "substantialistic" to the present theory, and at the same time as "phenomenological" to the unified theory we are aiming at the next stage*.

In fact, such limitation of our theory appears already in following forms. The first, our theory gives no answer to questions such as "What kind of Urmaterie should we take?", or "What kind of interaction should we introduce between them?". The second, the introduction of spinor Urmaterie as we did in § 6 means to introduce elements not completely analyzable in our stage.

At the present, we can say nothing to these questions. But it is our strong feeling that a series of facts we quoted in this paper as the evidences of the limitation of the present theory are problems not to be attacked separately, but should be regarded all as suggesting the necessity of introducing a substance of higher level. In this connection, we hope that our theory might play some role as the first step toward an approach of such line.

Acknowledgement

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Appendix I

A detailed discussion of solving (3·5) is given here. In order to get a solution defined in the interval $0 < z < 1$, we put

$$(y-1)/y = z$$

$$u = (1-z)^v$$

* In this paper we disregarded the degree of freedom concerning with charge states entirely. Of course this degree of freedom must be taken in the work of next approximation, and in particular the amalgamation of our theory with that of Pais\(^9\) seems very promising.
Then (3·5) turns into

\[ s(1-z) v'' + \{(\alpha + \beta + 1 - \gamma) - (\alpha + \alpha + 1 - \gamma + 1) \} v' - \alpha(\alpha + 1 - \gamma) v = 0, \]

where \( y ; 1 \to \infty \) corresponds to \( z ; 0 \to 1 \).

Expanding \( v \) into power series

\[ v = \sum_{k} a_k z^k, \]

we get two independent solutions in terms of hypergeometric series

\[ v_0 = F(\alpha, \alpha + 1 - \gamma, \alpha + \beta + 1 - \gamma, z) \quad \text{for } \rho = 0 \]
\[ v_1 = s^{1-\alpha-\beta} F(\gamma - \beta, 1 - \beta, \gamma + 1 - \alpha - \beta, z) \quad \text{for } \rho = \gamma - \alpha - \beta = 1/2, \]

both of which are convergent in the interval \( 0 < z < 1 \).

Thus, the general solution of (3·3) is given by

\[ \varphi = A \varphi_0 + B \varphi_1, \]

where

\[ \varphi_0 = (1 - z)^{1/2} \sum_{s=0}^{\infty} s^{s+1+\ast} F(\alpha, \alpha + 1 - \gamma, \alpha + \beta + 1 - \gamma, z) \]
\[ \varphi_1 = (1 - z)^{1/2} \sum_{s=0}^{\infty} z^{1-\alpha-\beta} F(\gamma - \beta, 1 - \beta, \gamma + 1 - \alpha - \beta, z), \]

and \( A \) and \( B \) arbitrary constants. These solutions must satisfy the condition of square integrability

\[ \int_{-\infty}^{\infty} |\varphi|^2 \sqrt{1 + x^2} dx = \text{finite}, \]

where \( \sqrt{1 + x^2} \) is the Jacobian that appeared in transforming variable from \( \theta \) into \( \varphi \) and \( x \).

In order that this condition be satisfied, the following is necessary and sufficient.

i) \( \varphi = \text{finite} \) at \( z = 0 \),

and

ii) \( \varphi = 0 \) at \( z = 1 \).

For i) it is easily seen that the condition is automatically satisfied since

\[ \gamma - \alpha - \beta = 1/2 > 0, \]

but for ii) a detailed discussion is needed. It is because although \( (1 - z)^{s+1+\ast} \) tends to zero since

\[ \frac{1}{2} (s+1) + \alpha = \frac{1}{2} + \frac{\sqrt{s(s+1) + 1 - m^2}}{2} > 0, \]

another factors \( F(\alpha, \alpha + 1 - \gamma, \alpha + \beta + 1 - \gamma, z) \) and \( F(\gamma - \beta, 1 - \beta, \gamma + 1 - \alpha - \beta, z) \) diverge.

The following identity is useful to investigate the behavior of \( \varphi_0 \) and \( \varphi_1 \) near \( z = 1 \);

\[ F(\alpha', \beta', \gamma', z) \equiv (1 - z)^{\gamma' - \alpha' - \gamma'} F(-\alpha' + \gamma', \gamma' - \beta', \gamma' + 1 - \alpha' - \beta', z). \]
Using this, (AI.1) is expressed as
\[
\varphi_0 = (1 - z)^{\frac{1}{2}(a+1)} F(\beta + 1 - \gamma, \beta, \alpha + \beta + 1 - \gamma, z)
\]
\[
\varphi_1 = (1 - z)^{\frac{1}{2}(a+1)} z^{-a-\delta} F(1 - \alpha, \gamma - \alpha, \gamma + 1 - \alpha - \beta, z)
\]
\[
(\text{AI}.2)
\]
Both \(F\) appearing in the right hand side of (AI.2) being finite at \(z=1\) on account of

\[
(\alpha + \beta + 1 - \gamma) - (\beta + \beta + 1 - \gamma) = \alpha - \beta = \sqrt{s(s+1)} + 1 - m^2 > 0
\]

and

\[
(\gamma + 1 - \alpha - \beta) - (\gamma - \alpha + 1 - \alpha) = \alpha - \beta > 0,
\]

the behavior of \(\varphi_0\) and \(\varphi_1\) near \(z=1\) is determined completely by whether the power index of \((1-z)\) is positive or is negative.

Thus, for \(m^2 \geq s(s+1)\), namely for

\[
Re\left[ \frac{s+1}{2} + \beta \right] = Re\left[ \frac{1}{2} (1 - \sqrt{s(s+1)} + 1 - m^2) \right] \geq 0
\]

both \(\varphi_0\) and \(\varphi_1\) converge. This yields continuous spectrum.

For \(m^2 < s(s+1)\), on the other hand, \((1-z)^{\frac{1}{2}(a+1)}\) diverges at \(z=1\). In order to get allowable solutions in this case, therefore, we must choose \(F\) in (AI.2) so that it tends to zero at \(z=1\). This means that we must choose \(F\) in (AI.1) so that it breaks at finite terms. Then we meet following two case;

i) For \(\varphi_0\)
\(\alpha\) must be negative integer;
\[
\alpha = \frac{1}{2} \{ - s + \sqrt{s(s+1)} + 1 - m^2 \} = - n,
\]
\[
n = 0, 1, 2, \ldots < \frac{s}{2}.
\]
Then the eigenvalue of the mass is given by
\[
m^2_{\varphi_0,n} = s + 1 + 4ns - 4n^2,
\]
with the corresponding eigenfunction
\[
\varphi_{\varphi_0,n} = (1 + s^2)^{\frac{-n-\beta+1}{2}} F\left( n, s - n + \frac{1}{2}, 1 + \frac{1}{2}, 1 + s^2 \right).
\]

ii) For \(\varphi_1\)
\(\gamma - \beta\) must be negative integer.
In this case the eigenvalue of the mass is given by
\[
m^2_{\varphi_1,n} = 3s + 4ns - 4n - 4n^2
\]
\[
n = 0, 1, 2, \ldots < \frac{s - 1}{2},
\]
with the corresponding eigenfunction

\[ \varphi_{s,n} = (1 + x^2)^{s-\gamma-n-\frac{1}{2}} x F\left(-n, s-n+\frac{1}{2}, \frac{3}{2}, \frac{x^2}{1+x^2}\right). \]

Other cases, for example, \(1 - \beta = -n\), or \(\alpha + 1 - \gamma = -n\) is not permitted, since it contradicts to \(m^2 < s(s+1)\). Moreover it is impossible that both \(\varphi_0\) and \(\varphi_1\) become eigenfunctions belonging to the same value of mass. This is easily seen from the fact that for \(\alpha = -n\), neither \(\gamma - \beta\) nor \(1 - \beta\) can be negative integer.

Above two cases are expressed in a compact form, if we introduce another parameter which is given by \(2n\) for case i) and \((2n+1)\) for case ii); Then

\[ m^2 = s+1 + 2s - l^2, \quad l = 0, 1, 2 \cdots < s, \]  

(AI·1)

and the eigenfunction is

\[ \varphi_{s,l} = (1 + x^2)^{s-l+1} \left(x^2 \frac{1}{1+x^2}\right)^{\frac{l}{2}} x F\left(-l, s+1 - l + \varepsilon, \frac{1}{2} + 2\varepsilon, \frac{x^2}{1+x^2}\right) \equiv (1 + x^2)^{s-l+1} \left(x^2 \frac{1}{1+x^2}\right)^{\frac{l}{2} - \varepsilon} G_{\frac{l}{2} - \varepsilon}^{\frac{1}{2} + \varepsilon} \left(s + \frac{1}{2} + 2\varepsilon, \frac{x^2}{1+x^2}\right). \]  

(AI·4)

where \(G_{\frac{l}{2} - \varepsilon}\) is the Jacobi's polynomial of \((l/2 - \varepsilon)\)th order and

\[ \varepsilon = \begin{cases} 
0 & \text{for even} \\
1/2 & \text{for odd.} 
\end{cases} \]

Although it seems apparently possible that the solution of the form \(\varphi = (A\varphi_0 + B\varphi_1)\) can tend to zero at \(x = \pm \infty\) even in the case when both \(\varphi_0\) and \(\varphi_1\) diverges (since their order of divergence is the same), we can see at once that this does not occur. It is because \(\varphi_0\) is an even and \(\varphi_1\) is an odd function of \(x\) as will be easily seen from (AI·1).

From it we can conclude at once that constants \(A\) and \(B\), non of which is zero, satisfying

\[ A\varphi_0(\infty) + B\varphi_1(\infty) = 0 \]

\[ A\varphi_0(-\infty) + B\varphi_1(-\infty) = A\varphi_0(\infty) - B\varphi_1(\infty) = 0 \]

do not exist.

Next we examine the orthogonality of eigenfunctions given by (AI·4). The first equation (3.5) can be written in a self-adjoint form

\[ \left( (1 + x^2)^{s/2} \varphi' \right)' - \frac{x^2}{\sqrt{1+x^2}} s(s+1) \varphi + m^2 \sqrt{1+x^2} \varphi = 0 \]

Then we get for the same value of \(s\)

\[ (m^2_{n,k} - m^2_{n,l}) \int_{-\infty}^{\infty} \varphi_{s,k} \varphi_{s,l} \sqrt{1+x^2} dx = \left[(1 + x^2)^{s/2} \left\{ \varphi_{s,k} \varphi_{s,l} - \varphi_{s,l} \varphi_{s,k} \right\} \right]_{-\infty}^{\infty}. \]

Substituting explicit form of \(\varphi\), we find at once
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\[ \psi_{s,t}^* = 2x^2 \left( \frac{1 - s - n + 1}{2} \right) (1 + x^2)^{-1} + \mathcal{C} \left( \frac{x^2}{1 + x^2} \right)^{-1} \frac{1}{1 + x^2} \psi_{s,t} \]

\[ + (1 + x^2)^{s-1} \left( \frac{x^2}{1 + x^2} \right)^{\frac{s-1}{2}} F \left( -\frac{l}{2} + \epsilon + 1, s + \frac{3}{2} - \frac{l}{2} + \epsilon, \frac{3}{2} + 2\epsilon, \frac{x^2}{1 + x^2} \right) \times \frac{2x}{1 + x^2} \]

and noticing that in the third term \( F \) does not contribute to divergence since it is a polynomial in \( x^2/(1 + x^2) \) and is finite everywhere, we see that the term that diverges most weakly in limits \( x \to \pm \infty \) is the first, which is of the order of \( 1/x \). Hence, putting,

\[ \psi_{s,t} = O(1/x) \psi_{s,t}, \]

we obtain

\[ (1 + x^2)^{s-\frac{3}{2}} \{ \psi_{s,t}^* \psi_{s,t} - \psi_{s,t}^\prime \psi_{s,t} \} \sim (1 + x^2)^{-1} (1 + x^2)^{-\frac{3}{2}} \times O(1/x) \times (1 + x^2)^{-\frac{3}{2}}. \]

The first term is \( O\{ (1/x)^{2(s+1)-(l+k)} \} \), the second \( O(1/x) \), and the third is \( O\{ (1/x)^{1} \} \), from which we get as the orthogonality condition for the same \( s \) and different \( l, k \),

\[ 2(s+1) - (l+k) + 1 - 3 > 0, \]

or

\[ 2s > l + k. \]

This condition is satisfied by \( s > l \), and \( s > k \), which was just the condition of square integrability.

Appendix II. Group theoretical investigation of the eigenvalue equation*

The eigenvalue problem for the mass spectrum can be solved group-theoretically. The result agrees with that of §3 as it should be, but it is useful in clarifying the mathematical feature of the problem.

The internal eigenfunction solved under the boundary condition of square integrability transform under Lorentz transformations according to a unitary representation of the Lorentz group. The unitary representation of the Lorentz group was studied in detail by Wigner\(^{33}\) and by Bargmann\(^{80}\) and in particular Bargmann gave an explicit solution of it. According to him, there are two such irreducible representations labeled by two parameters \( \xi \) and \( \eta \), which are eigenvalues of \( \xi \) and \( \eta \) of (3.9) respectively.

i) \( \xi = 0 \) and \( \eta \) is any positive number. In this case \( s \) takes all values of 0, 1, 2, \( \cdots \), where \( s \) is the eigenvalue of \( R_{ij} R_{ij}; \frac{1}{2} R_{ij} R_{ij} = s(s+1). \)

* The content of this appendix is due to Mr. Murai\(^{30}\).
ii) $\xi$ is any real number, and $\eta = 1 - k^2 + (\xi/k)^2$.

In this case $s = k, k+1, k+2, \ldots$ with

$k = 1/2, 1, 3/2, \ldots$

The representation space is a Hilbert space defined on the surface of a unit sphere except for $0 < \eta < 1$ of case i).

Our discrete case corresponds to $\xi = 0$, and $s =$ integer of ii)*. Thus, required eigenvalue is given by

$$M^2 = s(s+1) + 1 - k^2$$

$k = 1, 2, 3, \ldots$

$s = k + 1, k+2, k+3, \ldots$

or, if $s$ is specified first,

$$M^2 = s(s+1) + 1 - (s-l)^2$$

$s \geq 1, l = 0, 2, \ldots, s-1,$

which agrees with (3.8) completely.

Appendix III

In this appendix the calculations outlined in § 3 will be explained in details.

We first construct eigenfunctions of the mass and the spin operators in arbitrary reference system, and then derive Fierz's subsidiary conditions for each decomposed parts of the Urmaterie field.

In what follows quantities referring to the rest system of the external motion will be distinguished by primes. Thus, the eigenfunction belonging to the eigenvalue $s$ and $m$ of the spin and mass operators satisfies

$$S^m \Phi_{s,m,n} = s(s+1) \Phi_{s,m,n}$$

and

$$M^2 \Phi'_{s,m,n} = m^2 \Phi'_{s,m,n}$$

where

$$\Phi'_{s,m,n} = \varphi_{s,m}(x') P_s^n (\cos \theta') e^{i\phi'}; \ x' = \sinh \xi'.$$

The explicit form of $\varphi_{s,m}$ is given in Appendix I. (See eq. (AIV.4).) $S^3$ and $M^3$ being invariant under Lorentz-transformations, required eigenfunctions can be obtained by generalizing $\Phi'_{s,m,n}$ into an invariant form. This can be done at once with the help of the formula

* That $\kappa_{\mu\nu} \kappa_{\mu\nu}$ vanishes can be seen at once, for example, by taking $r_\mu = (r_1, 0, 0, 0)$. The appearance of such restriction is a little queer. We think that this related to the use of Yukawa's second equation, which stipulates $r_\mu$ to be space-like. A fuller study of it will appear elsewhere.
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\[ P_s^n (\cos \theta) e^{in\phi} = \frac{e(s, n)}{r^2} \int_{-\infty}^{\infty} du (r_3 + ir_1 \cos u + ir_2 \sin u) e^{inu}. \]  

(AlII. 2)

The result is

\[ \Phi_{s,m,n}(k_{\mu}, r_{\mu}) = \frac{e(s, n)}{r^2} \left( \frac{1}{k_{\mu} k_{\mu}} \right)^{1/2} \sum_{s'} \frac{\eta_{s'}(s'+1)}{\eta_{s}(s+1) \eta_{s'}(s'+1)} \left( \frac{k_{\mu} r_{\mu}}{-\lambda^2 r_{\mu}^2} \right)^{1/2} \times \]

\[ \times \int_{-\infty}^{\infty} du (a_{\mu \nu} r_{\mu} + ia_{\mu \nu} r_{\mu} \cos u + ia_{\mu \nu} r_{\mu} \sin u) e^{inu}, \]

where \( a_{\mu \nu} \) are coefficients of Lorentz-transformation given in (3.11); 

\[ a_{\mu \nu} k_{\lambda} = \begin{cases} 0 & \text{for } \mu = 1, 2, 3, \\ \text{im}(\hbar/c) & \text{for } \mu = 4, \end{cases} \]

and \( s \) and \( m \) are assumed as scalar.

It is clear that \( \Phi_{s,m,n}(k_{\mu}, r_{\mu}) \) reduces to \( \Phi_{s',m,n}(k'_{\mu}, r'_{\mu}) \) when written in terms of \( r'_{\mu} \), and therefore composes the complete set of eigenfunctions in arbitrary reference systems;

\[ S^s \Phi_{s,m,n}(k_{\mu}, r_{\mu}) = s(s+1) \Phi_{s,m,n}(k_{\mu}, r_{\mu}) \]

\[ M^m \Phi_{s,m,n}(k_{\mu}, r_{\mu}) = m^2 \Phi_{s,m,n}(k_{\mu}, r_{\mu}). \]

The orthogonality of \( \Phi_{s,m,n}(k_{\mu}, r_{\mu}) \) follows at once from that of \( \Phi_{s',m,n}(k'_{\mu}, r'_{\mu}) \).

As pointed out in §4, \( A_{\lambda_{\mu \nu} ...}(k_{\mu} ; s, m) \) must satisfy \( \{(s+1)(s+2)(s+3)/6 - (2s+1)\} \) subsidiary conditions. We try next to deduce explicit form of these subsidiary conditions. For this purpose we construct

\[ k_{\lambda} A_{\lambda_{\mu \nu} ...}(k_{\mu} ; s, m) = \sum_{|n| \leq s} c(s, n) A(k_{\mu} ; s, m, n) \times \]

\[ \times \int_{-\infty}^{\infty} du e^{inu} \left[ a_{\mu \nu} k_{\mu} + ia_{\mu \nu} k_{\mu} \cos u + ia_{\mu \nu} k_{\mu} \sin u \right] \]

\[ \times \left( a_{31} + ia_{11} \cos u + ia_{21} \sin u \right)^{a-1} \left( a_{32} + ia_{12} \cos u + ia_{22} \sin u \right)^{a} \]

\[ \times \left( a_{33} + ia_{13} \cos u + ia_{23} \sin u \right)^{a} \left( a_{34} + ia_{14} \cos u + ia_{24} \sin u \right)^{a}, \]

\[ \alpha \geq 1, \]

and

\[ A_{\lambda_{\mu \nu} ...}(k_{\mu} ; s, m) = \sum_{|n| \leq s} c(s, n) A(k_{\mu} ; s, m, n) \int_{-\infty}^{\infty} du e^{inu} \times \]

\[ \times \left\{ a_{\nu \mu} a_{\mu \nu} - a_{\mu \nu} a_{\mu \nu} \cos^2 u - a_{\nu \mu} a_{\mu \nu} \sin^2 u \right\} \]

\[ + 2 \left\{ ia_{\nu \mu} a_{\mu \nu} \cos u + ia_{\nu \mu} a_{\mu \nu} \sin u - a_{\nu \mu} a_{\mu \nu} \sin u \cos u \right\} \times \]
\[ x (a_{31} + ia_{11} \cos u + ia_{21} \sin u)^{s-2} (a_{32} + ia_{12} \cos u + ia_{22} \sin u)^s \]
\[ x (a_{33} + ia_{13} \cos u + ia_{23} \sin u)^{s-1} (a_{34} + ia_{14} \cos u + ia_{24} \sin u)^s, \]
\[ \alpha \geq 2. \]

Terms in \([\ldots]\) in the right hand side of (AIII·5) vanish on account of (AIII·4), and those in \([\ldots]\) in the right hand side of (AIII·6) also vanish which can be seen at once from
\[ a_{\alpha \lambda} a_{\nu \lambda} = \delta_{\mu \nu}. \]

Thus, we get
\[ \mathcal{K}_\lambda A_{\lambda \mu_\ldots} (\mathcal{K}_\mu; s, m) = 0, \quad (\text{AIII·7}) \]
and
\[ A_{\lambda \mu_\ldots} (\mathcal{K}_\mu; s, m) = 0. \quad (\text{AIII·8}) \]

(AIII·7) and (AIII·8) are nothing but the subsidiary conditions for particle of spin \(s\) and mass \(m(\hbar/c)\) as given by Fierz.

The number of independent subsidiary conditions resulting from (AIII·7) and (AIII·8) is \(s(s+1)(s+2)/6\) and \(s(s-1)/2\) respectively. This assures that other conditions can never appear.

### Appendix IV

In this appendix details of calculations not touched in the paper is given.

(a) The proof of (2·23)

(22·2) restricts the internal world to the surface of a hyperboloid, and \(u(r_\mu)\) to be of the form
\[ u(r_\mu) = \delta(r_\mu r_\mu' - \mathcal{L}^2) u'(r_\mu), \]
where it should be understood that \(r_\mu'\) in \(u'(r_\mu)\) is restricted by \(r_\mu^2 = \mathcal{L}^2\). This suggests a modification of quantum mechanics we assumed to the internal motion. Our new assumption is that the essential part of the internal wave function is \(u'(r_\mu)\), and the expectation values of various dynamical variables are given by
\[ \left \langle F(r_i, \frac{1}{i} \frac{\partial}{\partial r_i}, \ldots) \right \rangle = \int \delta^\ast (r_\mu r_\mu' - \mathcal{L}^2) \delta (r_\mu r_\mu' - \mathcal{L}^2) \delta (r_\mu n_\mu) \]
\[ \times u'^\ast(r_\mu) F\left(r_i, \frac{1}{i} \frac{\partial}{\partial r_i}, \ldots\right) u'(r_\mu) (d\nu_\mu). \]

This gives for \(m^2\)
\[ m^2 = -\int \delta^\ast (r_\mu r_\mu' - \mathcal{L}^2) \delta (r_\mu r_\mu' - \mathcal{L}^2) \delta (r_0) u'^\ast(r_\mu) \]
\[ \times \left(r_i \frac{\partial}{\partial r_i} + r_0 \frac{\partial}{\partial r_0}\right) u'(r_\mu) (d\nu_\mu), \quad (\text{AIV·1}) \]
where we have put \( n_\mu \) as \((0, 0, 0, i)\). Of terms of \((2.21)\) what remained were \( r \partial / \partial r \) and \( r^2 \partial^2 / \partial r_0^2 \).

The vanishing or \( r \partial / \partial r \) can be shown at once if we rewrite it in terms of polar coordinates introduced in \((2.26)\)

\[
\frac{\partial}{\partial r} = \cosh \xi \frac{\partial}{\partial \Lambda} - \sinh \xi \frac{\partial}{\partial \xi},
\]

(AIV·2)

where \( \Lambda \) should be regarded as variable. An important fact is that \( \Lambda \) does not appear in \( \nu'(r_\mu) \) owing to \((2.22)\). The first term of \((AIV·2)\) therefore vanishes, and the second also vanish by virtue of \( \partial (r_\mu) \) appearing in \((AIV·1)\).

Thus, we are left with

\[
m^2 = -\int \delta^*(r_\mu r_\mu - \vec{\lambda}^2) \delta(r_\mu r_\mu - \vec{\lambda}^2) \delta(r_\mu) u^*(r_\mu) \\
\times r^2 \frac{\partial}{\partial r_0} u'(r_\mu) (dr_\mu).
\]

Here \( r^2 \) can be replaced by \( \vec{\lambda}^2 \), and under the assumption that \( -\partial^2 / \partial r_0^2 \) corresponds to the operator of the square of the energy of the internal motion, we get finally

\[
m^2 = (\hbar / \lambda c)^2 (E_{\text{int}}^2).
\]

(b) Proof of \([M^2, S^2] = 0\).

It is sufficient to prove it in center of mass system since \( M^2 \) and \( S^2 \) are scalar.

1) The case of scalar.

\[
S^2 = S^2_1 + S^2_2 + S^2_3,
\]

and

\[
M^2 = \mu^2_1 + \mu^2_2 + \mu^2_3.
\]

Using the commutability relations in \((2.14)\), we get

\[
[S^2, \sum_i \mu_i^2] = \sum_i \{[S^2, \mu_i^2] + \mu_i^2[S^2, \mu_i^2]\} = \sum_i \{\mu^3_i \mu_j - \mu^3_j \mu_i + \mu^4_i \mu_j - \mu^4_j \mu_i\} = 0.
\]

Hence

\[
[S^2, \sum_i S^2_i, \sum_i \mu_i^2] = 0.
\]

2) The case of spinor.

\[
S^2 = L^2 + (\sigma L) + 3/4,
\]

and

\[
M^2 = \mu^2 - i\rho_4 (\mu \sigma) - 3/4.
\]

Then

\[
[S^2, M^2] = \sum_i \{-i\rho_4 [L, (\mu \sigma)] L_i - i\rho_1 L_0 [L_0, (\mu \sigma)] \}.
\]
\[-\sum \{ [\mu_\lambda (L\sigma)] \mu_\lambda + \mu_\lambda (L\sigma) \}
+ \rho_1 (\sigma, L \times \mu - \mu \times L),\]
terms in the second bracket vanish by virtue of (2.14), and the first is rewritten as
\[\rho_1 (\sigma, \mu \times L - L \times \mu),\]
and compensates with the third.

(c) The proof of \([S^2, r_\mu r_\mu] = 0\) and \([M^2, r_\mu r_\mu] = 0\).
\[[R_{\lambda \lambda}, r_\mu r_\mu] = 1/i \cdot \{ r_\mu [\partial/\partial r_{\lambda \lambda}, r_\mu r_\mu] - r_\lambda [\partial/\partial r_\mu, r_\mu r_\mu] \}
= 1/i \cdot \{ r_\mu r_\mu \delta_{\lambda \mu} - r_\mu r_\mu \delta_{\mu \lambda} \} = 0,\]
thus \(r_\mu r_\mu\) and \(S^2, M^2\) are commutable.

(d) The proof of \([S^2, R_{\mu\nu} R_{\mu\nu}] = 0\).
Again, we can evaluate it in the center of mass system utilizing the scalar character of various operators. Thus,
\[[S^2, R_{\mu\nu} R_{\mu\nu}] = 1/2 \cdot [L^2 + (L\sigma) + 3/4, (\sigma \mu) - i\rho_1 (\sigma L) - 3/2 \cdot i\rho_1]
= 1/2 \cdot \{ [L^2, (\mu \sigma)] + [(L\sigma), (\mu \sigma)] \}
= 1/2 \cdot \{ i(\sigma, \mu \times L - L \times \mu) + i(\sigma, L \times \mu - \mu \times L) \}
= 0.
[[M^2, R_{\mu\nu} R_{\mu\nu}] = 1/2 [\mu^2 - i\rho_1 (\mu \sigma) - 3/4, (\mu \sigma) - i\rho_1 (L\sigma) - 3/2 \cdot i\rho_1]
= 1/2 \cdot \{ [\mu^2, (\mu \sigma)] - [(\mu \sigma), (L\sigma)] \}
= 1/2 \cdot \{ i(\sigma, \mu \times L - L \times \mu) - i(\sigma, \mu \times L - L \times \mu) \}
= 0.

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