On the Renormalization in Tamm-Dancoff Approximation
for One-nucleon Problem, II

--- Subtraction of Divergences in the Generalized
Tamm-Dancoff Equations ---

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(Received July 17, 1954)

The subtraction of the divergences are carried out in the generalized Tamm-Dancoff equations
derived in the previous paper (Part I). The essence of the method is the extension of Fubini's
procedure to the arbitrarily higher order approximation. That is, first, we construct the formal solutions
which satisfy the equations in question; they contain the infinities and accordingly are quite meaningless.
Then, we separate the infinities (including the overlapping divergences) individually at each stage of
the construction of these formal solutions. The forms of the nucleon propagation function or the vertex
parts which have been made convergent by this method depend on the configurations to which they
refer. However, this seems to be inevitable for the present approximation method, i.e. the reduction
of the infinite set of coupled integral equations to the finite one.

§ 1. Introduction

In the previous paper (Part I), we have proposed the generalized Tamm-Dancoff
equations. The next problem is how to subtract the divergences from them. Fubini has
already given the answer to this problem in the approximation where the meson number
(in our sense) is restricted within two. However, since the generalized T-D equations reduce
to the essentially single equation in this order of approximation and the procedure of Fubini
rests on this specially simple circumstance, it is not applicable directly to the higher order
approximations.

That is, in this order of approximation, the starting equations are eqs. (19), (20) and (21) of Part I with

\[ K(p_1 - k_{1a} k_1 k_2 k_3 ; p_{\alpha}, k_{\alpha}) = 0. \]

Substituting eq. (I. 21)) into eq. (I. (20)) and performing the subtraction of the self-energy type divergence caused by this substitution, we get the following equations:

\[ K(p_1 ; p_{\alpha}, k_{\alpha}) = g(2\pi)^{-3} \int dk_1 S_{p1} (p_1) \gamma_s^a K(p_1 - k_1 ; k^a_{\alpha} ; p_{\alpha} k_{\alpha}) , \]

\[ K(p_1, k^a_{\alpha} ; p_{\alpha}, k_{\alpha}) = \delta(p_1 - p_{\alpha}) \delta(k_1 - k_{\alpha}) \delta_{a0} S'_{p1} (p_1) d_{p1} (k_1) \]

\[ + g(2\pi)^{-3} S'_{p1} (p_1) \gamma_s^a d_{p1} (k_1) K(p_1 + k_1 ; p_{\alpha} k_{\alpha}) \]

\[ + g(2\pi)^{-3} S_{p1} (p_1) \gamma_s^a d_{p1} (k_1) K(p_1 - k_1 ; p_{\alpha} k_{\alpha}) \]

\[ + g(2\pi)^{-3} S_{p1} (p_1) \gamma_s^a d_{p1} (k_1) K(p_1 - k_1 ; p_{\alpha} k_{\alpha}) \]
where \( S_p'(\rho_1) \) is the same as that of Fubini (see also § 3). However, the latter equation (2) can be solved (apart from the infinities contained) for \( K(\rho_1; p_0^0, k_0^0) \) without the knowledge of the form of the function \( K(\rho_1; p_0^0, k_0^0) \) because of the fact that the conservation of 4-momenta holds here (c.f. the paper of Fubini). But, in the next higher order approximation, our equations become coupled integral equations for \( K(\rho_1, k_1; p_0^0, k_0^0) \) and \( K(\rho_1, k_1, k_2; p_0^0, k_0^0) \), and the simple circumstance such as described above no longer exists for this case, i.e. the equations for \( K(\rho_1, k_1; p_0^0, k_0^0) \) and \( K(\rho_1, k_1, k_2; p_0^0, k_0^0) \) are essentially coupled ones. Accordingly, the method of Fubini is not applicable directly to higher order approximations. Moreover, he has mentioned nothing concerning with the subtraction of the overlapping divergences.

The main purpose of this paper is to show how to extend Fubini’s procedure to any higher order approximations. The outline of the method is as follows:

Analogously to the method of Fubini, we find the method for constructing the formal solutions which satisfy the generalized T-D equations in any higher order approximation. These solutions contain the various types of infinities and therefore they are meaningless for themselves. Then we perform the subtraction of those divergences at each stage of constructing the solutions. In this connection, it should be noted that the quantities appearing in our formal solutions generally contain the overlapping divergences in a very complicated form. In this paper, it will also be shown that the convergent parts of them can be consistently defined by the help of the method of Chiba and Tanaka and Ito.

For clearness, first, we investigate the nature of those formal solutions by the help of Feynman-Dyson diagrams (§ 2). Second, we show the method for constructing those solutions in the analytic form (§ 3), and finally perform the subtraction of the divergences (§ 4). For simplicity, these are all investigated in the approximation where the meson “number” is restricted within three. Section 5 will be devoted to the consideration concerning the case of arbitrarily higher order approximation.

§ 2. Consideration with the Feynman-Dyson diagrams

For simplicity, we shall consider the pion-nucleon scattering problem in the approximation where the meson “number” is restricted within three. Because we trace the events along the nucleon line, we shall draw the nucleon line as a straight line from the “initial” stage (below) to the “final” stage (above). Hence the word “meson number at a certain stage” in this article means the number of meson lines which intersect the straight line drawn perpendicularly to the nucleon line at the corresponding stage. As explained in Part I, we keep the order of vertices along the nucleon line as \( (x_0), \xi_1, \xi_2, \ldots, \xi_n(x_0) \), and omit the factor \((n!)^{-1}\) in the defining equations of the Feynman kernels (eqs. (7), (8) and (9) in Part 1). Therefore, we do not distinguish between the diagrams which can be transformed to each other by only changing the name of vertices.
Now, suppose that all the diagrams that contribute to the pion-nucleon scattering process are drawn out. It should be possible to classify these diagrams into 3 kinds as follows:

a) the diagrams that do not possess any intermediate stages of meson number 0 or 1.

b) the diagrams which neither pass any intermediate stages of meson number 0 nor belong to the class a).

c) all the diagrams except those belonging to the classes a) or b).

The assembly of all diagrams belonging to the class a) will represent the whole of the processes in which the meson number varies as

\[ 1 \rightarrow 2 \rightarrow 1 \]

or

\[ 1 \rightarrow 2 \rightarrow 3 \rightarrow 2 \rightarrow \text{(any times of } 2 \rightarrow 3 \rightarrow 2) \rightarrow 1 . \] (3)

In accordance with the notation of § 3, we shall call the whole of these processes as the \( G_2 \)-process and the assembly of the diagrams belonging to class a) as the \( G_2 \)-diagram. In other words, the \( G_2 \)-diagram can be constructed by the following procedure. First, suppose that all the 2-meson scattering diagrams that have no intermediate stage of meson number 0 or 1. We shall name the process corresponding to the assembly of these diagrams as \( R_2 \)-process. Next, close the open end of one meson line to the nucleon line at each of the "initial" and "final" stages in those diagrams (dotted line in Fig. 1). Then the assembly of all the diagrams thus constructed is the \( G_2 \)-diagram.

![Fig. 1 G2-diagram](https://example.com/fig1)

If we had succeeded in solving the equation for the \( R_2 \)-process, it would be very easy to construct the analytic expression of the function which represents the \( G_2 \)-process, i.e. the process corresponding to the class a).

As regards the diagrams of the class b), we redivide these into the subclasses \( b_1 \), \( b_2 \), \( \cdots \), \( b_n \), \( \cdots \), where the subclass \( b_n \) consists of all diagrams that possess \( n \) intermediate stages of meson number 1. It is obvious that we can regard the diagrams of subclass \( b_1 \), as a whole, as consisting of two \( G_2 \)-parts connected by one nucleon and one meson lines (cf. Fig. 2). Similarly, we can summarize the diagrams \( b_n \) into one entity consisting of three \( G_2 \)-parts, and the diagrams \( b_2 \) into that of consisting four \( G_2 \)-parts, and so on. Accordingly, if we sum up all the diagrams belonging to the classes a) and b), we shall get the whole of the diagrams which represent any times of repetition of the \( G_2 \)-process (Fig. 3). We shall call these diagrams belonging to a) or b), as a whole, as \( R_1 \)-diagram and the corresponding process as the \( R_1 \)-process. In other words, the \( R_1 \)-diagram corres-
ponds to the solution of an integral equation which has the $G_x$-function (the function which represents the $G_x$-process) as its kernel, i.e. symbolically

$$R_1 = G_2 + G_2 R_1,$$

Finally, if we divide each diagram belonging to the class c) into two parts at the "last" of the intermediate stages of 0 meson and sum up all of these diagrams analogously to the above cases, it will be found that all the diagrams of class c), as a whole, can be constructed by connecting adequately the $R_1$-part and the part corresponding to the Feynman kernel $K(p_0 + k_0; p_0, k_0)$. The existence of the unknown function $K(p_0 + k_0; p_0, k_0)$ does not cause any difficulty for solving the equations because of the 4-momentum conservation. (The momentum of a final nucleon in $K(p; p_0, k_0)$ has always a constant value $p = p_0 + k_0$ cf. the work of Fubini.)

Thus, if we had succeeded in solving the $R_2$ and $R_1$-equations, it would be not difficult to construct the formal solutions of the equations considered here. Moreover, it should be noted that the $R_2$-equation is quite divergence-free and $R_1$-equation can be made to be also
divergence-free provided only that the subtraction of divergences are carried out in its kernel \( G_2 \).

§ 3. The formal solutions of the generalized T-D equations

In this section, we shall construct the formal solutions of the generalized T-D equations in the approximation where the meson number is restricted within three. In this order of approximation, the generalized T-D equations are eqs. (19), (20), (21) and (22') of Part I. Substituting eq. (1.22') into eq. (1.21) and performing the subtraction of the self-energy type divergences caused by that substitution, we get the following equations:

\[
K(p_1; p_0, k_0) = g S_F(p_1) \int dk_1 K(p_1-k_1, k_1^\ast; p_0, k_0),
\]

\[
K(p_1, k_1^\ast; p_0, k_0) = \delta(p_1-p_0) \delta(k_1^\ast-k_0^\ast) S_F(p_1) \Delta(p_1)
+ g S_F(p_1) \int dk_1 K(p_1+k_1; p_0, k_0),
\]

\[
+ g S_F(p_1) \int dk_1 K(p_1-k_1^\ast, k_1^\ast; p_0, k_0),
\]

and

\[
K(p_1, k_1^\ast, k_2^\ast; p_0, k_0) = g S_F'(p_1) \int dk_1 K(p_1+k_1, k_2^\ast; p_0, k_0^\ast)
+ g S_F'(p_1) \int dk_1 S_F(p_1-k_1) \int dk_2 K(p_1+k_1-k_2, k_2^\ast; p_0, k_0^\ast)
\]

\[
+ g S_F'(p_1) \int dk_1 S_F(p_1-k_1) \int dk_2 S_F(p_1-k_1) \int dk_3 S_F(p_1-k_1-k_3, k_3^\ast; p_0, k_0^\ast),
\]

where

\[
S_F'(p_1) = [A_0(p_1)]^{-1} S_F(p_1),
\]

\[
\delta(k_1^\ast-k_0^\ast) = \delta(k_1-k_0) \delta_{ae}
\]

and \( A_0(p_1) \) is the finite part of \( A(p_1) \),

\[
A(p_1) = 1 - g^2 S_F(p_1) \int dk S_F(p_1-k) \int dk S_F(p_1-k) \Delta_F(k).
\]

We have rewritten here \( g(2\pi)^{-2} \) of Part I as \( g \).

The procedure to construct the solutions of these simultaneous integral equations (4), (5) and (6) would be obvious by the considerations of the preceding section. First, one must solve the \( R_2 \)-equation:

\[
R_2(p_1, k_1^\ast, k_2^\ast; p_0, k_0^\ast) = \delta(p_1-p_0) S_F'(p_1) \Delta_F(k_1) \Delta_F(k_2)
\times [\delta(k_1^\ast-k_0^\ast) \delta(k_2^\ast-k_0^\ast) + \delta(k_1^\ast-k_0^\ast) \delta(k_2^\ast-k_0^\ast)]
\]
Performing the subtraction of the self-energy type divergences caused by the processes corresponding to the diagram of Fig. 4 (e), we have taken $S_F' (\rho_i)$ instead of $S_F (\rho_i)$ for the "last" nucleon propagators in the right-hand side of eq. (9). This is in accordance to the similar circumstance in eq. (6). The terms of the right-hand side of eq. (9) correspond to the diagrams of Fig. 4 (a), (b), (c) and (d). Especially it should be remembered that the kernel of this integral equation is the same as that of eq. (6).

If we assume that the solution of eq. (9) has been found, our next task is to construct the $G_2$-function by enclosing each one of the "initial" and "final" meson lines of $R_2$ to the nucleon line (Fig. 1).

$$G_2 (\rho_i, k_1^a; \rho_o, k_3^T) = g^2 S_F (\rho_i) \gamma_5 \gamma_8 \int \frac{d k_2}{(2 \pi)^3} \frac{d k_4}{(2 \pi)^3} \times R_2 (\rho_i - k_2, k_2^a; \rho_o - k_4, k_4^T) \gamma_5 \gamma_8 [A_F (k_2)]^{-1}. \quad (10)$$

Taking into account that this function $G_2$ is to be used as the kernel of the $R_1$-equation in the next stage, the last factor $[A_F (k_2)]^{-1}$ in eq. (10) must be inserted to avoid overcounting the meson lines in $R_1$. Finally, we have to solve the $R_1$-equation (Fig. 3).

$$R_1 (\rho_1, k_1^a; \rho_o, k_3^T) = G_2 (\rho_1, k_1^a; \rho_o, k_3^T)$$

$$+ \int d \rho_2 \int d k G_2 (\rho_1, k_1^a; \rho_2, k_2^a) R_1 (\rho_2, k_2^a; \rho_o, k_3^T). \quad (11)$$

It should be noted that the diagrams corresponding to the $R_1$-function have not the nucleon and meson lines of their initial stages ($\rho_o, k_3^T$).

Now, if we had solved the $R_2$ and $R_1$-equations, the solutions of our starting integral equations (4), (5) and (6) were to be constructed from these functions $R_2$ and $R_1$ as considered in the preceding section. In fact, the solutions are given as follows:

$$K (\rho_i; \rho_o, k_3^T) = [A (\rho_i) - \Sigma_1 (\rho_i)]^{-1} \cdot g S_F (\rho_i) \times \gamma_5 \gamma_8 [\rho_i - \rho_o - k_2] S_F (\rho_o) A_F (k_2) + A_F (\rho_i; \rho_o, k_3^T) \quad (12)$$
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\[ K(p_1, k_1^*; p_0, k_0^*) = \delta(p_1 - p_0) \delta(k_1^* - k_0^*) S_\rho(p_1) \Delta_\rho(k_1) \]

\[ + R_1(p_1, k_1^*; \rho_0, k_0^*) S_\rho(p_0) \Delta_\rho(k_0) + g S_\rho(p_1) \Delta_\rho(k_1) \gamma_5 \tau_a K(p_1 + k_1; \rho_0, k_0^*) \]

\[ + \int dp A_1^*(p_1, k_1^*; p) K(p; \rho_0, k_0^*) , \]  

(13)

and

\[ K(p_1, k_1^*; \rho_0, k_0^*) = g \int dp_2 dk_2 d k_1 R_1(p_1, k_1^*; \rho_0, k_0^*) \]

\[ \times \gamma_5 \tau_a [\Delta_\rho(k_1)]^{-1} K(p_0 + k_0, k_0^*; \rho_0, k_0^*) , \]

(14)

where

\[ A_1^*(p_1; \rho_0, k_0^*) = g \int dk R_1(p_1, k^*; \rho_0, k_0^*) S_\rho(p_0) \Delta_\rho(k_0) , \]

(15)

\[ A_1^*(p_1, k_1^*; p) = g \int dk R_1(p_1, k_1^*; \rho, k^*; \rho_0, k_0^*) \Delta_\rho(k) S_\rho(p - k) \gamma_5 \tau_a , \]

(16)

and

\[ \sum_1(p) = g^2 S_\rho(p_1) \gamma_5 \tau_a \int dp_2 dk_2 d k_1 R_1(p_1 - k_1, k_1^*; \rho, k_0^*) \Delta_\rho(k_1) S_\rho(p) \gamma_5 \tau_a . \]

(17)

That this set of functions is actually the solution can easily be confirmed by substituting this into the starting equations. It is possible to deduce the expression (12) for \( K(p_1; \rho_0, k_0^*) \) from eqs. (4) and (13) by algebraic procedure only, if we take into account that the 4-momentum in \( R_1 \) always conserves.

In the purely mathematical language, this method can also be described as follows: First, we assume the form of \( K(p_1, k_1^*; \rho_0, k_0^*) \) to be given by eq. (14), where \( R_2 \) is the function to be determined later. Next, we substitute this \( K(p_1, k_1^*; \rho_0, k_0^*) \) into the right-hand side of eq. (6). Then, if we take into account that the first two terms of eq. (6) can be rewritten as

\[ g S_\rho(p_1) \gamma_5 \tau_a \Delta_\rho(k_1) K(p_1 + k_1, k_2^*; \rho_0, k_0^*) \]

\[ = g \int dp d p_1 dk d k_2 \delta(p_1 - p_2) S_\rho(p_1) \Delta_\rho(k_2) [\delta(k_1^* - k_2^*) \delta(k_2^* - k_4^*) \]

\[ + \delta(k_1^* - k_4^*) \delta(k_2^* - k_3^*)] \gamma_5 \tau_a [\Delta_\rho(k_2)]^{-1} \delta(p_2 + k_4 - p_0) K(p_0, k_3^*; \rho_0, k_0^*) , \]

(18)

it will be found to be sufficient that \( R_2 \) is the function which satisfies eq. (9). Finally, substituting \( K(p_1, k_1^*; \rho_0, k_0^*) \) of (14) into eq. (5), we get the following equation for \( K(p_1, k_1^*; \rho_0, k_0^*) \),

\[ K(p_1, k_1^*; \rho_0, k_0^*) = \delta(p_1 - p_0) \delta(k_1^* - k_0^*) S_\rho(p_1) \Delta_\rho(k_1) \]

\[ + g S_\rho(p_1) \gamma_5 \tau_a \Delta_\rho(k_1) K(p_1 + k_1; \rho_0, k_0^*) \]

\[ + \int dp d p_1 dk G_3(p_1, k_1^*; \rho_0, k_0^*) \gamma_5 \tau_a K(p_0 k_3^*; \rho_0, k_0^*) , \]

(19)

where \( G_3 \) is the function given by eq. (10). Eq. (19) differs from the equation studied...
by Fubini by only the form of its kernel $G_\rho$, and its solution can be written down according to the Fubini's procedure. Thus we get the equation (13) as the solution of (19). It should need no words as regards the solution of $K(p_1; \rho_0, k_0^\alpha)$.

§ 4. Subtraction of the divergences

In the preceding section, we have got the formal solutions of our equations. The next task is to subtract the divergences from these solutions according to the program described previously, so as to get significant results. However, the infinities related to self-energies and vertex parts are mixed up in our formalism (especially in $G_\rho$) in a very complicated manner, and therefore it is necessary to classify these infinities according to their types so as to be able to perform the usual subtraction techniques.

For example, let us consider the infinities in $G_\rho$. The reason why the function $G_\rho$ contains infinities in spite of the fact that $G_\rho$ is constructed from the quite divergence-free quantity $R_\rho$ lies in the $k_2$-and $k_4$-integrations in eq. (10). Moreover, that the infinities of various types are contained in $G_\rho$ in a mixed form is due to that $R_\rho$ itself corresponds to the assembly of the various Feynman-Dyson diagrams. Therefore, to classify these infinities according to their types, it is convenient to make use of the nature of the $R_\rho$-equation.

For convenience, we shall adopt the rule of matrix product in the following. Then, the $R_\rho$-equation (10) can be written as follows,

$$ R_\rho(p_1, k_1^\alpha, k_2^\beta; \rho_0, k_3^\gamma, k_4^\delta) = A(p_1, k_2^\beta; \rho_0, k_3^\gamma, k_4^\delta) \Delta_p(k_1) \delta(k_1^\alpha - k_2^\beta) + A(p_1, k_2^\beta; \rho_0, k_3^\gamma) \Delta_p(k_1) \delta(k_1^\alpha - k_4^\delta) + B(p_1, k_2^\beta; \rho_0, k_3^\gamma, k_4^\delta) R_\rho(p_0, k_4^\delta, k_3^\gamma; \rho_0, k_2^\beta) + B(p_1, k_2^\beta; \rho_0, k_3^\gamma, k_4^\delta) R_\rho(p_0, k_4^\delta, k_3^\gamma; \rho_0, k_2^\beta), $$

(20)

where

$$ A(p_1, k_2^\beta; \rho_0, k_3^\gamma, k_4^\delta) = S_\rho(p_1) \Delta_p(k_1) \delta(p_1 - \rho_0)(k_2^\beta - k_3^\gamma), $$

(21)

$$ B(p_1, k_2^\beta; \rho_0, k_3^\gamma, k_4^\delta) = g^2 S_\rho(p_1) \gamma_5 \gamma_5 \gamma_5 S_\rho(p_1 - k_2) \Delta_p(k_1) \delta(p_1 + k_2 - \rho_0 - k_4), $$

(22)

and $B(p_1, k_2^\beta; \rho_0, k_3^\gamma, k_4^\delta) R_\rho(p_0, k_4^\delta, k_3^\gamma; \rho_0, k_2^\beta)$ for example, means

$$ \sum_\epsilon \int dp dz B(p_1, k_2^\beta; \rho_0, k_3^\gamma, k_4^\delta) R_\rho(p_0, k_4^\delta, k_3^\gamma; \rho_0, k_2^\beta). $$

Now, the "renormalization" procedure is usually performed by considering certain parts of the various diagrams as the radiative corrections inserted at the vertices or the propagators of their skeletons. Therefore, also in our case, it would be natural to classify the $G_\rho$-diagrams according to the type of skeleton, i.e. according to whether they are the corrections to the propagator (self-energy type), or the vertex where the meson $k_1^\alpha$ (or $k_2^\beta$) is produced (or annihilated), or of other types. At any rate, the history of the meson $k_1^\alpha$ or $k_2^\beta$ would be one of keys for the desired classification of $G_\rho$. 

If we, first, pay attention to the vertex where the "final" meson \( k_1^a \) is produced, all of the \( R_2 \)-diagrams could be divided into three kinds, that is, (1) the diagrams where the meson \( k_1 \) is identical with the meson \( k_3 \), (2) the diagrams where \( k_1^a \) is identical with \( k_4^a \) and (3) others. According to this classification of the \( R_2 \)-diagrams, we put

\[
R_2(p_1, k_1^a, k_2^p; p_0, k_3^\gamma, k_4^\delta) = Q(p_1, k_2^p; p_0, k_4^\delta) \delta(k_1^a - k_3^\gamma) + Q'(p_1, k_2^p; p_0, k_3^\gamma) \delta(k_1^a - k_4^\delta) + S(p_1, k_2^p; p_0, k_3^\gamma, k_4^\delta),
\]

where \( S \) is the quantity which contains neither the factor \( \delta(k_1^a - k_3^\gamma) \) nor \( \delta(k_1^a - k_4^\delta) \), and therefore corresponds to the kind (3) of the above classification. Substituting this expression (23) into the \( R_2 \)-equation (20) and equating the terms which contain the factor \( \delta(k_1^a - k_3^\gamma) \) or \( \delta(k_1^a - k_4^\delta) \) or the remaining terms, respectively, we get the equations for \( Q \) and \( Q' \) or \( S \). The resulting equations for \( Q \) and \( Q' \) are identical and of the following form,

\[
Q(p_1, k_2^p; p_0, k_4^\delta) = A(p_1, k_2^p; p_0, k_4^\delta) + B(p_1, k_2^p; p_0, k_3^\gamma) Q(p_0, k_3^\gamma; p_0, k_4^\delta).
\]

(24)

This is easily understood by the symmetry of \( R_2 \) with respect to \( k_3 \) and \( k_4 \). From the above equation for \( Q \), it is obvious that the quantity \( Q \) represents the one-meson scattering process where the meson number varies as \( 1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow \cdots \rightarrow 1 \rightarrow 2 \rightarrow 1 \). The equation for \( S \) is as follows:

\[
S(p_1, k_1^a, k_2^p; p_0, k_3^\gamma, k_4^\delta)
= B(p_1, k_1^a; p_0, k_3^\gamma) Q(p_0, k_2^p; p_0, k_4^\delta) \delta(k_1^a - k_3^\gamma) + B(p_1, k_1^a; p_0, k_3^\gamma) Q(p_0, k_2^p; p_0, k_4^\delta) \delta(k_1^a - k_4^\delta) + B(p_1, k_1^a; p_0, k_3^\gamma) S(p_0, k_1^a, k_2^p; p_0, k_3^\gamma, k_4^\delta) + B(p_1, k_1^a; p_0, k_3^\gamma) S(p_0, k_1^a, k_2^p; p_0, k_3^\gamma, k_4^\delta).
\]

(25)

Finally, paying attention to the vertex at which the meson \( k_1^a \) is produced and taking into account that the scattering process where the meson number varies as \( 1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow \cdots \rightarrow 1 \rightarrow 2 \rightarrow 1 \) is represented by \( Q \), we put \( S \) as follows:

\[
S(p_1, k_1^a, k_2^p; p_0, k_3^\gamma, k_4^\delta)
= Q(p_1, k_1^a; p_0, k_2^p) [S_y'(p_0)]^{-1} [\delta_y(p)]^{-1} \times B(p_0, k_1^a; p_0, k_2^p) T(p_0, k_2^p, k_3^\gamma, k_4^\delta).
\]

(26)

Taking into account that

\[
B(p_1, k_1^a; p_0, k_2^p) Q(p_0, k_2^p; p_0, k_2^p)
= Q(p_1, k_1^a; p_0, k_2^p) [S_y'(p_2)]^{-1} [\delta_y(p)]^{-1} B(p_0, k_2^p; p_0, k_2^p) S_y'(p_0) \delta_y(k_2^p),
\]

(27)

it follows from eqs. (5) and (26) that
Therefore, it is found to be sufficient that we take $T$ as $R_2$ itself. Thus we have get an alternative equation for $R_2$ in the following form.

\[
R_2(p_1, k_1^a; p_o, k_1^b) = Q(p_1, k_1^a; p_o, k_2^a) \cdot S_1(p_1, k_2^a) - B(p_1, k_1^a; p_2, k_2^a). \tag{28}
\]

The above derivation of eq. (29) is useful to explain the meaning of such a modification of $R_2$ equation. However, there is an alternative method for the derivation of eq. (29), and the latter is more simple and convenient especially for the application to the higher order approximations. That is, if we take into account eq. (2), it will be found that eq. (29) can also be written as follows,

\[
R_2(p_1, k_1^a; p_o, k_2^a) = Q(p_1, k_1^a; p_o, k_2^a) \cdot S_1(p_1, k_2^a) - B(p_1, k_1^a; p_2, k_2^a). \tag{30}
\]

Such a relationship as eq. (30) is the one due to the nature of $Q$, and generally holds for any function $F$. That is, from the fact that the quantity $Q$ which is the solution of eq. (24) also satisfies the equation

\[
Q(p_1, k_1^a; p_o, k_2^a) = A(p_1, k_1^a; p_o, k_2^a) + Q(p_1, k_1^a; p_o, k_2^a) \cdot S_1(p_1, k_2^a) - B(p_1, k_1^a; p_2, k_2^a). \tag{31}
\]

it follows that

\[
F(p_1, k_1^a, k_2^a, \ldots) = Q(p_1, k_1^a; p_o, k_2^a) \cdot S_1(p_1, k_2^a) - B(p_1, k_1^a; p_2, k_2^a). \tag{32}
\]

for any function $F$. If we notice that the divergences appearing in $dk_2 R_2(p_1, k_1^a; k_1^a, \ldots)$ in the defining equation (10) of $G_2$ is entirely due to the term $B(p_1, k_1^a; p_o, k_2^a) R_2(p_o, k_1^a, k_2^a; \ldots)$ in the right-hand side of $R_2$-equation (20), and that the second term in \{ \} in eq. (30) corresponds to the former, it would be understood that the modification (29) or (30) of $R_2$ is useful for the desired classification of $G_2$. 

\[
Q(p_1, k_1^a; p_o, k_2^a) \cdot S_1(p_1, k_2^a) - B(p_1, k_1^a; p_2, k_2^a) = 0 \tag{28}
\]
Now, the original $R_2$-equation (20) can also be written as

$$
R_2(p_1, k_1^a; p_0, k_0^a) = \Delta(p_1, k_1^a; p_0, k_0^a) \Delta_x(k_0) \delta(k_2^a - k_3^a)
$$

$$
+ A(p_1, k_1^a; p_0, k_0^a) \Delta_x(k_0) \delta(k_1^a - k_3^a)
$$

$$
+ R_2(p_1, k_1^a, k_2^a; p_0, k_0^a, k_3^a) [S'_{y'}(p_3)]^{-1} [\Delta_y(k)]^{-1}
$$

$$
\times B(p_0, k_0^a) S_y'(p_3) \Delta_y(k_0)
$$

$$
+ R_2(p_1, k_1^a, k_2^a; p_0, k_0^a, k_3^a) [S'_{y'}(p_3)]^{-1} [\Delta_y(k)]^{-1}
$$

$$
\times B(p_0, k_0^a; p_0, k_3^a) S_y'(p_3) \Delta_y(k_0),
$$

(33)

by the same reason as that of eq. (27) or (31), i.e.

$$
B(p_1, k_1^a; p, k^0) A(p, k^0; p_0, k_0^a)
$$

$$
= A(p_1, k_1^a; p, k^0) [S'_{y'}(p)]^{-1} [\Delta_y(k)]^{-1} B(p, k^0; p_0, k_0^a) S_y'(p_0) \Delta_y(k_0).
$$

If we apply a procedure similar to that used in the above derivation of eq. (29) from eq. (20), we can easily derive the following equation for $R_2$ from eq. (33).

$$
R_2(p_1, k_1^a, k_2^a; p_0, k_0^a, k_3^a) = Q(p_1, k_1^a; p_0, k_0^a) \Delta_y(k_0) \delta(k_2^a - k_3^a)
$$

$$
+ Q(p_1, k_2^a; p_0, k_0^a) \Delta_y(k_0) \delta(k_1^a - k_3^a) + R_2(p_1, k_1^a, k_2^a; p_0, k_0^a, k_3^a)
$$

$$
\times [S'_{y'}(p_3)]^{-1} [\Delta_y(k)]^{-1} B(p_0, k_0^a; p_0, k_3^a) \Delta_y(k_0) .
$$

(34)

Substituting eq. (35) into eq. (29), we get

$$
R_2(p_1, k_1^a, k_2^a; p_0, k_0^a, k_3^a) = Q(p_1, k_1^a; p_0, k_0^a) \Delta_y(k_0) \delta(k_2^a - k_3^a)
$$

$$
+ Q(p_1, k_2^a; p_0, k_0^a) \Delta_y(k_0) \delta(k_1^a - k_3^a) + Q(p_1, k_1^a; p_0, k_0^a, k_3^a)
$$

$$
\times [S'_{y'}(p_3)]^{-1} [\Delta_y(k)]^{-1} B(p_0, k_0^a; p_0, k_3^a) \Delta_y(k_0) .
$$

(35)

The important fact is that, in such a form of $R_2$-equation as (36), the variables in question, $k_2$ and $k_0$, appear only in the function $Q$. The meanings of the terms in the right-hand side of eq. (36) are very obvious. (c.f. Fig. 5)

With the expression (36) for $R_2$ and the defining equation
of $G_2$, the classification of the infinities in $G_2$ can be accomplished as follows;

\[ G_2 = G_2^{(a)} + G_2^{(b)} + G_2^{(c)} + G_2^{(d)}, \]

where

\[ G_2^{(a)} (p_1, k_1^*; p_0, k_2^\alpha) = g^2 S_F (p_1) \delta (k_1^* - k_2^\alpha), \]

\[ G_2^{(b)} (p_1, k_1^*; p_0, k_2^\alpha) = g^2 S_F (p_1) \int dp_2 A_2 (p_1; p_0, k_2^\alpha) [S_F' (p_0)]^{-1} \]

\[ \times [A_F (k_2^\alpha)]^{-1} A_2^* (p_0, k_1^*; p_0), \]

\[ G_2^{(c)} (p_1, k_1^*; p_0, k_2^\alpha) = g^2 S_F (p_1) A_F (k_1) \int dp_2 dp_3 dk A_2 (p_1; p_2, k_2^\beta) \]

\[ \times [A_F (k_2^\beta)]^{-1} [C_5^\tau] - 1 \tau_A^2 \tau_A^2 A_2^* (p_2 + k_1^* - k_2^\beta; p_0), \]

and

\[ G_2^{(d)} (p_1, k_1^*; p_0, k_2^\alpha) = g^2 S_F (p_1) \int dp_2 dp_3 dk A_2 (p_1; p_2, k_2^\beta; p_0, k_2^\alpha) \]

\[ \times [A_F (k_2^\beta)]^{-1} [C_5^\tau] - 1 \tau_A^2 \tau_A^2 A_2^* (p_2 + k_1^* - k_2^\beta, k_2^\alpha; p_0), \]

with

\[ A_2 (p_1; p_2, k_2^\beta) = \tau_5 \tau_6 \int dk_4 Q (p_1 - k_2^\beta, k_2^\alpha; p_2, k_2^\beta), \]

\[ A_2^* (p_1, k_2^\beta; p_0) = \int dk_4 Q (p_1, k_2^\beta; p_0 - k_2^\beta, k_2^\alpha) \tau_5 \tau_6, \]

and

\[ \sum_2 (p_1; p_0) = \tau_5 \tau_6 \int dk_4 dk_5 Q (p_1 - k_2^\beta, k_2^\alpha; p_0 - k_2^\beta, k_2^\alpha) \tau_5 \tau_6. \]

For the subtraction of the divergences in $G_2$, it is obviously sufficient that we perform the subtraction procedure only for the quantities $A_2, A_2^*$ and $\sum_2$. These contain the divergences of the so-called overlapping type, and the treatments of them are complicated. However, Chiba and Tanaka-Ito have given a method for obtaining reasonable convergent parts of such quantities as given by eqs. (42)-(44). For example, let us consider the quantity $A_2 (p_1; p_2, k_2^\beta)$. This obeys the following integral equation:

\[ A_2 (p_1; p_2, k_2^\beta) = \tau_5 \tau_6 S_F (p_2) A_F (k_2^\beta) \delta (p_1 - p_2 - k_2^\beta) \]

\[ + A_2 (p_1; p_2, k_2^\alpha) [S_F (p_2)]^{-1} [A_F (k_2^\beta)]^{-1} B (p_2, k_2^\alpha; p_2, k_2^\beta) S_F (p_2) A_F (k_2^\beta). \]

Now, if we construct the finite quantity $A_2 (p_1; p_2, k_2^\beta)$ by
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\[ A_{0c}(p_1; \rho_0, k^c) = A_0^0(p_1; \rho_0, k^c) + A_2^0(p_1; \rho_0, k'^c)[S_{\rho}(\rho_0)]^{-1} [A_{\rho}(k')]^{-1} \]
\[ \times \{ B(\rho_0, k'^c; \rho_0, k''^c) - [B(\rho_0, k'^c; \rho_0, k''^c)]_{\rho_0 + k' = \rho_0} \} Q(p_1, k''^c; \rho_0, k^c), \]

(46)

with the solution \( A_2^0 \) of the integral equation

\[ A_2^0(p_1; \rho_0, k^c) = \gamma_{\rho} S_{\rho}(p_1) A_{\rho}(k) \delta(p_1 - \rho_0 - k) + A_0^0(p_1; \rho_0, k'^c) \]
\[ \times [S_{\rho}'(p_1)]^{-1} [A_{\rho}'(k')]^{-1} \{ B(\rho_0, k'^c; \rho_0, k^c) - B(\rho_0, k'^c; \rho_0, 0) \} \rho_0 + k' = \rho_0 \]
\[ \times S_{\rho}'(p_2) A_{\rho}(k), \]

(47)

then this \( A_{0c} \) according to Tanaka and Ito\(^6\), is just identical with the convergent part of the quantity \( A_2 \) derived by applying the procedure of Salam\(^6\) to the perturbational expansion (iteration) of eq. (45). In eqs. (46) and (47), \( \rho^0 \) is the 4-momentum which satisfies the relation

\[ (\rho^0)^2 + m^2 = 0. \]

(48)

The convergent part of \( A_{2c}^* \) can also be constructed by the similar procedure. The authors quoted above have also shown the possibility of the subtraction equivalent to Salam’s method as to the quantity \( \Sigma_{2c} \). In this case, it is convenient to rewrite eq. (44) as follows: First, consider the \( Q \)-equation which can be written symbolically as (24). If both sides of this equation are multiplied by \( Q(p, k'^c; \rho_0, k'^c)[S_{\rho}'(p_1)]^{-1} [A_{\rho}(k_2)]^{-1} \), it follows that

\[ Q(p, k'^c; \rho_0, k'^c)[S_{\rho}'(p_1)]^{-1} [A_{\rho}(k_2)]^{-1} \]
\[ = Q(p, k'^c; \rho_0, k'^c) + Q(p, k'^c; \rho_0, k'^c)[S_{\rho}'(p_1)]^{-1} [A_{\rho}(k_2)]^{-1} \]
\[ \times \{ B(p_1, k'^c; \rho_0, k'^c) \} Q(p_1, k'; \rho_0, k'^c). \]

(49)

Therefore, eq. (44) can be rewritten as follows:

\[ \Sigma_{2c}(p_1; \rho_0) \]
\[ = \int dpdk A_2(p_1; p, k^c)[S_{\rho}'(p)]^{-1} [A_{\rho}(k)]^{-1} A_{2c}^*(p, k^c; \rho_0) \]
\[ - \int dpdp'ddkd'k A_2(p_1; p, k^c)[S_{\rho}'(p)]^{-1} [A_{\rho}(k)]^{-1} \]
\[ \times \{ B(p, k^c; p', k'^c) \} A_{2c}^*(p', k'^c; \rho_0). \]

(50)

Then, according to Chiba\(^3\), the reasonable convergent part of \( \Sigma_{2c} \) is obtained by applying the usual subtraction technique to the quantity \( \Sigma_{2c}' \) defined as

\[ \Sigma_{2c}'(p_1; \rho_0) = \int dpdk A_{0c}(p_1; p, k^c)[S_{\rho}'(p)]^{-1} [A_{\rho}(k)]^{-1} A_{2c}^*(p, k^c; \rho_0) \]
\[ - \int dpdp'ddkd'k A_{0c}(p_1; p, k^c)[S_{\rho}'(p)]^{-1} [A_{\rho}(k)]^{-1} \]
\[ \times \{ B(p, k^c; p', k'^c) \} A_{0c}(p', k'^c; \rho_0). \]

(51)
Thus, using the procedure of Chiba and Tanaka-Ito, we have got the convergent part of the kernel \( G_2 \). Finally, it should be noted that the effect of the subtraction of the divergence in \( G_2 \) is merely to replace eq. (19) by

\[
K(p_1, k_1^a; p_0, k_0^a) = \delta(p_1 - p_0) \delta(k_1^a - k_0^a) S''_{\sigma}(p_1) A_\sigma(p_1) \\
+ g S_{\sigma}''(p_1) \gamma_5 a D_{\sigma}(k_1) K(p_1 + k_1^a; p_0, k_0^a) \\
+ \int dp_2 dk_3 G_{2e}(p_1, k_1^a; p_0, k_0^a) K(p_1 + k_1^a; p_0, k_0^a),
\]

where

\[
G_{2e} = G_{2e}^{(b)} + G_{2e}^{(c)} + G_{2e}^{(d)},
\]

\[
S_{\sigma}''(p_1) = [1 - g^2 S_{\sigma}(p_1) \Sigma_{2e}(p_1)]^{-1} S_{\sigma}(p_1),
\]

and \( G_{2e}^{(b)}, G_{2e}^{(c)}, G_{2e}^{(d)} \) are the kernels which are obtained by replacing \( A_\sigma, A_\sigma^* \) and the "final" \( S_{\sigma}(p_1) \) in (39), (40) and (41) by \( A_\sigma, A_\sigma^* \) and \( S_{\sigma}''(p_1) \), respectively.

As regards the subtraction of divergences after this stage, there occur no more difficulties and divergences can all be subtracted by a similar or more simple procedure as above. Thus, we have succeeded, at least in principle, in deriving a convergent and significant result in the approximation where the meson "number" is restricted within three.

§ 5. Higher order approximations

In this section, we shall consider briefly the possibility of extending the subtraction procedure investigated so far to higher order approximations.

For example, let us consider the next order of approximation, i.e. the approximation where the meson number is restricted within 4. In this case, eqs. (4) and (5) are kept unchanged. However, we must take

\[
K(p_1, k_1^a, k_2^a; p_0, k_0^a) = g S_{\sigma}(p_1) \gamma_5 a D_{\sigma}(k_1) K(p_1 + k_1^a, k_2^a; p_0, k_0^a) \\
+ g S_{\sigma}(p_1) \gamma_5 a D_{\sigma}(k_2) K(p_1 + k_2^a, k_1^a; p_0, k_0^a) \\
+ g S_{\sigma}(p_1) \gamma_5 a \int d k_3 K(p_1 - k_3^a, k_1^a, k_2^a, k_3^a; p_0, k_0^a),
\]

and

\[
K(p_1, k_1^a, k_2^a, k_3^a; p_0, k_0^a) = [g S_{\sigma}''(p_1) \gamma_5 a D_{\sigma}(k_1) K(p_1 + k_1^a, k_2^a, k_3^a; p_0, k_0^a) \\
+ g^2 S_{\sigma}''(p_1) \gamma_5 a \int d k_3 S_{\sigma}(p_1 - k_3^a) \gamma_5 a D_{\sigma}(k_1) \\
\times K(p_1 + k_3^a - k_4, k_2^a, k_3^a, k_4^a; p_0, k_0^a)] \\
+ [\text{terms with } k_1^a \text{ and } k_2^a \text{ interchanged}] \\
+ [\text{terms with } k_1^a \text{ and } k_3^a \text{ interchanged}],
\]
instead of eq. (6). But, formal solutions of these equations can easily be obtained by a procedure similar to that of § 3. In fact, if we put

\[ K(p_1, k_1^a, k_2^b, k_3^c; p_0, k_0^d) = \frac{1}{2} g^2 \int dp dq dk dl dk' \]

\[ \times R_3(p_1, k_1^a, k_2^b, k_3^c; p_0, k_0^d, k_1'^a, k_2'^b, k_3'^c) \gamma_5 \gamma_\lambda \]

\[ \times [\Delta_x(k')^{-1} [\Delta_x(k)^{-1} K(p_0, k_0^d; p_0, k_0^d)] \right) \]

it is easily found that the quantity \( R_3 \) represents the 3-meson scattering process where the meson number varies as \( 3 \rightarrow 4 \rightarrow 3 \rightarrow 4 \rightarrow 3 \rightarrow \cdots \rightarrow 3 \rightarrow 4 \rightarrow 3 \).

\[ R_3(p_1, k_1^a, k_2^b, k_3^c; p_0, k_0^d, k_1'^a, k_2'^b, k_3'^c) \]

\[ = S_x(p_1) A_x(k_1) A_x(k_2) A_x(k_3) \delta (p_1 - p_0) \sum \delta (k_1^a - k_0^b) \delta (k_2^b - k_1'^a) \delta (k_3^c - k_2'^b) \]

\[ + B(p_1, k_1^a; p_0, k_1'^a) R_3(p_0, k_2^b, k_3^c; p_0, k_0^d, k_1'^a, k_2'^b, k_3'^c) \]

\[ + B(p_1, k_2^b; p_0, k_2'^b) R_3(p_0, k_1^a, k_3^c; p_0, k_0^d, k_1'^a, k_2'^b, k_3'^c) \]

\[ + B(p_1, k_3^c; p_0, k_3'^c) R_3(p_0, k_1^a, k_2^b; p_0, k_0^d, k_1'^a, k_2'^b, k_3'^c), \]

(57)

where the summation in the first term is taken over all the permutations of \( k_1^a, k_2^b, \) and \( k_3^c. \)

The factor \( 1/2 \) in the right-hand side of eq. (57) corresponds to the fact that the mesons \( k_1^a, k_2^b, \) and \( k_3^c. \) in the intermediate stage can not be distinguished. With this expression (57) for \( K(p_1, k_1^a, k_2^b, k_3^c; p_0, k_0^d) \), eq. (55) can be written as

\[ K(p_1, k_1^a, k_2^b; p_0, k_0^d) = g S_x(p_1) \gamma_5 \gamma_\lambda A_x(k_1) K(p_1, k_1^a, k_2^b; p_0, k_0^d) \]

\[ + g S_x(p_1) \gamma_5 \gamma_\lambda A_x(k_2) K(p_1, k_2^b, k_3^c; p_0, k_0^d) \]

\[ + G_3(p_1, k_1^a, k_2^b; p_0, k_0^d, k_3^c) K(p_0, k_0^d, k_3^c; p_0, k_0^d), \]

(59)

where

\[ G_3(p_1, k_1^a, k_2^b; p_0, k_0^d, k_3^c) = \frac{1}{2} g^2 S_x(p_1) \gamma_5 \gamma_\lambda \int dk dk' \]

\[ \times R_3(p_1 - k_0, k_1^a, k_2^b; p_0, k_0^d, k_1'^a, k_2'^b, k_3'^c) \gamma_5 \gamma_\lambda [\Delta_x(k')^{-1} [\Delta_x(k)^{-1} K(p_0, k_0^d; p_0, k_0^d)] \right) \]

(60)

After this stage, the procedure for constructing the formal solutions is quite analogous to that of § 3, because the difference between eqs. (6) and (59) is only the form of their kernels. Thus, it would easily be concluded that we can, at least in principle, construct the formal solutions of the generalized T-D equations in any desired order of approximation only by solving several number of single (non-coupled) integral equations.

As regards the subtraction of the divergences, it would be sufficient only to consider briefly the kernel \( G_3. \) Analogously to the classification of \( G_2 \) in § 4, we first rewrite the \( R_3 \)-equation by the first method of § 4 as follows:

\[ R_3(p_1, k_1^a, k_2^b, k_3^c; p_0, k_0^d, k_1'^a, k_2'^b, k_3'^c) = R_3(p_1, k_1^a, k_3^c; p_0, k_0^d, k_1'^a, k_2'^b) \]

\[ \times A_x(k_1) \delta (k_1^a - k_1'^a) + R_3(p_1, k_2^b, k_3^c; p_0, k_0^d, k_1'^a, k_2'^b) A_x(k_2) \delta (k_2^b - k_2'^b) \]

\[ \times [\Delta_x(k_1)^{-1} [\Delta_x(k)^{-1} K(p_0, k_0^d; p_0, k_0^d)] \right) \]

\[ + R_3(p_1, k_1^a, k_2^b; p_0, k_0^d, k_3^c, k_1'^a, k_2'^b) A_x(k_3) \delta (k_3^c - k_3'^c) \]

\[ \times \gamma_5 \gamma_\lambda [\Delta_x(k_1)^{-1} [\Delta_x(k)^{-1} K(p_0, k_0^d; p_0, k_0^d)] \right) \]

\[ + R_3(p_1, k_1^a, k_3^c; p_0, k_0^d, k_1'^a, k_2'^b) A_x(k_2) \delta (k_2^b - k_2'^b) \]

\[ \times [\Delta_x(k_1)^{-1} [\Delta_x(k)^{-1} K(p_0, k_0^d; p_0, k_0^d)] \right) \]

\[ + R_3(p_1, k_2^b, k_3^c; p_0, k_0^d, k_1'^a, k_2'^b) A_x(k_1) \delta (k_1^a - k_1'^a) \]

\[ \times [\Delta_x(k_1)^{-1} [\Delta_x(k)^{-1} K(p_0, k_0^d; p_0, k_0^d)] \right) \]

\[ + R_3(p_1, k_2^b, k_3^c; p_0, k_0^d, k_1'^a, k_2'^b) A_x(k_3) \delta (k_3^c - k_3'^c) \]

\[ \times [\Delta_x(k_1)^{-1} [\Delta_x(k)^{-1} K(p_0, k_0^d; p_0, k_0^d)] \right) \]

\[ + R_3(p_1, k_1^a, k_2^b; p_0, k_0^d, k_3^c, k_1'^a, k_2'^b) A_x(k_3) \delta (k_3^c - k_3'^c) \]

\[ \times [\Delta_x(k_1)^{-1} [\Delta_x(k)^{-1} K(p_0, k_0^d; p_0, k_0^d)] \right) \]
where $R_2$ is the function appearing in § 3, i.e. the solution of eq. (9). This equation can be derived by noticing the vertex at which the meson $k'_1$ is produced, the method being quite analogous to that used when we have derived the modified $R_2$ equation (29).

With this modification (61) of the $R_2$ equation, we can easily derive the equation for $R_3$ analogous to eq. (36) for $R_2$. By substituting the equation for $R_2$ thus derived, into eq. (60), it is easily seen that the problem of defining the convergent part of $G_3$ can be reduced to that of defining the convergent part of $\int dk_3 R_2 (P_1 - k_3, k_1, k_2; \cdots)$ or $\int dk_3 R_2 \times (\cdots; p_3 - k_3, k_3)$ which can easily be solved by the technique described in § 4.

For the modification of $R_3$ equation it is also possible and more simple to apply the second method explained in § 4. Taking into account that the source of the divergences in $\int dk_3$ in the defining equation (60) of $G_3$ is the last term in the right-hand side of the $R_3$-equation (58), and thus applying the rule (32) with $k'_1$ instead of $k_1$ to the quantity $R_3$, we get the following equation for $R_3$:

$$
R_3 (p_{10}, k_1, k_2, k_3; p_2, k_4, k_5, k_6) = \left[ \sum Q (P_1; p_2, k_4) \delta (k_4 - k_5) \delta (k_5 - k_6) \right] A_3 (k_1) A_3 (k_2)
+ Q (p_1, k_1; p_2, k_4) [S'_3 (p_2)]^{-1} [A'_3 (k_2)]^{-1}
\times \left\{ B (p_3, k_3; p_4, k_4) R_3 (p_3, k_4, k_6, k_4; p_4, k_6, k_4, k_6) + B (p_3, k_3; p_4, k_4) R_3 (p_3, k_3, k_6, k_3; p_4, k_6, k_3, k_6) \right\},
$$

(62)

where the summation ($\sum$) in the first term is taken over all permutations of $k_4, k_5$ and $k_6$.

In this form of $R_3$, it would easily be understood that $\int dk_3 R_3 (p_1 - k_3, k_3; \cdots)$ can be made convergent provided only that $\int dk_3 Q (p_1 - k_3, k_3; \cdots)$ is made convergent. The divergences caused by the integration with respect to $k'_1$ in the defining equation (60) of $G_3$ can also be separated by the similar procedure with the $R_3$-equation corresponding to the equation (33) for $R_2$.

Thus we see that the convergent contribution of $G_3$ can, at least in principle, be defined. In the approximation where the meson number is restricted within 4, it is necessary after this stage to construct the solution $\tilde{R}_2$ of the integral equation which has $G_3$ (made convergent by the above procedure) as its kernel, and moreover to construct the kernel $\tilde{G}_2$ by enclosing one of the meson lines at each of the initial and final stages of $\tilde{R}_2$. Again, this kernel $\tilde{G}_2$ would contain the overlapping divergences. However, this would also be made convergent by the procedure explained so far. In fact, the source of the overlapping divergences in $\int dk_3 \tilde{R}_2 (p_1 - k_3, k_3; \cdots)$ is only the term (in the $\tilde{R}_2$-equation) which contains the kernel shown in Fig. 6 (one term of $G_3$), and therefore our method is directly applicable also to this case.
Thus far, we have considered the case where the meson number is restricted within 4. However, our method is directly applicable to any higher approximation and the significant convergent solution of our formalism can, at least in principle, be derived. The remaining question is only whether our method of defining the convergent contribution can be interpreted as the “renormalization” procedure or not.

§ 6. Discussion and conclusion

In this and the previous papers, we have shown that, by extending the notion of meson number, it is possible to formulate a covariant generalization of the Tamm-Dancoff equations for one-nucleon problem (though in the approximation where the nucleon closed loops are entirely omitted) and that it is also possible, at least in principle, to define the convergent solution of those generalized T-D equations.

Now, it should be noted that, in such an approximation as the T-D method (generalized or not), it is probably impossible to eliminate all divergences only by changing the scales of the mass and the coupling constant as in the case of the perturbational treatment. $^7$ Usual renormalization technique is intimately related to the perturbation expansion, and it would be very difficult to apply it directly to other expansion methods, except very special cases. For example, if one wishes to eliminate the divergences, in the present formalism, by the method analogous to that of the perturbational treatment, the renormalized coupling constant would depend on the configuration of that stage at which it appears. This is easily realized by the example shown in Fig. 7. In the approximation where the meson number is restricted within three, there is no correction to the vertex of Fig. 7 (a) and therefore the coupling constant $g_a$ is unchanged (except the change by the contribution from the self-energy type diagrams). However, in the same approximation, the process shown in Fig. 7 (c) contributes as the correction to the vertex $g_b$ shown in Fig. 7 (b), and for the vertex $g_d$ of Fig. 7 (d) there are more complicated corrections. Thus, after the “renormalization”, the modified coupling constants $g'_a$, $g'_b$, and $g'_d$ are all different.

However, these “renormalized” coupling constants would become identical with each other as we proceed to infinitely higher order approximation, and, in this limit, our subtraction technique would give the correct result. In other words, what we want in such an approximate theory is the approximation in a certain sense to the correct answer of the problem in question. Therefore, supposing that we borrow a certain contribution to $g_a$ or
from the higher order configurations, we put $g_a' = g_6' = g_d'$. The circumstance is quite similar as for the "renormalized" nucleon mass terms. This is a justification for merely omitting the various divergences in our theory. At any rate, the convergent result derived by the procedure of this paper would have the significance as the approximation to the correct result, corresponding to a certain restriction of meson number, and the important one is not $g'$ but the convergent part of our formal solution.

Concluding this paper, the author wishes to thank to Professors K. Nakabayasi and I. Sato for their kind encouragement. A part of this work was studied at the Research Institute for Fundamental Physics (Kyoto) where the author was also indebted very much to Messrs. K. Nishijima and Z. Maki for their helpful suggestions and discussions.

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