Comments on Gauge Invariant Overlaps for Marginal Solutions in Open String Field Theory

Isao Kishimoto

Theoretical Physics Laboratory, RIKEN, Wako 351-0198, Japan

(Received August 14, 2008)

We calculate the gauge invariant overlaps for Schnabl/Kiermaier-Okawa-Rastelli-Zwiebach’s marginal solution with nonsingular current. The obtained formula is the same as that for Fuchs-Kroyter-Potting/Kiermaier-Okawa’s marginal solution, which was already computed by Ellwood. Our result is consistent with the expectation that these two solutions may be gauge equivalent. We also comment on a gauge invariant overlap for rolling tachyon solutions in cubic open string field theory.

§1. Introduction

Since Schnabl’s construction of an analytic solution for tachyon condensation in bosonic cubic open string field theory, there have been new developments. (See Ref. 2) and references therein.) In Refs. 3) and 4), gauge invariants $O_V(\Psi)$ specified by on-shell closed string states, which we will call gauge invariant overlaps, were computed for some solutions. The evaluation of gauge invariant overlaps, in addition to that of the action, can be used to check the gauge equivalence of apparently different string fields. In Ref. 4), the values of gauge invariant overlaps for Schnabl’s analytic solution and the numerical solution in the Siegel gauge (Ref. 5) and references therein) for tachyon condensation were computed and compared. The result was consistent with the expectation of their gauge equivalence. In Ref. 3), gauge invariant overlaps for Schnabl’s solution and one type of marginal solutions given in Refs. 6) and 7) were evaluated and interesting formulas were found. In this work, we will compute gauge invariant overlaps for another type of marginal solutions constructed in Refs. 8) and 9) and find the same formula for marginal solutions as obtained in Ref. 3). This is consistent with the expectation that the marginal solutions given in Refs. 6) and 7) and in Refs. 8) and 9) are gauge equivalent. We also apply our result to a rolling tachyon solution investigated in Ref. 10), which is an example of the marginal solutions in Refs. 8) and 9), and comment on the large deformation limit.

Let us begin by reviewing marginal solutions and gauge invariant overlaps briefly. There are two types of marginal solutions.** One was constructed by Schnabl and Kiermaier-Okawa-Rastelli-Zwiebach, which we abbreviate as Schnabl/KORZ’s marginal solution. The other one was constructed by Fuchs-Kroyter-Potting and...
generalized by Kiermaier-Okawa,\textsuperscript{7} which we denote as FKP/KO’s marginal solution in the following. The former can be applied only to the case of nonsingular marginal current $J$, namely, the operator product expansion (OPE) among $J$’s is nonsingular: $J(y)J(z) \sim \text{finite} \ (y \to z)$. For the latter, we can apply more general currents using a particular regularization, although we treat only nonsingular current $J$ in this paper for simplicity. Both solutions have one parameter $\lambda_m$ and the same form, $\lambda_m J(0)|0\rangle$, for the lowest term with respect to $\lambda_m$, but the higher terms are different.

Schnabl/KORZ’s marginal solution $\Psi_{\lambda_m}^{S/KORZ}$ is given by\textsuperscript{*)}

$$\Psi_{\lambda_m}^{S/KORZ} = \sum_{n=1}^{\infty} \lambda_m^n \psi_{m,n}, \quad (1.1)$$

$$\psi_{m,k+1} = \left(\frac{\pi}{2}\right)^k \int_0^1 dr_1 \cdots \int_0^1 dr_k \hat{U}_{\gamma(k)+1} \prod_{m=0}^{k} \hat{J}(\tilde{x}_m^{(k)})$$

$$\times \left[ -\frac{1}{\pi} (B_0 + B_0^*) \tilde{c}(\tilde{x}_0^{(k)}) \tilde{c}(\tilde{x}_k^{(k)}) + \frac{1}{2} \left( \tilde{c}(\tilde{x}_0^{(k)}) + \tilde{c}(\tilde{x}_k^{(k)}) \right) \right] |0\rangle, \quad (1.2)$$

where $U_r \equiv (2/r)^{L_0}, \hat{U}_r \equiv U_r^\dagger U_r, B_0 \equiv b_0 + \sum_{k=1}^{\infty} \frac{2(1-k)^{k+1}}{dk^2-1} b_{2k}$ and $L_0 = \{Q_B, B_0\}$. The arguments of fields $\tilde{c}$ and $\tilde{J}$, where $\tilde{c}\tilde{z}$ in the sliver frame is given by $(\cos \tilde{z})^{-2h} \phi(\tan \tilde{z})$ using $\phi(\tilde{z})$ in the upper half-plane for a primary field with dimension $h$, are specified by

$$\gamma^{(k)} = 1 + \sum_{l=1}^{k} r_l, \quad \tilde{x}_m^{(k)} = \frac{\pi}{4} \left( \gamma^{(k)} - 1 - 2 \sum_{l=1}^{m} r_l \right). \quad (1.3)$$

On the other hand, FKP/KO’s solution is given by (Appendix B)

$$\Psi_{\lambda_m}^{FKP/KO} = \sum_{n=1}^{\infty} \lambda_m^n \psi_{L,n}, \quad (1.4)$$

$$\psi_{L,k+1} = \hat{U}_{k+2} \tilde{c}\tilde{J} \left( \frac{\pi}{4} \right)$$

$$\times (-1)^k \int_{\tilde{x}_1^{(k-2)}}^{\tilde{x}_1} d\tilde{x}_1 \int_{\tilde{x}_2^{(k-4)}}^{\tilde{x}_2} d\tilde{x}_2 \cdots \int_{\tilde{x}_{k-1}^{(k-4)}}^{\tilde{x}_{k-1}} d\tilde{x}_{k-1} \hat{J}(\tilde{x}_1) \hat{J}(\tilde{x}_2) \cdots \hat{J}(\tilde{x}_k)|0\rangle. \quad (1.5)$$

In order to satisfy the reality condition, we should apply a gauge transformation by

$$U = \mathcal{I} + \sum_{n=1}^{\infty} \lambda_m^n U_n, \quad (1.6)$$

\textsuperscript{*)} We use the notation in Ref. 1). $\Psi_{\lambda_m}^{S/KORZ}$ is essentially obtained from a BRST-invariant and nilpotent string field $\psi_m = \hat{U}_1 \tilde{c}\tilde{J}(0)|0\rangle$ as\textsuperscript{13)} $\Psi_{\lambda_m}^{(\alpha,\beta)} = P_\alpha \ast (1 + \lambda_m \hat{\tilde{v}}_m \ast A^{(\alpha+\beta)}_{m-1}) \ast \lambda_m \hat{\tilde{v}}_m \ast P_\beta$, where $P_\alpha = \hat{U}_{\alpha+1}|0\rangle$, $A^{(\gamma)} = \frac{1}{2} \int_0^1 d\alpha B_1^\dagger P_\alpha$, and $B_1^\dagger = \frac{1}{2} (b_1 + b_{-1}) + \frac{1}{4} (Q_B + B_0^*)$. In this paper, we set $\alpha = \beta = 1/2$ for simplicity. Other solutions, except for $\alpha = \beta = 0$, can be obtained using the relation $\Psi_{\lambda_m}^{(\alpha,\beta)} = e^{\pi(\beta-m)K_1} (\alpha + \beta)(C_0 - C_0^*)^2 \Psi_{\lambda_m}^{(1/2,1/2)}$, which gives the same value of gauge invariant overlaps thanks to Eq. (1.10).
\[ U_n = \hat{U}_{n+1}(1)^n \int_{\bar{x}(-n-1)}^{\bar{x}(n-1)} \bar{d}x_1 \int_{\bar{x}(n-1)}^{\bar{x}(n-1)} \bar{d}x_2 \cdots \int_{\bar{x}_{n-1}}^{\bar{x}(n-1)} d\hat{x}_n \hat{J}(\hat{x}_1) \hat{J}(\hat{x}_2) \cdots \hat{J}(\hat{x}_n)|0\rangle, \]

where \( \mathcal{I} = \hat{U}_1|0\rangle \) is the identity state, then

\[ \psi_{\lambda_m}^U \equiv \frac{1}{\sqrt{U}} * \psi_\text{FKP/KO}^U * \sqrt{U} + \frac{1}{\sqrt{U}} * Q_B \sqrt{U}. \]

The gauge invariant overlap \( \mathcal{O}_\gamma(\hat{V}) \) is defined by

\[ \mathcal{O}_\gamma(\hat{V}) = \langle \mathcal{I}|\hat{V}(i)|\Psi\rangle = \langle \hat{\gamma}(1_c, 2)|V_c\rangle_{1c}. \]

where \( \langle \mathcal{I}|\hat{V}(i) = \langle \hat{\gamma}(1_c, 2)|V_c\rangle_{1c} \) corresponds to an on-shell closed string state and \( \langle \hat{\gamma}(1_c, 2) \rangle \) is Shapiro-Thorn’s vertex,\(^\text{(14)}\) which relates the closed string Hilbert space \( (1_c) \) to the open string Hilbert space \( (2) \). \( |V_c\rangle = \langle \gamma_c = \lambda_1 V_{m}(0, 0)|0\rangle \) is given by a matter primary field \( V_{m}(z, \bar{z}) \) with dimension \( (1, 1) \). We consider \( \psi_{\lambda_m}^U \) instead of \( \psi_{\lambda_m}^U \) because gauge invariant overlap \( \mathcal{O}_\gamma(\hat{V}) \) is invariant under gauge transformations. In particular, the on-shell closed string state in the open string Hilbert space, \( \langle \mathcal{I}|\hat{V}(i) = \langle \hat{\gamma}(1_c, 2)|V_c\rangle_{1c} \), has the following symmetries (Appendix C):

\[ \langle \mathcal{I}|\hat{V}(i)|K_n = 0, \quad K_n \equiv L_n - (-1)^n L_{-n}, \]

\[ \langle \mathcal{I}|\hat{V}(i)|b_n - (-1)^n b_{-n} = 0, \]

\[ \langle \mathcal{I}|\hat{V}(i)|c_n + (-1)^n c_{-n} = 0. \]

In the first line, \( L_n \) denotes the total Virasoro generator, which has zero central charge. The first line can be derived from the second line and BRST invariance: \( \langle \mathcal{I}|\hat{V}(i)|Q_B = 0 \).

The rest of this paper is organized as follows. In the next section, we compute gauge invariant overlaps for two types of marginal solutions by rewriting string fields appropriately. In §3, we comment on gauge invariant overlaps for lightlike and timelike rolling tachyon solutions using the result in §2. In Appendix A, we rewrite Schnabl’s solution for tachyon condensation in the same way as marginal solutions in §2. In Appendix B, we briefly review FKP/KO’s marginal solution and fix our conventions. In Appendix C, we derive symmetries for on-shell closed string states using the Shapiro-Thorn vertex.

**§2. Evaluation of gauge invariant overlaps for marginal solutions**

We will rewrite the marginal solutions in order to evaluate gauge invariant overlaps easily using symmetries (1-10), (1-11) and (1-12) of on-shell closed string states.

Schnabl/KORZ’s marginal solution can be decomposed in the same way as \( \psi_\gamma^S \) (Appendix A) because the ghost sector of \( \psi_{m,k+1} \) (1-2) is similar to \( \psi_r \) (A-2). From (A-4), (A-5) and \( r\mathcal{L}_0 \phi(r\tilde{z}) = r^h \tilde{\phi} \) for a primary field \( \phi \) with dimension \( h \), we rewrite \( \psi_{m,k+1} \) as

\[ \psi_{m,k+1} = \left(-\frac{\pi}{2}\right)^k \int_0^1 dr_1 \cdots \int_0^1 dr_k (\gamma(k)(\mathcal{L}_0 - \mathcal{L}_0^\dagger)/2) \prod_{m=0}^k (\gamma(k) -1, j(\frac{\pi}{\gamma(k)})^m) \]
\[ \times \left[ \frac{\gamma^{(k)}}{\pi} \left( \mathcal{B}_0 - \mathcal{B}_0^{\dagger} \right) \tilde{c} \left( \frac{\tilde{x}^{(k)}}{\gamma^{(k)}} \right) \tilde{c} \left( \frac{\tilde{x}^{(k)}}{\gamma^{(k)}} \right) + \frac{1}{2} \left( \tilde{c} \left( \frac{\tilde{x}^{(k)}}{\gamma^{(k)}} \right) + \tilde{c} \left( \frac{\tilde{x}^{(k)}}{\gamma^{(k)}} \right) \right) \right] |0\rangle. \] (2.1)

Then we have

\[ \psi^{S/KORZ}_{\lambda_m} = \sum_{k=0}^{\infty} \lambda_m^{k+1} \int_0^1 dr_1 \cdots \int_0^1 dr_k \left( \frac{-\pi}{2} \right)^k (\gamma^{(k)})^{-k-1} \prod_{m=0}^{k-1} \tilde{J} \left( \frac{\tilde{x}^{(k)}}{\gamma^{(k)}} \right) c_1 |0\rangle + O(\mathcal{L}_0 - \mathcal{L}_0^{\dagger}, \mathcal{B}_0 - \mathcal{B}_0^{\dagger}, c_n + (-1)^n c_{-n}) \]

\[ = \sum_{k=0}^{\infty} \lambda_m^{k+1} \int_0^1 dr_1 \cdots \int_0^1 dr_k \left( \frac{-\pi}{2} \right)^k (\gamma^{(k)})^{-k-1} \prod_{m=0}^{k-1} \tilde{J} \left( \frac{\tilde{x}^{(k)}}{\gamma^{(k)}} \right) c_1 |0\rangle + O(K_1, \mathcal{L}_0 - \mathcal{L}_0^{\dagger}, \mathcal{B}_0 - \mathcal{B}_0^{\dagger}, c_n + (-1)^n c_{-n}), \] (2.2)

where we have used the relations: \( e^{\alpha K_1} \tilde{J}(\tilde{z}) e^{-\alpha K_1} = \tilde{J}(\tilde{z} + \alpha) \), (A.6), (A.7) and

\[ e^{\alpha K_1} c_1 |0\rangle = \tilde{c}(\alpha) |0\rangle = \tilde{c}_1 |0\rangle + \sum_{k=0}^{\infty} (\alpha^{2k+2} \tilde{c}_{-1-2k} |0\rangle + \alpha^{2k+1} \tilde{c}_{-2k} |0\rangle), \] (2.3)

\[ \tilde{c}_{-2k} |0\rangle = \sum_{l=0}^{k-1} C_l^{(k)} (c_{1-2k+l} - c_{-1+2k-l}) |0\rangle, \quad \tilde{c}_1 |0\rangle = c_1 |0\rangle, \quad \tilde{c}_0 |0\rangle = c_0 |0\rangle, \] (2.4)

\[ C_l^{(k)} \equiv \oint_0 \frac{dz}{2\pi i} \left( \arctan z \right)^{-2k-1} \sum_{l=0}^{k-1} C_l^{(k)} (c_{l-2k+l} + c_{2k-2l}) |0\rangle \] (2.5)

\[ \tilde{c}_1 |0\rangle = c_1 |0\rangle, \quad \tilde{c}_0 |0\rangle = c_0 |0\rangle, \] (2.6)

\[ \tilde{c}_{-2k} |0\rangle = \left( \sum_{l=0}^{k-1} C_l^{(k)} (c_{l-2k+l} + c_{2k-2l}) + C_k^{(k)} c_0 \right) |0\rangle, \quad \tilde{c}_1 |0\rangle = c_1 |0\rangle, \] (2.7)

\[ C_l^{(k)} \equiv \oint_0 \frac{dz}{2\pi i} \left( \arctan z \right)^{-2k-2} \sum_{l=0}^{k-1} C_l^{(k)} (c_{l-2k+l} + c_{2k-2l}) \left( \frac{-1}{2} \right)^{2k-2l} \] (2.8)

The boundary operator \( \tilde{J}(\tilde{z}) \) on the sliver frame with dimension 1 is related to the one on the unit disk \( w = e^{2iz} \), which we denote by \( J_w(w) \), as \( \tilde{J}(\tilde{z}) = |2ie^{2iz}|J_w(e^{2iz}) \). Using \( J_w \) and inserting \( 1 = U^{-1}_1 U_1 \) in front of the first term, \( \psi_{\lambda_m}^{S/KORZ} \) (2.2) can be rewritten as

\[ \psi_{\lambda_m}^{S/KORZ} = -2U^{-1}_1 \sum_{k=0}^{\infty} (-\lambda_m)^{k+1} \int_D d\varphi_1^{(k)} \cdots d\varphi_k^{(k)} J_w(1) \prod_{l=1}^{k} J_w(e^{i\varphi_l^{(k)}}) c_1 |0\rangle \]

\[ + O(K_1, \mathcal{L}_0 - \mathcal{L}_0^{\dagger}, \mathcal{B}_0 - \mathcal{B}_0^{\dagger}, c_n + (-1)^n c_{-n}), \] (2.9)

where the arguments are given by changing the variables as

\[ \varphi_l^{(k)} \equiv \frac{4(\tilde{x}_l^{(k)} - \tilde{x}_0^{(k)})}{\gamma^{(k)}} = -2\pi \frac{\sum_{m=1}^l r_m}{1 + \sum_{m=1}^k r_m}. \] (2.10)
It induces the Jacobian
\[
\left| \frac{\partial (\varphi_1^{(k)}, \ldots, \varphi_k^{(k)})}{\partial (r_1, \ldots, r_k)} \right| = \frac{(2\pi)^k}{(\gamma(k)+1)} = \frac{(2\pi)^k}{(1 + \sum_{l=1}^k r_l)^{k+1}},
\]
which cancels the extra factor of the first term of (2.2). The integration region \(\mathcal{D}\) is determined by \(0 \leq r_l \leq 1\) (\(l = 1, 2, \ldots, k\) or
\[
0 \leq -\varphi^{(k)}_1 \leq 2\pi - (-\varphi^{(k)}_k), \quad 0 \leq -\varphi^{(k)}_l - (-\varphi^{(k)}_{l-1}) \leq 2\pi - (-\varphi^{(k)}_k), \quad (l = 2, 3, \ldots, k)
\]
and the volume is computed as
\[
\int_{\mathcal{D}} d\varphi_1^{(k)} \cdots d\varphi_k^{(k)} = \int_0^1 dr_1 \cdots \int_0^1 dr_k \frac{(2\pi)^k}{(1 + \sum_{l=1}^k r_l)^{k+1}} = \frac{(2\pi)^k}{(k+1)!}.
\]
We note that the nonsingular currents \(J_w\) can be exchanged without singular behavior. Therefore, by noting (C.4) and using \(U_1^{-1} = (\int_0^{2\pi} d\theta \, e^{\theta K_1} + \int_0^{2\pi} d\theta (1 - e^{\theta K_1}))U_1^{-1} = \int_0^{2\pi} d\theta e^{\theta K_1} + O(K_1)\), the integration in (2.9) can be rewritten as that on the unit circle:
\[
\Psi_{\lambda_m}^{S/KORZ} = -\frac{1}{\pi} U_1^{-1} \sum_{k=0}^{\infty} (-\lambda_m)^{k+1} \int_0^{2\pi} d\theta \int_{\mathcal{D}} d\varphi_1^{(k)} \cdots d\varphi_k^{(k)}
\]
\[
\times J_w(e^{i\theta}) \prod_{l=1}^k J_w(e^{i(\theta + \varphi^{(k)}_l)})c_1|0\rangle
\]
\[
+ O(K_1, \mathcal{L}_0 - \mathcal{L}_0^\dagger, B_0 - B_0^\dagger, c_n + (-1)^n c_{-n})
\]
\[
= -\frac{1}{\pi} U_1^{-1} \left( e^{-\lambda_m \int_0^{2\pi} d\theta J_w(e^{i\theta})} - 1 \right) c_1|0\rangle
\]
\[
+ O(K_1, \mathcal{L}_0 - \mathcal{L}_0^\dagger, B_0 - B_0^\dagger, c_n + (-1)^n c_{-n}).
\]
(2.14)
In the last equality, we note that any \(k + 1\) points specified by coordinates \(\theta_l (\in \mathbb{R} \mod 2\pi)\) (\(l = 0, 1, \ldots, k\)) can be chosen, such as \((\theta_0 = \theta, \theta_1 = \theta + \varphi^{(k)}_1, \ldots, \theta_k = \theta + \varphi^{(k)}_k)\), which satisfy (2.12) by shifting the origin and exchanging them for each other appropriately, and we have used (2.13) for the \(O(\lambda_m^{k+1})\) term on multiplicity.
In the case of FKP/KO’s marginal solution (1.4), we can also rewrite \(\Psi_{\lambda_m}^{FKP/KO}\) in the same way as above:
\[
\Psi_{\lambda_m}^{FKP/KO} = \sum_{n=1}^{\infty} \lambda_m^n J \left( \frac{\pi(n-1)}{4n} \right) \left( -\frac{1}{n} \right)^{n-1} \int_0^{\pi(n-1)} d\tilde{x}_1 \int_0^{\pi(n-3)} d\tilde{x}_2 \cdots \int_0^{\pi(n-1)} d\tilde{x}_n \cdot \int_0^{\pi(n-1)} d\tilde{x}_n - 1
\]
\[
\times J(\tilde{x}_1/n, \tilde{x}_2/n) \cdots J(\tilde{x}_n-1/n)c_1|0\rangle
\]
\[
+ O(\mathcal{L}_0 - \mathcal{L}_0^\dagger, c_n + (-1)^n c_{-n})
\]
\[
= -\frac{1}{\pi} U_1^{-1} \sum_{n=1}^{\infty} (-\lambda_m)^n \int_0^{2\pi} d\theta_0 \int_0^{\theta_0} \frac{2\pi}{n} d\theta_1 \int_{\theta_1}^{\theta_0} \frac{4\pi}{n} d\theta_2 \cdots \int_{\theta_{n-2}}^{\theta_{n-2}} \frac{2\pi(n-1)}{n} d\theta_{n-1}
\]
\[
\times J(\tilde{x}_1/n, \tilde{x}_2/n) \cdots J(\tilde{x}_n-1/n)c_1|0\rangle.
\]
In this case, noting the volume
\[
\int_0^{2\pi} d\theta_0 \int_{\theta_0 - \frac{2\pi}{n}}^{\theta_0} d\theta_1 \int_{\theta_0 - \frac{2\pi}{n}}^{\theta_0} d\theta_2 \cdots \int_{\theta_0 - \frac{2\pi(n-1)}{n}}^{\theta_0} d\theta_{n-1} = \frac{(2\pi)^n}{n!}
\] (2.16)
and using the fact\(^3\) that any \(n\) points specified by coordinates \(\theta_1(\in \mathbb{R} \mod 2\pi) \,(l = 0, 1, \cdots, n-1)\) can be chosen, such as \(\theta_0 - \theta_l \leq \frac{2\pi}{n}\), \(\theta_0 \geq \cdots \geq \theta_{n-1}\), by shifting the origin and exchanging them for each other appropriately, (2.15) can be rewritten as
\[
\psi_{\lambda_m, L}^{\text{FKP/KO}} = -\frac{1}{\pi} U_1^{-1} \left( e^{-\lambda_m \int_0^{2\pi} d\theta J_w(e^{i\theta})} - 1 \right) c_1 |0\rangle + O(K_1, \mathcal{L}_0 - \mathcal{L}_0^i, c_n + (-1)^n c_{-n}) \). (2.17)

From decompositions of Schnabl/KORZ’s (2.14) and FKP/KO’s (2.17) marginal solutions, the symmetries of the on-shell closed string state \(\langle \mathcal{I} | V(i) \rangle: (1.10), (1.11), (1.12), \) and \(\langle \mathcal{I} | V(i) | \varphi \rangle = \langle V(i) f_{\mathcal{I}} \circ \varphi \rangle\), where the conformal map \(f_{\mathcal{I}}(z) = 2z/(1 - z^2)\) corresponds to \(U_1\), we have
\[
\mathcal{O}_V(\psi_{\lambda_m}^{S/KORZ}) = \mathcal{O}_V(\psi_{\lambda_m, L}^{\text{FKP/KO}})
= -\frac{1}{2\pi i} \left\langle e^{\omega \bar{e} w} V_{w \bar{w}, m}(0, 0) e^{\bar{w} w} (1) \left( e^{-\lambda_m \int_0^{2\pi} d\theta J_w(e^{i\theta})} - 1 \right) \right\rangle_{\text{disk}}. (2.18)
\]
The second equality was already shown in Ref. 3). The first equality means that for the same parameter \(\lambda_m\) and nonsingular current \(J\), Schnabl/KORZ’s marginal solution and FKP/KO’s one give the same value for the gauge invariant overlap \(\mathcal{O}_V(\psi)\). This is consistent with the expectation that \(\psi_{\lambda_m}^{S/KORZ}\) and \(\psi_{\lambda_m, L}^{\text{FKP/KO}}\) are gauge equivalent. The above value (2.18) is related to the closed string one-point function\(^3\), such as \(A_{\lambda_m}(V) - A_{\lambda_m = 0}(V)\), where \(A_{\lambda_m}(V)\) is the disk amplitude for a closed string vertex \(V\) with the boundary condition deformed by \(\lambda_m J\).

§3. Comments on rolling tachyon solutions

Let us consider the gauge invariant overlap \(\mathcal{O}_V(\psi)\) with the zero momentum graviton \(V_m = \zeta_{\mu\nu} \partial X^\mu \partial X^\nu\) for Hellerman-Schnabl’s solution\(^10\), which we denote by \(\psi_{\lambda_m}^{\text{HS}}\). \(\psi_{\lambda_m}^{\text{HS}}\) is given by Schnabl/KORZ’s solution \(\psi_{\lambda_m}^{S/KORZ}\) with the lightlike rolling tachyon operator \(J = e^{\beta X^+}\) (\(\beta \equiv 1/(\alpha' V^+)\)), on the linear dilaton background \(\Phi(x) = V_{\mu} x^\mu, V^+ > 0, 26 = D + 6 \alpha' V_{\mu} V^\mu, \mu = 0, 1, \cdots, D - 1\). On the linear dilaton background, the matter Virasoro operator is deformed by \(V^\mu\) as \(L^{(m)}_n = \frac{1}{2} \sum_k : \alpha_{k, \mu} \alpha_{n-k}^\mu : + i \sqrt{\alpha'} (n + 1) V_{\mu} \alpha^\mu_n\). Therefore, polarization \(\zeta_{\mu\nu}\) for the on-shell
closed string state should satisfy the transversality condition \( \zeta_{\mu\nu} V^\nu = V^\mu \zeta_{\mu\nu} = 0 \). Applying the formula for gauge invariant overlap (2.18), we have

\[
O_{V_\zeta}(\Psi_{\lambda_m}^{HS}) = \frac{1}{2\pi i} \zeta_{\mu\nu} \left( \partial_w X^\mu \partial_w X^\nu \left( e^{-\lambda_m \int_0^{2\pi} d\theta J_w(e^{i\theta})} - 1 \right) \right)_{\text{disk}}^\text{mat} \\
= \int d^D x \frac{1}{2\pi i} \zeta_{\mu\nu} (A^{\mu\nu}(x) |_{\lambda=2\pi\lambda_m} - A^{\mu\nu}(x) |_{\lambda=0}) .
\]

Here, we have used

\[
A^{\mu\nu}(x) \equiv \langle \partial_w X^\mu \partial_w X^\nu ; (0,0) | e^{-\lambda \int_0^{2\pi} d\theta J_w(e^{i\theta})} \rangle_{\text{disk},x} ,
\]

which is a CFT correlator in the linear dilaton background on a disk with a fixed zero mode such as \( x^\mu = \frac{1}{2\pi} \int_0^{2\pi} d\theta X^\mu(e^{i\theta}) \). Substituting a concrete expression for \( A^{\mu\nu}(x) \), which was explicitly computed in §5 in Ref. 10), the gauge invariant overlap is evaluated as

\[
O_{V_\zeta}(\Psi_{\lambda_m}^{HS}) = \frac{\alpha'}{4\pi i} \int d^D x e^{-V(x)} \left( \zeta_{\mu\nu} \eta^{\mu\nu} (e^{-2\pi\lambda_m e^{\beta x^+}} - 1) - 4\pi \beta^2 \alpha' \lambda_m \zeta_{-} e^{\beta x^+ - 2\pi\lambda_m e^{\beta x^+}} \right) .
\]

On the other hand, the gauge invariant overlap \( O_{V_\zeta}(\Psi) \) for Schnabl’s solution for tachyon condensation \( \Psi^S_{\lambda=1} \) can be evaluated using the result in Eq. (3.28) in Ref. 4) with the normalization

\[
C_{V_\zeta} = (2\pi)^D \delta^D(iV) \zeta_{\mu\nu} \eta^{\mu\nu} \frac{-\alpha'}{2} = -\frac{\alpha'}{4\pi i} \zeta_{\mu\nu} \eta^{\mu\nu} \int d^D x e^{-V(x)}
\]

in this case. Namely, we have

\[
O_{V_\zeta}(\Psi_{\lambda_m}^{S}) = -\frac{\alpha'}{4\pi i} \zeta_{\mu\nu} \eta^{\mu\nu} \int d^D x e^{-V(x)} .
\]

Comparing the above expression and (3.3), we obtain the relation

\[
\lim_{\lambda_m \to +\infty} O_{V_\zeta}(\Psi_{\lambda_m}^{HS}) = O_{V_\zeta}(\Psi_{\lambda_m}^{S}) ,
\]

at least formally. The result is consistent with the limit of the string field itself, \( \lim_{x^0 \to +\infty} \Psi_{\lambda_m}^{HS} = \lim_{\lambda_m \to +\infty} \Psi_{\lambda_m}^{S} = \Psi^S_{\lambda=1} \), which was proved in Ref. 10) in terms of the \( \mathcal{L}_0 \) basis.

Next, let us consider the ordinary timelike rolling tachyon solution, namely, Schnabl/KORZ’s solution \( \Psi_{\lambda_m}^{S/KORZ} \) with the timelike rolling tachyon operator \( J = e^{X^0} \) on the flat background. It is known\(^8,9\) that the tachyon component given by the coefficient function for \( c_1 |0\) in \( \Psi_{\lambda_m}^{S/KORZ} \) wildly oscillates for \( x^0 \to +\infty \) (or \( \lambda_m \to +\infty \)) numerically. (See also Refs. 10, 15) and 16).) This seems to imply \( \lim_{\lambda_m \to +\infty} \Psi_{\lambda_m}^{S/KORZ} \neq \Psi_{\lambda=1}^S \) for \( J = e^{X^0} \). On the other hand, one can formally evaluate the gauge invariant overlap with the zero momentum graviton
$V_m = \zeta_{\mu\nu} 2\partial X^\mu \tilde{\partial} X^\nu$ in the same way as in the above lightlike case using the formula (2.18) and $A^{\mu\nu}(x)$ computed in Ref. 17):

$$\mathcal{O}_V(\Psi^{S/KORZ}_{\lambda_m}) = \frac{1}{2\pi i} \int d^d x \zeta_{\mu\nu} \eta^{\mu\nu}(f(x^0) - 1), \quad f(x^0) \equiv \frac{1}{1 + 2\pi \lambda_m e^{x^0}}. \quad (3.7)$$

If one adopts the limit $\lambda_m \rightarrow +\infty$ in the integrand naively, it seems to converge to the value for Schnabl’s solution for tachyon condensation: $\mathcal{O}_V(\Psi^{S}_{\lambda=1}) = \frac{1}{2\pi i} \zeta_{\mu\nu} \eta^{\mu\nu} \int d^d x$. However, the limit for the flat space may be too naive because $\lambda_m$-dependence should be absorbed by shifting the origin of $x^0$ as an integration value of $\mathcal{O}_V(\Psi^{S/KORZ}_{\lambda_m})$. It is desired to define and evaluate local gauge invariant quantities in string field theory in order to investigate such a limit.

### Acknowledgements

The author would like to thank Koji Hashimoto, Hiroyuki Hata, Teruhiko Kawano, Michael Kiermaier, Yuji Okawa and Tomohiko Takahashi for helpful discussions and the Yukawa Institute for Theoretical Physics at Kyoto University for providing a stimulating atmosphere. Discussions during the YITP workshop YITP-W-08-04 on “Development of Quantum Field Theory and String Theory” were useful in completing this work. The work was supported in part by the Special Post-doctoral Researchers Program at RIKEN and a Grant-in-Aid for Young Scientists (#19740155) from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

### Appendix A

---

**On Schnabl’s Solution for Tachyon Condensation**

Schnabl’s solution for tachyon condensation\(^1\) is similar to Schnabl/KORZ’s marginal solutions but simpler than them. Hence, it is instructive to investigate the decomposition of the solution for tachyon condensation in order to simplify the computation of gauge invariant overlaps.

Schnabl’s solution $\Psi^{S}_{\lambda}$ with parameter $\lambda$ is given by

$$\psi_r = \frac{2\pi}{\lambda} \left[ \frac{1}{2} \int d^d x \zeta_{\mu\nu} \eta^{\mu\nu}(f(x^0) - 1) \right] |0\rangle.$$

The string field $\psi_r$ can be rewritten as

$$\psi_r = \frac{2\pi}{\lambda} \left[ \frac{1}{2} \left( \tilde{c} \left( \frac{\pi r}{4(1 + r)} \right) + \tilde{c} \left( -\frac{\pi r}{4(1 + r)} \right) \right) + \frac{1 + r}{\pi} \left( B_0 - B_0^\dagger \right) \tilde{c} \left( \frac{\pi r}{4(1 + r)} \right) \tilde{c} \left( -\frac{\pi r}{4(1 + r)} \right) \right] |0\rangle. \quad (A.2)$$

\[^1\] The string field $\psi_r$ can be rewritten as
using
\[
\hat{U}_{r+2} = e^{-\frac{r}{2}(\mathcal{L}_0 + \mathcal{L}_0^\dagger)} = (1 + r)^{-\mathcal{L}_0 \mathcal{L}_0^\dagger} (1 + r)^{-\mathcal{L}_0}, \tag{A.4}
\]
\[
\{ \mathcal{B}_0, \tilde{c}(\tilde{z}) \} = \tilde{z}, \quad [\mathcal{L}_0, \mathcal{B}_0 + \mathcal{B}_0^\dagger] = \mathcal{B}_0 + \mathcal{B}_0^\dagger. \tag{A.5}
\]
Furthermore, noting the anticommutation relation \( \{ b_p, \tilde{c}(x) \} = (1/2) \sin 2x (\tan x)^p \), we have
\[
\frac{1}{2} (\tilde{c}(x) + \tilde{c}(-x)) |0\rangle = c_1 |0\rangle + \cos^2 x \sum_{k=1}^{\infty} (\tan x)^{2k} (c_{1-2k} - c_{2k-1}) |0\rangle, \tag{A.6}
\]
\[
\tilde{c}(x) \tilde{c}(-x) |0\rangle = \left( c_0 + \sum_{l=1}^{\infty} (\tan x)^{2l} (c_{-2l} + c_{2l}) \right) \sin x \left( c_1 + \cos^2 x \sum_{k=1}^{\infty} (\tan x)^{2k} (c_{1-2k} - c_{2k-1}) \right) |0\rangle, \tag{A.7}
\]
which imply that \( \psi_r \) \((A.3)\) can be rewritten as
\[
\psi_r = \frac{2}{\pi} c_1 |0\rangle + O(\mathcal{L}_0 - \mathcal{L}_0^\dagger, \mathcal{B}_0 - \mathcal{B}_0^\dagger, c_k + (-1)^k c_{-k}). \tag{A.8}
\]
Here, \( O(\mathcal{L}_0 - \mathcal{L}_0^\dagger, \mathcal{B}_0 - \mathcal{B}_0^\dagger, c_k + (-1)^k c_{-k}) \) denotes some linear combination of terms comprising \( \mathcal{L}_0 - \mathcal{L}_0^\dagger, \mathcal{B}_0 - \mathcal{B}_0^\dagger, \) and \( c_k + (-1)^k c_{-k} \), where at least one of them is multiplied on the conformal vacuum \(|0\rangle\). The first term \((2/\pi) c_1 |0\rangle\) does not depend on \( r \). Using this fact and \((A.1)\), we have
\[
\Psi_\lambda^S = \begin{cases} 
\frac{2}{\pi} c_1 |0\rangle + O(\mathcal{L}_0 - \mathcal{L}_0^\dagger, \mathcal{B}_0 - \mathcal{B}_0^\dagger, c_k + (-1)^k c_{-k}), & (\lambda = 1) \\
O(\mathcal{L}_0 - \mathcal{L}_0^\dagger, \mathcal{B}_0 - \mathcal{B}_0^\dagger, c_k + (-1)^k c_{-k}), & (\lambda \neq 1) 
\end{cases} \tag{A.9}
\]
Because \( \mathcal{L}_0 - \mathcal{L}_0^\dagger \) and \( \mathcal{B}_0 - \mathcal{B}_0^\dagger \) are linear combinations of \( K_n \) and \( b_n \rightarrow (-1)^n b_{-n} \), respectively, the terms in \( O(\mathcal{L}_0 - \mathcal{L}_0^\dagger, \mathcal{B}_0 - \mathcal{B}_0^\dagger, c_k + (-1)^k c_{-k}) \) give no contribution to the gauge invariant overlaps thanks to symmetries \((1\cdot10), (1\cdot11)\) and \((1\cdot12)\) of on-shell closed string states \( \langle \mathcal{I} | \mathcal{V}(i) \rangle \). The first term \( \psi_0 = \frac{2}{\pi} c_1 |0\rangle \) of \( \Psi_\lambda^S \) only contributes to the gauge invariant overlaps, which is consistent with the result in Refs. 3) and 4), and it reproduces the ordinary boundary state by contracting the Shapiro-Thorn vertex with projection \( \mathcal{P} b_0^- \) in the closed string sector. 18)

**Appendix B**

—— On FKP/KO’s Marginal Solution ——

Here, we review FKP/KO’s marginal solution with nonsingular current \( J \). Let us construct a solution with parameter \( \lambda_m \), such as
\[
\Psi = \sum_{n=1}^{\infty} \lambda_m^n \psi_n, \quad \psi_m = Q_B \phi_m + \sum_{k=1}^{m-1} \psi_k \star \phi_{m-k}, \quad (m \geq 2); \quad \psi_1 = Q_B \phi_1. \tag{B.1}
\]
In fact, using (B.1) for any $\phi_m$ with ghost number 0, we can check the equation of motion $Q_B \Psi + \Psi \ast \Psi = 0$ formally order by order in $\lambda_m$:

$$Q_B \psi_1 = 0, \quad Q_B \psi_n + \sum_{k=1}^{n-1} \psi_{n-k} \ast \psi_k = 0. \quad (n \geq 2) \quad (B.2)$$

By choosing $\phi_n$, such as

$$\phi_n = \frac{(-1)^{n-1}}{n!} X^n(0)|0 \rangle \ast \hat{U}_n|0 \rangle = \frac{(-1)^{n-1}}{n!} \hat{U}_{n+1} \hat{X}^n \left( \frac{\pi}{4} (n-1) \right) |0 \rangle, \quad (n \geq 1) \quad (B.3)$$

with $X \equiv \zeta_{\mu} X^\mu, (\zeta_{\mu} \zeta^\mu = 0)$, one can show that the obtained solution $\Psi$ is independent of the zero mode of $X(z)$. Noting the BRST transformation $[Q_B, X(z)] = c J(z)$ with $J \equiv \partial X$, which is a primary field with dimension 1, we can obtain $\psi_n$ such as $\psi_1 = c J(0)|0 \rangle, \psi_2 = -\hat{U}_2 \hat{c} \hat{J}(\frac{\pi}{4}) \int_{-\pi}^{\pi} d\bar{x} \hat{J}(\bar{x}) |0 \rangle$ and

$$\psi_n = -\frac{(-1)^{n-2}}{(n-1)!} c J X^{n-1}(0)|0 \rangle \ast \hat{U}_n|0 \rangle + c J(0)|0 \rangle \ast \phi_{n-1} + \sum_{k=2}^{n-1} \psi_k \ast \phi_{n-k}, \quad (B.4)$$

for $n \geq 3$. Taking the ansatz for $\psi_n$ without $X$ itself:

$$\psi_n = \hat{U}_{n+1} \hat{c} \hat{J}(\frac{\pi}{4} (n-1)) X_{1,2}^{(n)} \psi_{n-2}(X_{1,2}^{(n)}, X_{1,3}^{(n)}, \cdots, X_{1,n}^{(n)})|0 \rangle, \quad (B.5)$$

$$X_{i,j}^{(n)} = \int \frac{\pi}{4} (n-2j+1) d\bar{x} \hat{J}(\bar{x}), \quad (B.6)$$

where $f_{n-2}(x_1, \cdots, x_{n-1})$ is a homogeneous polynomial of degree $n-2$ with respect to $x_i$, and using the star product formula developed in Ref. 1), we find the recurrence equation

$$f_{n-2}(x_1, x_2, \cdots, x_{n-1}) = -\frac{(x_1)^{n-2}}{(n-1)!} - \sum_{k=0}^{n-3} f_k(x_1, x_2, \cdots, x_{k+1})(x_{k+2})^{n-k-2} \frac{(n-k-2)!}{(n-1)!} \quad (B.7)$$

$(n \geq 3)$ with $f_0 \equiv -1$ from (B.4) and (B.3). This equation can be solved as

$$x_1 f_{n-2}(x_1, x_2, \cdots, x_{n-1})$$

$$= -\frac{(x_1)^{n-1}}{(n-1)!} + \sum_{s=1}^{n-2} \sum_{k_1=1}^{n-2-(s-1)} \cdots \sum_{k_l=1}^{n-2-(s-p)-\sum_{i=1}^{p-1} k_i} \frac{(-1)^{s-1}}{(k_1)!(\prod_{q=1}^{s} (x_{\sum_{m=1}^{s-1} k_{m+1}})^{k_{q+1}})} \frac{(x_{\sum_{m=1}^{s} k_{m+1}})^{n-1-\sum_{m=1}^{s} k_{m}}}{(n-1-\sum_{m=1}^{s} k_{m})!} \quad (B.8)$$

Using the above result, we have

$$X_{1,2}^{(n)} f_{n-2}(X_{1,2}^{(n)}, X_{1,3}^{(n)}, \cdots, X_{1,n}^{(n)})$$

$$= (-1)^{n-1} \int_{-\pi}^{\pi} d\bar{x} \int_{\pi}^{\bar{x}} d\bar{x}_1 \int_{\pi}^{\bar{x}_2} d\bar{x}_2 \cdots d\bar{x}_{n-2} d\bar{x}_{n-1} \hat{J}(\bar{x}_1) \hat{J}(\bar{x}_2) \cdots \hat{J}(\bar{x}_{n-1}), \quad (B.9)$$
which corresponds to the expression given in Ref. 7). If we use this formula or (1.4) with any matter primary field \( J \), which has dimension 1 and nonsingular OPE, we can check that the obtained string field \( \Psi_{\lambda m, L}^{FKP/KO} \) satisfies the equation of motion.

### Appendix C

**Relation to the Shapiro-Thorn Vertex**

It is convenient to use the Shapiro-Thorn vertex \( \langle \hat{\gamma}(1, 2) \rangle \) to find formulas related to the gauge invariant overlaps, because they can be expressed using \( \langle \hat{\gamma}(1, 2) \rangle \), as in (1.9). On the closed and open string sides, \( \langle \hat{\gamma}(1, 2) \rangle \) is specified by maps \( h_1(w_1) = -i(w_1 - 1)/(w_1 + 1) \) and \( h_2(w_2) = (w_2 - 1/w_2)/2 \), respectively. (See Appendix B in Ref. 4) for details.) Using these maps or explicit formulas for Neumann coefficients, we can derive

\[
\langle \hat{\gamma}(1, 2) \rangle \left( K_n^{(2)} - (-1)^n \frac{n}{4} \delta_{n, even} \right) = \langle \hat{\gamma}(1, 2) \rangle \left( -2i^n \right) \sum_{m=0} \left( -1 \right)^m \left( \eta_{2m+1} - \eta_{2m-1} \right) (L_m^{(1)} + (-1)^n \bar{L}_m^{(1)}),
\]

\[
\langle \hat{\gamma}(1, 2) \rangle \left( b_n^{(2)} - (-1)^n b_n^{(2)} \right) = \langle \hat{\gamma}(1, 2) \rangle \left( -2i^n \right) \sum_{m=0} \left( -1 \right)^m \left( \eta_{2m+1} - \eta_{2m-1} \right) (b_m^{(1)} + (-1)^n \bar{b}_m^{(1)}),
\]

\[
\langle \hat{\gamma}(1, 2) \rangle \left( c_m^{(2)} + (-1)^m c_{-m}^{(2)} \right) = \langle \hat{\gamma}(1, 2) \rangle \left( -i^n \right) \sum_{m=1} \left( -1 \right)^m \left( \eta_{2m+1} - \eta_{2m-1} \right) (c_m^{(1)} + (-1)^m \bar{c}_m^{(1)}),
\]

where \( \delta_{n, even} = 1(0) \) for \( n \): even (odd) and \( c \) is the central charge for the Virasoro algebra in the first line. \( \eta_n^k \) is defined by the generating function \( \left( \frac{1 + x}{1 - x} \right)^k = \sum_{n=0} \eta_n^k x^n \).

By contracting the above with the closed string state \( |V_\epsilon\rangle_{1c} = c_1 \bar{c}_1 V_m(0, 0)|0\rangle_{1c} \) where \( V_m(z, \bar{z}) \) is a matter primary field with dimension (1, 1), we obtain formulas (1.10), (1.11) and (1.12).

We note that the level-matching projection \( \mathcal{P} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta(L_0 - \bar{L}_0)} \) for closed string states corresponds to \( \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta} K_1 \) for the open string side on the Shapiro-Thorn vertex because of the identity

\[
\langle \hat{\gamma}(1, 2) \rangle (L_0^{(1)} - \bar{L}_0^{(1)}) = \langle \hat{\gamma}(1, 2) \rangle \frac{i}{4} K_1^{(2)},
\]

which follows from (C.1).

### References