Monopole Excitation to Cluster States

Taiichi YAMADA,1,* Yasuro FUNAKI,2 Hisashi HORIUCHI,3,4 Kiyomi IKEDA2 and Akihiro TOHSAKI3

1Laboratory of Physics, Kanto Gakuin University, Yokohama 236-8501, Japan
2Nishina Center for Accelerator-Based Science, The Institute of Physical and Chemical Research (RIKEN), Wako 351-0098, Japan
3Research Center for Nuclear Physics, Osaka University, Osaka 567-0047, Japan
4International Institute for Advanced Studies, Kizugawa 619-0225, Japan

(Received September 1, 2008)

We discuss strength of monopole excitation of the ground state to cluster states in light nuclei. We clarify that the monopole excitation to cluster states is in general strong as to be comparable with the single particle strength and shares an appreciable portion of the sum rule value in spite of large difference of the structure between the cluster state and the shell-model-like ground state. We argue that the essential reasons of the large strength are twofold. One is the fact that the clustering degree of freedom is possessed even by simple shell model wave functions. The detailed feature of this fact is described by the so-called Bayman-Bohr theorem which tells us that SU(3) shell model wave function is equivalent to cluster model wave function. The other is the ground state correlation induced by the activation of the cluster degrees of freedom described by the Bayman-Bohr theorem. We demonstrate, by deriving analytical expressions of monopole matrix elements, that the order of magnitude of the monopole strength is governed by the first reason, while the second reason plays a sufficient role in reproducing the data up to the factor of magnitude of the monopole strength. Our explanation is made by analysing three examples which are the monopole excitations to the 0_2^+ and 0_3^+ states in 16O and the one to the 0_2^+ state in 12C. The present results imply that the measurement of strong monopole transitions or excitations is in general very useful for the study of cluster states.

§1. Introduction

The monopole transitions from cluster states to ground states in light nuclei are rather large in comparison with the single particle strength. For example in 16O the monopole matrix elements M(E0) between the ground state and the first and second excited 0^+ states at E_x = 6.05 MeV and 12.05 MeV which are known to have 12C+α cluster structure1–3) are 3.55 ± 0.21 fm^2 and 4.03 ± 0.09 fm^2, respectively.4) Also in 12C the M(E0) value between the ground state and the first excited 0^+ state at E_x = 7.66 MeV (so-called Hoyle state5)) which is known to have a 3α cluster structure1) is 5.4 ± 0.2 fm^2.4) A rough estimate of the single particle strength \langle u_f(r)|r^2|u_i(r)\rangle is (3/5)R^2 ~ 5.4 fm^2 for p- and sd-shell nuclei (R ~ 3.0 fm). This estimation formula is obtained under the uniform-density approximation of u(r) ~ \sqrt{3/R^3} for u_f(r) and u_i(r) with R standing for the nuclear radius. The energy weighted strengths of the above mentioned monopole transitions give an appreciable portion of the sum rule values: in 16O they are about 3 % and 8 % for 0_2^+ and 0_3^+, respectively, and in 12C...
T. Yamada, Y. Funaki, H. Horiuchi, K. Ikeda and A. Tohsaki

about 16 % for $0^+_2$ (see Appendix A). Recently Kawabata and his collaborators have studied the excited states of $^{11}$B by performing $^{11}$B($d, d'$) reaction and they concluded that the third $3/2^-\text{state at } E_x = 8.56 \text{ MeV has a } 2\alpha + t \text{ cluster structure}$.$^6$ Among many reasons for this conclusion, one is a large monopole strength for the third $3/2^-\text{state which is of similar value to the monopole strength for the second } 0^+\text{ state in } ^{12}\text{C, and another is that the AMD (antisymmetrized molecular dynamics) calculation}$.$^6$ as well as the $2\alpha + t \text{ OCM (orthogonality condition model) calculation}$.$^7$ have reproduced the large monopole strength and have assigned loosely bound $2\alpha + t \text{ cluster structure to the third } 3/2^-\text{state}$.

The single particle estimate of the monopole transition is based on the assumption that the excited state has a one-particle one-hole excitation from the ground state. However, the cluster structure is very different from the shell-model-like structure of the ground state, and its state is described as a superposition of many-particle many-hole configurations when it is expanded by shell model configurations. This means that in the excited state with a cluster structure, the component of a one-particle one-hole excitation from the ground state configuration is expected to be very small. Therefore the observation of rather large monopole strengths for cluster states which are comparable with single particle strength looks not to be easy to explain. The $^{12}$C+$\alpha \text{ OCM calculation}$.$^3$ for $^{16}$O and $3\alpha \text{ RGM (resonating group method) calculation}$,$^8,9$ however, have reproduced rather well the experimental data of the monopole transitions. No explicit and detailed analyses of the reason why the cluster models reproduce plausibly the experimental data have been presented so far as long as we know. There should exist underlying physics in the monopole transition strengths in light nuclei.

The purpose of this paper is to clarify the basic reasons why monopole transition strength between a cluster state and the ground state in light nuclei is generally rather large in comparison with the single particle strength and shares an appreciable portion of the sum rule value, in spite of the large difference of structure between the initial and final states. We analyse the above-mentioned three cases of monopole transitions in $^{16}$O and $^{12}$C, namely the monopole transitions between the ground state and the first and second excited $0^+$ states in $^{16}$O, and the monopole transition between the ground state and the first excited $0^+$ state in $^{12}$C. Using these analyses we will show that there are two basic reasons for the generally large strength of monopole transitions. The first reason is the fact that the clustering degree of freedom is possessed even by simple shell model wave functions. The detailed feature of this fact is described by the so-called Bayman-Bohr theorem.$^{10}$ This theorem tells us that the $SU(3) \text{ shell model wave function}$.$^{11}$ describing the ground state is in most cases equivalent to the cluster-model wave function discussed by Wildermuth and Kanellopoulos.$^{12}$ Thus we can see what kinds of clustering degrees of freedom are embedded in the ground state. For example the doubly closed-shell wave function of the $^{16}$O ground state (total quanta $N_{TOT} = 12$) which is just the $SU(3) \text{ shell model wave function with } (\lambda, \mu) = (0, 0) \text{ is equivalent to a } ^{12}$C+$\alpha \text{ cluster-model wave function with } N_{TOT} = 12$. This means that the ground-state wave function of $^{16}$O originally has a $^{12}$C+$\alpha \text{ clustering degree of freedom. The second reason is the ground state correlation induced by the activation of the cluster degrees of freedom}.
Monopole Excitation to Cluster States

described by the Bayman-Bohr theorem. In the case of the above example of the 16O ground state, the ground state correlation is due to the 12C + α clustering degrees of freedom. As was explained, the first and second excited 0+ states of 16O are the cluster states with 12C + α structure. These cluster states are formed just by the excitation of the 12C + α clustering degree of freedom which is already existent in the ground state. Therefore it is quite reasonable that the large strength of the monopole transition between the ground state and the first and second excited 0+ states is explained by the above-mentioned first and second reasons. We will demonstrate, by deriving analytical expressions of monopole matrix elements, that the order of magnitude of the monopole strength is governed by the first reason, while the second reason plays a sufficient role in reproducing the data up to the factor of magnitude of the monopole strength.

In the present paper we discuss the details of the first and second reasons for 16O and 12C. In the case of 16O, we make use of the microscopic 12C + α cluster wave function, while in the case of 12C, we discuss the problem by using the so-called THSR wave function. Our results mean that the measurement of strong monopole transitions provides us in general with a very useful tool for the experimental study of cluster states as has been practiced in Ref. 6).

The present paper is organized as follows. In §2 we derive analytical expressions of the monopole matrix elements between the ground state and 12C+α cluster states in 16O and those between the ground state and 3α cluster state in 12C, by using the Bayman-Bohr theorem. In §3 we discuss the effect of the ground state correlation on the monopole transitions from the 12C+α cluster states in 16O, and those from the 3α cluster state in 12C. In §4 we give discussions and summary.

§2. Monopole transition and Bayman-Bohr theorem

2.1. Monopole transition from two-cluster states in 16O

We discuss the following two observed values of the monopole transition matrix element in 16O: One is \( M(E0) = 3.55 \pm 0.21 \) fm\(^2\) between the ground state (0\(^+_1\)) and the first excited 0\(^+\) state (0\(^+_2\)) at \( E_x = 6.05 \) MeV, and the other is \( M(E0) = 4.03 \pm 0.09 \) fm\(^2\) between the ground state and the excited 0\(^+\) state (0\(^+_3\)) at \( E_x = 12.05 \) MeV. These excited 0\(^+\) states are known to have 12C+α structures. In this section, we explain that the order of magnitude of these \( M(E0) \) values comparable with the single nucleon strength is explained to come from the fact that the doubly closed shell wave function already contains in it the 12C+α clustering degree of freedom. For this purpose we derive analytical expressions of these monopole matrix elements by the use of the Bayman-Bohr theorem.

The nuclear \( SU(3) \) model or Elliott model\(^{11} \) is known to describe well ground states of light nuclei. The ground state of 16O has a doubly closed shell structure of 0s and 0p orbits which belongs to the \( SU(3) \) irreducible representation \( (\lambda, \mu) = (0, 0) \). This doubly closed shell model wave function with the nucleon size parameter \( \nu_N = M \omega / 2 \hbar \) (\( M \): nucleon mass) is equivalent to a cluster wave function of 12C +
configuration, according to the Bayman-Bohr theorem,

$$\frac{1}{\sqrt{16!}} \det |(0s)^4(0p)^{12}| = N_g \frac{1}{\sqrt{16 C_4}} A \left( \mathcal{R}_4(r, 3\nu_N)\phi(12C) \right) \phi_G(r_G), \quad (2.1)$$

$$[\mathcal{R}_4(r, 3\nu_N)\phi(12C)](\lambda, \mu) = \sum_{L=0,2,4} C_L \left[ \mathcal{R}_{4L}(r, 3\nu_N)\phi_L(12C) \right]_{J=0}, \quad (2.2)$$

$$\phi_G(r_G) = \left( \frac{32\nu_N}{\pi} \right)^{3/4} \exp \left( -16\nu_N r_G^2 \right), \quad r_G = \frac{1}{16} \sum_{i=1}^{16} r_i, \quad (2.3)$$

$$C_L = \langle (4,0)L, (0,4)L | (0,0)0 \rangle, \quad 16 C_4 = \frac{16!}{12!4!}. \quad (2.4)$$

Here $\phi(\alpha)$ and $\phi_L(12C)$ stand for the internal wave function of $\alpha$ cluster with the $(0s)^4$ configuration and internal wave function of $^{12}\text{C}$ with angular momentum $L$, respectively. $\phi_G$ denotes the center-of-mass wave function of $^{16}\text{O}$, which can be separated from the internal wave function as is written in Eq. (2.1). The relative wave function between the $\alpha$ and $^{12}\text{C}$ clusters is presented by the harmonic oscillator wave function $\mathcal{R}_{NLM}(r, \beta) = \mathcal{R}_{NL}(r)Y_{LM}(\hat{\boldsymbol{r}})$ with the oscillator quanta $N = 4$ [nodal number $n = (N - L)/2$] and size parameter $\beta = 3\nu_N$, where $\hat{\boldsymbol{r}}$ is the relative coordinate between the center-of-masses of $\alpha$ and $^{12}\text{C}$ clusters. It is noted that $\mathcal{R}_{4L}(r, 3\nu_N)$ and $\phi_L(12C)$ belong to the $SU(3)$ irreducible representations $(\lambda, \mu) = (4,0)$ and $(0,4)$, respectively. Equation (2.2) means that these representations are coupled to the $SU(3)$ scalar representation $(\lambda, \mu) = (0,0)$. $A$ is the nucleon antisymmetrizer between $^{12}\text{C}$ and $\alpha$ cluster, $N_g$ is the normalization constant, and $C_L$ is the reduced Clebsch-Gordan coefficient of $SU(3)$ group for the $SU(3)$ vector coupling $(4,0) \times (0,4) \rightarrow (0,0)$.

The doubly closed shell model wave function of $^{16}\text{O}$ has the total number of the oscillator quanta $N_{TOT} = 12$ and is only one possible wave function allowed for $N_{TOT} = 12$. Since all three wave functions of $^{16}\text{O}$, $A \left( \mathcal{R}_{4L}(r, 3\nu_N)\phi_L(12C) \right)_{J=0}\phi(\alpha)$ for $L=0$, 2, and 4, have the total quanta $N_{TOT} = 12$, they are necessarily equivalent to the doubly closed shell model wave function $\Phi_{CS}$ and represent the internal wave function of the $^{16}\text{O}$ ground state (see also Appendix B),

$$|0^+\rangle = \Phi_{CS} \equiv \left( \frac{1}{\sqrt{16!}} \det |(0s)^4(0p)^{12}| \times [\phi_G(r_G)]^{-1} \right) \quad (2.5)$$

$$= N_g^0 \frac{1}{\sqrt{16 C_4}} A \left( \mathcal{R}_{40}(r, 3\nu_N)\phi_L(12C) \right)_{J=0}\phi(\alpha) \quad (2.6)$$

$$= N_g^2 \frac{1}{\sqrt{16 C_4}} A \left( \mathcal{R}_{42}(r, 3\nu_N)\phi_L(12C) \right)_{J=0}\phi(\alpha) \quad (2.7)$$

$$= N_g^4 \frac{1}{\sqrt{16 C_4}} A \left( \mathcal{R}_{44}(r, 3\nu_N)\phi_L(12C) \right)_{J=0}\phi(\alpha) \quad (2.8)$$

where $N_g^0$, $N_g^2$, and $N_g^4$ denote the normalization constants. It is important to recognize the implication of these relations of Eqs. (2.1), and (2.6)–(2.8). They imply that the ground state of $^{16}\text{O}$ can be excited not only through single particle degrees of freedom by promoting nucleons from $0s$ and $0p$ orbits to higher orbits, but
also through cluster degrees of freedom by exciting the $^{12}\text{C} - \alpha$ relative motion from $R_{4L}(r, 3\nu_N)$ state to higher nodal states. The latter characteristic is an essential point to understand why the monopole transition matrix elements to cluster states are in general large.

2.1.1. Monopole transition between $0^+_1$ and $0^+_2$ states

The $0^+_2$ state of $^{16}\text{O}$ is known to have a loosely bound $^{12}\text{C} + \alpha$ structure, in which the dominant component of $^{12}\text{C}$ is the ground state.$^{1-3}$ Thus we express the $0^+_2$ wave function as

$$|0^+_2\rangle = N_I \frac{1}{\sqrt{16C_4}} \mathcal{A}\{\chi_0(r)\phi_{L=0}(^{12}\text{C})\phi(\alpha)\}, \quad (2.9)$$

where $N_I$ represents the normalization constant. By expanding $\chi_0(r)$ in terms of harmonic oscillator functions, we have

$$|0^+_2\rangle = \sum_{N=6}^{\infty} \eta_N \Phi_N, \quad (2.10)$$

$$\Phi_N = \frac{1}{\sqrt{\tau_{0,N}}} \frac{1}{\sqrt{16C_4}} \mathcal{A}\{R_{N0}(r, 3\nu_N)Y_{00}(\hat{r})\phi_{L=0}(^{12}\text{C})\phi(\alpha)\}, \quad (2.11)$$

$$\tau_{0,N} \equiv \langle R_{N0}(r, 3\nu_N)\phi_{L=0}(^{12}\text{C})\phi(\alpha)\mathcal{A}\{R_{N0}(r, 3\nu_N)\phi_{L=0}(^{12}\text{C})\phi(\alpha)\}\rangle. \quad (2.12)$$

It should be noted that $\Phi_N$ are normalized. Also it should be noted that $\Phi_{N=4}$ is just the doubly closed shell wave function as is seen in Eq. (2.6), $|0^+_1\rangle = \Phi_{N=4}$.

Since both $0^+_1$ and $0^+_2$ states have the total isospin $T = 0$, the monopole transition matrix element $M(E0)$ is

$$M(E0, 0^+_2 - 0^+_1) = \langle 0^+_1 | \sum_{i=1}^{16} \frac{1}{2} (1 + \tau_{3i})(r_i - r_G)^2 | 0^+_2\rangle$$

$$= \langle 0^+_1 | \frac{1}{2} \sum_{i=1}^{16} (r_i - r_G)^2 | 0^+_2\rangle$$

$$= \eta_6 \langle \Phi_{N=4} | \frac{1}{2} \sum_{i=1}^{16} (r_i - r_G)^2 | \Phi_{N=6}\rangle. \quad (2.13)$$

Here $r_G$ stands for the total center-of-mass coordinate, $r_G = (1/16) \sum_{i=1}^{16} r_i$. The last equality in Eq. (2.13) is because $\sum_{i=1}^{16} (r_i - r_G)^2 \Phi_{N=4}$ cannot have more than $2\hbar \omega$ excitation than $\Phi_{N=4}$. Then we have

$$M(E0, 0^+_2 - 0^+_1) = \frac{\eta_6}{2} \frac{1}{\sqrt{\tau_{0,4}^{20}}} \sqrt{\tau_{0,4}^{20}}$$

$$\times \langle \mathcal{R}_{40}(r, 3\nu_N)\phi_{L=0}(^{12}\text{C})\phi(\alpha)\mathcal{A}\{\left(\sum_{i=1}^{16} (r_i - r_G)^2\right) \mathcal{R}_{60}(r, 3\nu_N)\phi_{L=0}(^{12}\text{C})\phi(\alpha)\}\rangle$$

$$= \frac{\eta_6}{2} \frac{1}{\sqrt{\tau_{0,4}^{20}}}$$
\[ \times \left\langle \mathcal{R}_{40}(r, 3\nu_N)\phi_{L=0}(^{12}C)\phi(\alpha) \left| A \left\{ \frac{12 \times 4}{16} r^2 \mathcal{R}_{60}(r, 3\nu_N)\phi_{L=0}(^{12}C)\phi(\alpha) \right\} \right. \right\rangle. \] (2.14)

In obtaining Eq. (2.14), we first used the identity
\[
\sum_{i=1}^{16} (r_i - r_G)^2 = \sum_{i \in ^{12}C} (r_i - r_C)^2 + \sum_{i \in \alpha} (r_i - r_\alpha)^2 + \frac{12 \times 4}{16} r^2, \tag{2.15}
\]
where \( r_C \) and \( r_\alpha \) express the center-of-mass coordinate of \(^{12}C\) and \( \alpha \), respectively. We then used the following relations,
\[
\left\langle \mathcal{R}_{40}(r, 3\nu_N)\phi_{L=0}(^{12}C)\phi(\alpha) \left| A \left\{ \mathcal{R}_{60}(r, 3\nu_N) \right. \right. \right. \nonumber
\times \left( \sum_{i \in ^{12}C} (r_i - r_C)^2 \right) \phi_{L=0}(^{12}C)\phi(\alpha) \left. \right\} \right. \right\} = 0, \tag{2.16}
\]
\[
\left\langle \mathcal{R}_{40}(r, 3\nu)\phi_{L=0}(^{12}C)\phi(\alpha) \left| A \left\{ \mathcal{R}_{60}(r, 3\nu_N)\phi_{L=0}(^{12}C) \right. \right. \right. \nonumber
\times \left( \sum_{i \in \alpha} (r_i - r_\alpha)^2 \right) \phi(\alpha) \left. \right\} \right. \right\} = 0. \tag{2.17}
\]

These relations can be easily proved by counting the total numbers of oscillator quanta of the bra and ket functions. First, the number of the oscillator quanta of \( \mathcal{R}_{60}(r, 3\nu_N) \) is larger than that of \( \mathcal{R}_{40}(r, 3\nu_N) \) by 2. Second, the number of the oscillator quanta of \( \left( \sum_i (r_i - r_C)^2 \right)\phi_{L=0}(^{12}C) \) cannot be smaller than that of \( \phi_{L=0}(^{12}C) \) because \( \phi_{L=0}(^{12}C) \) has the smallest number of the oscillator quanta in the \(^{12}C\) \((N = Z = 6)\) system. Similarly, the number of the oscillator quanta of \( \left( \sum_i (r_i - r_\alpha)^2 \right)\phi(\alpha) \) cannot be smaller than that of \( \phi(\alpha) \). Therefore in each of Eqs. (2.16) and (2.17), the ket function has larger total number of the oscillator quanta than that of the bra function at least by 2, which leads to the orthogonality of the bra and ket functions.

Now we expand \((12 \times 4/16) r^2 \mathcal{R}_{60}(r, 3\nu_N)\) in Eq. (2.14) in terms of the harmonic oscillator function
\[
\frac{12 \times 4}{16} r^2 \mathcal{R}_{60}(r, 3\nu_N) = \sum_N \left\langle \mathcal{R}_{N0}(r, 3\nu_N) \left| \frac{12 \times 4}{16} r^2 \mathcal{R}_{60}(r, 3\nu_N) \right. \right. \right. \nonumber
\times \left. \left. \left. \mathcal{R}_{N0}(r, 3\nu_N) \right. \right. \right. \tag{2.18}
\]
Here we note the following relation:

$$\langle R_{40}(r, 3\nu_N) \vert \frac{12 \times 4}{16} r^2 \vert R_{60}(r, 3\nu_N) \rangle = \langle R_{40}(r, \nu_N) \vert r^2 \vert R_{60}(r, \nu_N) \rangle,$$  \hfill (2.20)

where \( R_{NO}(r, \nu_N) \) is the harmonic oscillator radial function of single nucleon with the nucleon size parameter \( \nu_N \). It is noted here that the matrix elements for calculating the single particle E0 matrix element in \( ^{16}\text{O} \) are \( \langle R_{00}(r, \nu_N) \vert r^2 \vert R_{20}(r, \nu_N) \rangle \) and \( \langle R_{11}(r, \nu_N) \vert r^2 \vert R_{31}(r, \nu_N) \rangle \) which are a few times smaller than the present \( \langle R_{40}(r, \nu_N) \vert r^2 \vert R_{60}(r, \nu_N) \rangle \) as shown below,

\[
\begin{align*}
\langle R_{00}(r, \nu_N) \vert r^2 \vert R_{20}(r, \nu_N) \rangle &= \sqrt{\frac{3}{8} \frac{1}{\nu_N}}, \\
\langle R_{11}(r, \nu_N) \vert r^2 \vert R_{31}(r, \nu_N) \rangle &= \sqrt{\frac{5}{8} \frac{1}{\nu_N}}, \\
\langle R_{40}(r, \nu_N) \vert r^2 \vert R_{60}(r, \nu_N) \rangle &= \sqrt{\frac{21}{8} \frac{1}{\nu_N}}.
\end{align*}
\]

(2.21)

(2.22)

The reason why the number of oscillator quanta of the relative wave function is higher than those of the single particle wave functions is due to the Fermi statistics of nucleons.

The final analytical formula of \( M(E0, 0^+_2 - 0^+_1) \) is expressed as follows:

$$M(E0, 0^+_2 - 0^+_1) = \frac{1}{2} \sqrt{\frac{\tau_{04}}{\tau_{06}}} \eta_6 \langle R_{40}(r, \nu_N) \vert r^2 \vert R_{60}(r, \nu_N) \rangle.$$  \hfill (2.23)

This analytical expression of \( M(E0, 0^+_2 - 0^+_1) \) is our desired result. It explains clearly why \( M(E0, 0^+_2 - 0^+_1) \) has a comparable magnitude as the single nucleon E0 matrix element. The factor \( \langle R_{40}(r, \nu_N) \vert r^2 \vert R_{60}(r, \nu_N) \rangle \) is a few times larger than the single nucleon E0 matrix elements \( \langle R_{00}(r, \nu_N) \vert r^2 \vert R_{20}(r, \nu_N) \rangle \) and \( \langle R_{11}(r, \nu_N) \vert r^2 \vert R_{31}(r, \nu_N) \rangle \), while the factor \( \eta_6 \) works to make the E0 value smaller.

2.1.2. Monopole transition between \( 0^+_1 \) and \( 0^+_3 \) states

The \( 0^+_2 \) state of \( ^{16}\text{O} \) at \( E_x = 12.05 \text{ MeV} \) is known to have also a \( ^{12}\text{C} + \alpha \) structure like the \( 0^+_2 \) state.\(^{1,3}\) The \( ^{12}\text{C} \) cluster in the \( 0^+_3 \) state, however, is not mainly in its ground state like in \( 0^+_2 \) state but dominantly in its excited \( 2^+ \) state at \( E_x = 4.44 \text{ MeV} \). Thus we can express the \( 0^+_3 \) wave function in a good approximation as

$$|0^+_3\rangle = N_{II} \frac{1}{\sqrt{16C_4}} A[\chi_2(r)\phi_{L=2}^{(12\text{C})}\vert J=0\phi(\alpha)].$$  \hfill (2.24)

Like in the case of \( 0^+_2 \) state, we expand \( \chi_2(r) \) in terms of harmonic oscillator wave functions and we obtain

$$|0^+_3\rangle = \sum_{N=6}^{\infty} \zeta_N \Psi_N,$$  \hfill (2.25)

$$\Psi_N = \frac{1}{\sqrt{\tau_{2N}}} \frac{1}{\sqrt{16C_4}} A[\{R_{N2}(r, 3\nu_N)\phi_{L=2}^{(12\text{C})}\vert J=0\phi(\alpha)],$$  \hfill (2.26)

$$\tau_{2N} = \langle [R_{N2}(r, 3\nu_N)\phi_{L=2}^{(12\text{C})}\vert J=0\phi(\alpha)] A[\{R_{N2}(r, 3\nu_N)\phi_{L=2}^{(12\text{C})}\vert J=0\phi(\alpha)] \rangle.$$  \hfill (2.27)
It should be noted that $\Psi_N$ are normalized. Also it should be noted that $\Psi_{N=4}$ is just the doubly closed shell wave function as is seen in Eq. (2.7), $|0^+_1\rangle = \Psi_{N=4}$.

The calculation of the monopole transition matrix element $M(E0, 0^+_3 - 0^+_1)$ can be made in the same manner as that of $M(E0, 0^+_2 - 0^+_1)$ in the previous section, although we use Eq. (2.7) for the $0^+_1$ state of $^{16}$O,

$$M(E0, 0^+_3 - 0^+_1) = \langle 0^+_1| \frac{1}{2} \sum_{i=1}^{16} (r_i - r_G)^2 |0^+_3\rangle$$

$$= \frac{1}{2} \sqrt{\frac{72.4}{72.6}} \zeta_6 \langle R_{42}(r, \nu_N)|r^2|R_{62}(r, \nu_N)\rangle,$$  \hspace{1cm} (2.28)

where

$$\langle R_{42}(r, \nu_N)|r^2|R_{62}(r, \nu_N)\rangle = \frac{3}{2} \frac{1}{\nu_N}. \hspace{1cm} (2.29)$$

The analytical expression of $M(E0, 0^+_3 - 0^+_1)$ in Eq. (2.28) is our another desired result. Like in the case of $M(E0, 0^+_2 - 0^+_1)$, it explains clearly why $M(E0, 0^+_3 - 0^+_1)$ has also a comparable magnitude as the single nucleon E0 matrix element.

2.1.3. Wave function which absorbs total monopole strength from the doubly closed shell

The wave function $\Phi_{(2,0)}$ which absorbs total monopole strength from the doubly closed shell wave function $\Phi_{CS}$ in Eq. (2.5) is given by

$$\Phi_{(2,0)} = N_{(2,0)}(1 - |\Phi_{CS}\rangle \langle 0^+_1|O_M \Phi_{CS} = 2.30$$

$$O_M = \frac{1}{2} \sum_{i=1}^{16} (r_i - r_G)^2, \hspace{1cm} (2.31)$$

where $N_{(2,0)}$ is the normalization constant and is presented as

$$\frac{1}{N_{(2,0)}} = \sqrt{\langle \Phi_{CS}|O_M^2|\Phi_{CS}\rangle - \langle \Phi_{CS}|O_M|\Phi_{CS}\rangle^2} = \sqrt{\frac{69}{32} \frac{1}{\nu_N}}. \hspace{1cm} (2.32)$$

Any wave function $\Phi$ which is orthogonal to both $\Phi_{CS}$ and $\Phi_{(2,0)}$ has zero monopole strength from $\Phi_{CS}$, namely $\langle \Phi|O_M|\Phi_{CS}\rangle = 0$. This fact is easily derived from the orthogonality of $\Phi$ to $\Phi_{CS}$ and $\Phi_{(2,0)}$. Then, the monopole strength of the wave function $\Phi_{(2,0)}$ from $\Phi_{CS}$ is given by

$$\langle \Phi_{(2,0)}|O_M|\Phi_{CS}\rangle = \frac{1}{N_{(2,0)}} = \sqrt{\frac{69}{32} \frac{1}{\nu_N}} = \frac{1.47}{\nu_N}. \hspace{1cm} (2.33)$$

Reminding of the relation of $\langle \Phi_{CS}|O_M^2|\Phi_{CS}\rangle = \sum_k |\langle \Phi_k|O_M|\Phi_{CS}\rangle|^2$ ({$\Phi_k$} denoting a complete set of wave functions) in Eq. (2.32), one finds that the monopole strength in Eq. (2.33) corresponds to the squared root of the non-energy-weighted sum rule ($\sum_{k\neq CS} |\langle \Phi_k|O_M|\Phi_{CS}\rangle|^2$) of the monopole operator $O_M$ with respect to $\Phi_{CS}$, i.e. exhausting the total monopole strength from $\Phi_{CS}$. 

Let us denote by $\Phi_{(2,0)}^{cl}$ the $^{12}\text{C} + \alpha$ cluster wave function which absorbs the total monopole strength from $\Phi_{CS}$ within the $^{12}\text{C} + \alpha$ cluster model space. $\Phi_{(2,0)}^{cl}$ is not equal to $\Phi_{(2,0)}$ in Eq. (2.30). It is because the monopole operator of $^{12}\text{C}$ cluster, $(1/2) \sum_{i \in ^{12}\text{C}} (r_i - r_C)^2$, and that of the $\alpha$ cluster, $(1/2) \sum_{i \in \alpha} (r_i - r_\alpha)^2$, which are contained in the total monopole operator $O_M$ as seen in Eq. (2.15) do excite the $^{12}\text{C}$ and $\alpha$ clusters when $O_M$ operates on $\Phi_{CS}$. These excitations of clusters imply that the wave function $\Phi_{(2,0)}$ contains components out of the $^{12}\text{C} + \alpha$ cluster model space. The explicit form of $\Phi_{(2,0)}^{cl}$ is given as

$$
\Phi_{(2,0)}^{cl} = N_{(2,0)}^{cl} \frac{1}{\sqrt{16 C_4}} A[\{\mathcal{R}_6(r, 3\nu_N)\phi(12\text{C})\}_{(2,0)}\phi(\alpha)],
$$

$$
[\mathcal{R}_6(r, 3\nu_N)\phi(12\text{C})]_{(2,0)} = 
\sum_{L=0,2,4} \langle (6,0)L, (0,4)L | (2,0)0 \rangle [\mathcal{R}_6L(r, 3\nu_N)\phi_L(12\text{C})]_{J=0}.
$$

Here $\langle (6,0)L, (0,4)L | (2,0)0 \rangle$ is the reduced Clebsch-Gordan coefficient of the $SU(3)$ group for the $SU(3)$ vector coupling $(6,0) \times (0,4) \rightarrow (2,0)$. This relation is proved as follows. Since the nucleon coordinate $r_i$ is the sum of the creation ($a_i^\dagger$) and annihilation ($a_i$) operators of oscillator quanta, $r_i \propto a_i^\dagger + a_i$, the monopole operator $O_M$ consists of three parts, $O_M = O_M^{(2,0)} + O_M^{(0,2)} + O_M^{(0,0)}$. The number of the oscillator quanta is raised by 2 by $O_M^{(2,0)}$, lowered by 2 by $O_M^{(0,2)}$, and kept unchanged by $O_M^{(0,0)}$. The superfix $(\lambda, \mu)$ of the operator $O_M^{(\lambda,\mu)}$ expresses its $SU(3)$ tensor character. Thus the $2\hbar \omega$-excited wave function created by operating $O_M$ on $\Phi_{CS}$ necessarily has the $SU(3)$ symmetry $(2,0)$. Within the $^{12}\text{C} + \alpha$ cluster model space, $\Phi_{(2,0)}^{cl}$ is the only one wave function which is $2\hbar \omega$-excited and has $(2,0)$ symmetry. Thus $\Phi_{(2,0)}^{cl}$ absorbs all the monopole strength from $\Phi_{CS}$ and other excited wave functions orthogonal to $\Phi_{(2,0)}^{cl}$ all have zero monopole strength. The monopole strength of $\Phi_{(2,0)}^{cl}$ is given by (see Ref. 13)

$$
\langle \Phi_{(2,0)}^{cl} | O_M | \Phi_{CS} \rangle = \sqrt{\frac{45}{32 \nu_N}} = \frac{1.19}{\nu_N}.
$$

This magnitude of $\langle \Phi_{(2,0)}^{cl} | O_M | \Phi_{CS} \rangle$ is about 80% of the total monopole strength $\langle \Phi_{(2,0)} | O_M | \Phi_{CS} \rangle$. We now know, from the studies in previous subsections, that the reason of this large value is just because of the $^{12}\text{C} + \alpha$ clustering character embedded in the doubly closed shell wave function which is described by the Bayman-Bohr theorem. Namely, $\langle \Phi_{(2,0)}^{cl} | O_M | \Phi_{CS} \rangle$ can be expressed as

$$
\langle \Phi_{(2,0)}^{cl} | O_M | \Phi_{CS} \rangle = \sum_{L=0,2,4} E_L \frac{1}{2} \sqrt{\frac{\omega L}{\tau_{L,4}}} \langle R_{4L}(r, \nu_N) | r^2 | R_{6L}(r, \nu_N) \rangle,
$$

$$
\frac{1}{2} \sqrt{\frac{\omega_{0,4}}{\tau_{0,6}}} \langle R_{40}(r, \nu_N) | r^2 | R_{60}(r, \nu_N) \rangle = \frac{0.784}{\nu_N},
$$

(2.38)
The values of $\langle \Phi \rangle$ can be calculated by using their analytical expressions presented in Refs. 3) and 16). The definition of $\tau$ is as follows:

$$\tau = 0, 2, 4, \ldots$$

The coefficient $E_L$ is expressed as follows:

$$E_L = N_{cl}^2 \sqrt{\frac{\tau_L}{\tau_6}} \langle (6, 0) L, (0, 4) L | (2, 0) 0 \rangle,$$

$$\left( \frac{1}{N_{cl}^2 (2, 0)} \right)^2 \langle [R_6(r, 3\nu_N)\phi_L^{(12)C}](2, 0) \phi(\alpha) | A\{ [R_6(r, 3\nu_N)\phi_L^{(12)C}](2, 0) \phi(\alpha) \} \rangle = 112/81 = 1.38.$$  

The values of $\langle (6, 0) L, (0, 4) L | (2, 0) 0 \rangle$ are $\sqrt{1/10}$, $\sqrt{1/7}$, and $\sqrt{33/70}$, for $L = 0, 2,$ and 4, respectively. The value of $(1/N_{cl}^2 (2, 0))^2$ is given in Ref. 16) with the notation $\mu^6_{(2, 0)}$. In Eqs. (2-38) - (2-40) we used the values of $\tau_{L,N}$ calculated by the use of their analytical expressions presented in Refs. 3) and 16). The values of $\tau_{0,N}$ and $\tau_{2,N}$ are given in Table I. As we already emphasized, each term $(1/2)\sqrt{\tau_{L,A}/\tau_{L,6}} (R_{4L}(r, \nu_N)|r^2| R_{6L}(r, \nu_N))$ is all large comparable with single nucleon strength.

The monopole matrix elements between the $0^+_2$ and $0^+_3$ states and the $0^+_1$ state can be calculated by using $\Phi_{cl}^{(2,0)}$ as follows:

$$M(E_0, 0^+_2 - 0^+_1) = \langle 0^+_2 | \Phi_{cl}^{(2,0)} | O_M | \Phi_{CS} \rangle = \langle 0^+_2 | \Phi_{cl}^{(2,0)} \rangle \sqrt{45/32} \nu_N,$$

$$M(E_0, 0^+_3 - 0^+_1) = \langle 0^+_3 | \Phi_{cl}^{(2,0)} | O_M | \Phi_{CS} \rangle = \langle 0^+_3 | \Phi_{cl}^{(2,0)} \rangle \sqrt{45/32} \nu_N.$$
2.2. Monopole transition from three-cluster state in $^{12}\text{C}$

The calculation of the monopole transition from three-cluster state in $^{12}\text{C}$ can be made essentially in the same way as in the case of two-cluster state. We explain this point by calculating the monopole transition matrix element from the second $0^+$ state at $E_x = 7.66$ MeV to the ground state. The experimental data is $M(E0, 0^+_2 - 0^+_1) = 5.4 \pm 0.2$ fm$^2$. In the previous section we described the ground state $(0^+_1)$ of $^{12}\text{C}$ by the $SU(3)$ shell model wave function $\phi_{L=0}(^{12}\text{C})$ which belongs to the $SU(3)$ irreducible representation $(\lambda, \mu) = (0, 4)$. This wave function is known of course to be a rather good approximation. According to the Bayman-Bohr theorem the internal wave function of the $^{12}\text{C}$ ground state can be expressed in terms of the $3\alpha$ cluster wave function,

$$ |0^+_1\rangle = |(0s)^4(0p)^8(0, 4)J = 0\rangle_{\text{internal}} = \sqrt{\frac{4!4!4!}{12!}} A\{g_{(04)0}(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)\}, \quad (2.47) $$

where $s$ and $t$ are the Jacobi coordinates defined by

$$ s = X_2 - X_1, \quad t = X_3 - \frac{X_2 + X_1}{2}, \quad X_k = \frac{1}{4} \sum_{i \in \alpha_k} r_i, \quad (2.48) $$

and $A$ is antisymmetrizer among nucleons belonging to different $\alpha$ clusters. The relative wave function $g_{(04)0}(s, t)$ is expressed as follows:

$$ g_{(04)0}(s, t) = \sum_{L=0,2,4} \langle (4, 0)L, (4, 0)L || (0, 4)J = 0 \rangle \mathcal{R}^{8,J=0}_{4,4,L}(s, t), \quad (2.49) $$

$$ \mathcal{R}_{N_1,N_2,L}(s, t) = \left[ \mathcal{R}_{N_1,L}(s, 2\nu_N) \mathcal{R}_{N_2,L}(t, \frac{8\nu_N}{3}) \right]_{J=0}, \quad (N_1 + N_2 = N) \quad (2.50) $$

where $\mathcal{R}_{NL}(u, \beta)$ stands for the harmonic oscillator function of the size parameter $\beta$ of the coordinate $u$ with the oscillator quantum number $N$ and angular momentum $L$.

The $SU(3)$ symmetry $(0, 4)$ for $(0s)^4(0p)^8$ configuration is equivalent to the spatial symmetry [44] for $(0s)^4(0p)^8$ configuration. Since there is only one state with $J = 0$ for the $(0s)^4(0p)^8[44]$ configuration, the following identities hold (see also Appendix B),

$$ |0^+_1\rangle = \begin{align*}
&\sqrt{\frac{4!4!4!}{12!}} A\{\mathcal{R}^{8,J=0}_{4,4,L=0}(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)\} \quad (2.51) \\
&= \sqrt{\frac{4!4!4!}{12!}} A\{\mathcal{R}^{8,J=0}_{4,4,L=2}(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)\} \quad (2.52) \\
&= \sqrt{\frac{4!4!4!}{12!}} A\{\mathcal{R}^{8,J=0}_{4,4,L=4}(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)\}. \quad (2.53)
\end{align*} $$


2.2.1. Monopole transition between the ground and Hoyle states

The second $0^+ \ (0^+_2)$ state is known to have $3\alpha$ structure,\(^1\) and so we express its wave function as

$$|0^+_2\rangle = \hat{N}_H \sqrt{\frac{4!4!4!}{4!4!4!}} A\{\tilde{\chi}_H(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)\}. \quad (2.54)$$

In the expansion of the relative wave function $\tilde{\chi}_H(s, t)$ in terms of the harmonic oscillator wave functions, the number of the total oscillator quanta of these oscillator wave functions is larger than 8 which is the number of total oscillator quanta of relative wave function of the ground state. Just in the same manner as in the previous section, we can express the monopole transition matrix element $M(E_0, 0^+_2 - 0^+_1)$ as follows:

$$M(E_0, 0^+_2 - 0^+_1) = \langle 0^+_1 | \frac{1}{2} \sum_{i=1}^{12} (r_i - r_G)^2 | 0^+_2 \rangle$$

$$= \frac{1}{2} \sum_L \frac{\hat{N}_H}{\hat{N}_{g_L}} (R_{L, 4, 4}^+_1(s, t)|\left(2s^2 + \frac{8}{3}t^2\right)|\tilde{\chi}_H(s, t)\rangle. \quad (2.55)$$

Here we used the following relation:

$$\sum_{i=1}^{12} (r_i - r_G)^2 = \sum_{k=1}^{3} \sum_{i \in \alpha_k} (r_i - X_k)^2 + 2s^2 + \frac{8}{3}t^2. \quad (2.56)$$

It is noted that the first term in Eq. (2.56) does not contribute to the monopole transition matrix element like in the previous case of $^{16}\text{O}$.

The second $0^+$ state in $^{12}\text{C}$ which is known as the Hoyle state has been studied by many authors with $3\alpha$ cluster model and its structure is now regarded as being mainly composed of weakly interacting $3\alpha$ clusters mutually in $S$-wave.\(^1,8,9,17\) Therefore we write $\tilde{\chi}_H(s, t)$ as follows:

$$\tilde{\chi}_H(s, t) = \tilde{\chi}(s, t)Y_{00}(\tilde{s})Y_{00}(\tilde{t}). \quad (2.57)$$

As we already mentioned, the expansion $\tilde{\chi}_H(s, t)$ in terms of the harmonic oscillator function does not contain the components whose numbers of oscillator quanta are less than or equal to 8,

$$\tilde{\chi}_H(s, t) = \sum_{N_1, N_2} D_{H, N_1, N_2} R_{N_1, 0}(s, 2\nu_N) R_{N_2, 0}(t, \frac{8}{3}\nu_N), \quad (2.58)$$

$$D_{H, N_1, N_2} = 0 \quad \text{for} \quad N_1 + N_2 \leq 8. \quad (2.59)$$

Substituting Eq. (2.58) into Eq. (2.55), we have the following simple result:

$$M(E_0, 0^+_2 - 0^+_1) = \frac{1}{2} \frac{\hat{N}_H}{N_{g_0}} (D_{H, 6, 4} + D_{H, 4, 6}) \langle R_{40}(r, \nu_N)|r^2|R_{60}(r, \nu_N)\rangle. \quad (2.60)$$
This analytical expression explains why $M(E0, 0^+_2 - 0^+_1)$ has a comparable magnitude as the single nucleon E0 matrix element. The factor $\langle R_{40}(r, \nu_N)|r^2|R_{60}(r, \nu_N)\rangle$, which appears also in the case of $^{16}$O, is a few times larger than the single nucleon E0 matrix elements $\langle R_{00}(r, \nu_N)|r^2|R_{20}(r, \nu_N)\rangle$ and $\langle R_{11}(r, \nu_N)|r^2|R_{31}(r, \nu_N)\rangle$, while the other factors work to make the E0 value smaller. The reason why we have this formula of Eq. (2.60) is just the $3\alpha$ clustering character of the shell model wave function of the $^{12}$C ground state which is described by the Bayman-Bohr theorem.

2.2.2. Description of the Hoyle state as a $3\alpha$ condensate

Recently the structure of the Hoyle state has been studied from a new point of view that this state is the Bose-condensed state of $3\alpha$ particles.\textsuperscript{14,15,18} It has been demonstrated that both of the $3\alpha$ wave functions of Refs. 8 and 9 which are the full solutions of $3\alpha$ Resonating Group Method (RGM) equation of motion have large overlaps close to 100 % with the $3\alpha$ Bose-condensed wave functions.\textsuperscript{15} Therefore we here adopt as $\hat{\chi}_H(s, t)$ the following form:

$$\hat{\chi}_H(s, t) = (1 - P) \hat{\chi}_{HG}(s, t),$$

$$\hat{\chi}_{HG}(s, t) \equiv \left( \frac{8\gamma}{\sqrt{3}\pi} \right)^{3/4} \exp \left\{ -4\gamma \sum_{k=1}^3 (X_k - r_G)^2 \right\}$$

$$= \left( \frac{4\gamma}{\pi} \right)^{3/4} \left( \frac{16\gamma}{3\pi} \right)^{3/4} \exp \left\{ -\gamma \left( 2s^2 + \frac{8}{3}t^2 \right) \right\},$$

$$P = \sum_{N \leq 8} \sum_{N_1+N_2=N} \sum_{L} |\mathcal{R}_{N_1,N_2,L}(s, t)||\mathcal{R}_{N_1,N_2,L}(s, t)|,$$  

where $\gamma$ denotes the width parameter which characterizes the $3\alpha$ condensate wave function. $P$ is the projection operator onto the state of SU(3) relative motion of the ground state and the states forbidden by the antisymmetrization. Then, the analytical expression of the monopole transition matrix element in Eq. (2.60) is given as follows:

$$M(E0, 0^+_2 - 0^+_1) = \sqrt{\frac{7}{6}} \sqrt{\frac{\langle F_4 \rangle}{\langle F_5 \rangle}} \xi_5 \langle R_{40}(r, \nu_N)|r^2|R_{60}(r, \nu_N)\rangle,$$

$$\xi_5 \equiv \sqrt{\frac{\langle F_5 \rangle}{\langle F_5 \rangle + \sum_{n=6}^{\infty} \left( \frac{\nu_N - \gamma}{\nu_N + \gamma} \right)^{2(n-5)} \langle F_n \rangle}},$$

where the definitions of $F_n$ and $\langle F_n \rangle$ are

$$\hat{\chi}_H(s, t) = \left( \frac{2\sqrt{\nu_N\gamma}}{\nu_N + \gamma} \right)^{3/4} \sum_{n=5}^{\infty} \left( \frac{\nu_N - \gamma}{\nu_N + \gamma} \right)^n F_n(s, t),$$

$$F_n(s, t) = \sum_{n_1+n_2=n} \sqrt{\frac{(2n_1+1)!!(2n_2+1)!!}{(2n_1)!!(2n_2)!!}} \mathcal{R}_{2n_1,2n_2,L=0}(s, t),$$

$$\langle F_n \rangle = \langle F_n(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)|A(F_n(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)) \rangle.$$
We see that the dependence of $M(E_0, 0^+_2 - 0^+_1)$ on the parameter $\gamma$ is contained only in the factor $\xi_5$. The derivation of the above analytical expression of $M(E_0, 0^+_2 - 0^+_1)$ is given in Appendix C.

§3. Ground state correlations

We first study in this section the numerical values of the monopole matrix elements in $^{16}$O and $^{12}$C, by using the formulae obtained in the previous section. As is expected from the analytical forms, the numerical values are shown to have the same order of magnitude as the observed values which are comparable with the single nucleon strength. However, the calculated values are found to be smaller by a few times in $^{16}$O and by several times in $^{12}$C. Therefore we next study in this section the effect of the ground state correlation on the magnitude of the monopole matrix elements. We will see that the ground state correlation largely improves the reproduction of the observed values up to the factor of magnitude. The ground state correlation we consider is due to the activation of the clustering degree of freedom which is described by the Bayman-Bohr theorem.

3.1. Monopole transition matrix elements in $^{16}$O

We calculate $M(E_0, 0^+_2 - 0^+_1)$ and $M(E_0, 0^+_3 - 0^+_1)$ in $^{16}$O by using their analytical expressions given in Eqs. (2.23) and (2.28). The nucleon size parameter $\nu_N (\nu_N = 0.151$ fm$^{-2}$) is chosen so as to reproduce the experimental rms radius of $^{16}$O whose wave function is described with the doubly closed shell configuration $\Phi_{CS}$ in Eq. (2.5). We also use the expressions given in Eqs. (2.38) and (2.39). The realistic values of $\eta_6$ and $\zeta_6$, of course, should be obtained from the structure calculation. A representative structure calculation is that of Ref. 3) in which the $^{12}$C + $\alpha$ OCM was adopted. For the sake of the study of this paper, we repeated the same calculation as Ref. 3). According to the results, the $0^+_2$ wave function of $^{16}$O has the predominant component of $^{12}$C$(0^+) + \alpha$ channel and small components of $^{12}$C$(2^+) + \alpha$ and $^{12}$C$(4^+) + \alpha$ channels, as mentioned already. Similarly, the $0^+_3$ wave function of $^{16}$O has the large component of $^{12}$C$(2^+) + \alpha$ channel and small components of $^{12}$C$(0^+) + \alpha$ and $^{12}$C$(4^+) + \alpha$ channels.

By using the results in Table I of the first paper of Ref. 3), we get approximate values for $\eta_6$ and $\zeta_6$, and they are $\sqrt{0.144} = 0.379$ and $\sqrt{0.325} = 0.570$, respectively. We explain the details in Appendix D. Then, the monopole matrix elements are estimated to be

\begin{align*}
M(E_0, 0^+_2 - 0^+_1) &= 0.379 \times \frac{0.784}{\nu_N} = 1.97 \text{ fm}^2, \\
M(E_0, 0^+_3 - 0^+_1) &= 0.570 \times \frac{1.03}{\nu_N} = 3.89 \text{ fm}^2,
\end{align*}

(3.1)

where the formulae in Eqs. (2.38) and (2.39) are used. The result for $M(E_0, 0^+_2 - 0^+_1)$ gives about 60% of the experimental value, and the result for $M(E_0, 0^+_3 - 0^+_1)$ amounts to 96% of the experimental one.

In estimating the above values for $M(E_0, 0^+_2 - 0^+_1)$ and $M(E_0, 0^+_3 - 0^+_1)$, we used Eqs. (2.23) and (2.28), respectively. These formulae are based on the assump-
tion that the wave functions of the $0^+_2$ and $0^+_3$ states are purely of the structure of $^{12}\text{C}(0^+) + \alpha$ and $^{12}\text{C}(2^+) + \alpha$, respectively. However, of course, the OCM calculation of Ref. 3) shows the small contamination of other channel components than the dominant component. So we here also give values which take into account these small contamination of other non-dominant channel components. Such values can be obtained by using the formulae given in Eqs. (2.45) and (2.46). The values $\langle 0^+_2 | \Phi_{\text{cl}}^{(2,0)} \rangle$ and $\langle 0^+_3 | \Phi_{\text{cl}}^{(2,0)} \rangle$ are given in Table I of the second paper of Ref. 3), and they are 0.171 and 0.321, respectively. Then, the monopole matrix elements are estimated to be

$$M(E_0, 0^+_2 - 0^+_1) = 0.171 \times \frac{1.19}{\nu_N} = 1.35 \text{ fm}^2,$$

(3.3)

$$M(E_0, 0^+_3 - 0^+_1) = 0.321 \times \frac{1.19}{\nu_N} = 2.53 \text{ fm}^2.$$

(3.4)

Now this result for $M(E_0, 0^+_2 - 0^+_1)$ gives about 38 % of the experimental value, while the result for $M(E_0, 0^+_3 - 0^+_1)$ gives about 63 % of the experimental one. These values are smaller than the former values in Eqs. (3.1) and (3.2), but still they are different from the observed values only by a few factors. The reason why the former values in Eqs. (3.1) and (3.2) are larger than the corresponding present values is that the magnitudes of the $\Phi_{\text{cl}}^{(2,0)}$ component in the pure channel wave functions, $^{12}\text{C}(0^+) + \alpha$ for $0^+_2$ and $^{12}\text{C}(2^+) + \alpha$ for $0^+_3$, are larger than those of the OCM wave functions for the $0^+_2$ and $0^+_3$ states, respectively. We give detailed explanation in Appendix D.

As we see above the calculated values of the monopole matrix elements surely reproduce the order of magnitude of the observed values which are comparable with single nucleon strength. But when compared with the data in detail, they are a few times smaller than the observed values. We below show that when we take into account the ground state correlation theoretical values are improved as to attain the reproduction of the data within 10–20 % accuracy.

The ground state correlation we consider is the one caused by the activation of the clustering degree of freedom described by Bayman-Bohr theorem. In previous sections we demonstrated that the clustering degree of freedom described by Bayman-Bohr theorem is the very reason why the monopole strengths of excited cluster states are so large as to be comparable with single nucleon strength. However, we only considered the clustering degree of freedom rather in a static way. Namely we did not consider the dynamical effect of the clustering degree of freedom which excites the ground state configuration toward including higher quantum configurations. We know that the clustering degree of freedom described by Bayman-Bohr theorem has the physical reality because we observe many excited cluster states which are formed by exciting the clustering degree of freedom embedded in the ground state. Therefore taking into account the ground state correlation caused by the clustering degree of freedom described by Bayman-Bohr theorem is very natural and should be studied.

In order to study the effect of the ground state correlation we make use of the $^{12}\text{C} + \alpha$ OCM calculation. We repeat the same calculation as Ref. 3). Of course we adopt the same effective nuclear force. We express by $| 0^+_2 \rangle$ and $| 0^+_3 \rangle$ the obtained OCM wave functions of the $0^+_2$ and $0^+_3$ states, respectively. Next, by $| 0^+_1; N \rangle$ we
express the wave function of the ground state ($0^+_1$) which is calculated within the limited cluster model space where the highest number of the total oscillator quanta of the basis wave function is $N$ ($N = 4, 6, \cdots, 30$). The wave function $|0^+_1; N = 4\rangle$ is just the closed shell wave function without any ground state correlation. As $N$ becomes larger the wave function $|0^+_1; N\rangle$ contains more amount of ground state correlation. Since $|0^+_2\rangle$ and $|0^+_3\rangle$ are not orthogonal to $|0^+_1; N\rangle$, we construct the orthogonalized wave functions as follows:

$$\tilde{|0^+_k\rangle} = N_k (1 - |0^+_1; N\rangle \langle 0^+_1; N|) |0^+_k\rangle, \quad (k = 2, 3)$$

(3.5)

where $N_k$ is the normalization constant. We calculate the monopole matrix element between $\tilde{|0^+_k\rangle}$ and $|0^+_1; N\rangle$

$$M_N(E_0, 0^+_1 - 0^+_k) = \langle 0^+_1; N| \frac{1}{2} \sum_{i=1}^{16} (r_i - r_G)^2 |0^+_k\rangle, \quad (3.6)$$

and study the dependence on $N$ of this quantity.

Figure 1 shows the calculated results of $M_N(E_0, 0^+_1 - 0^+_k)$, $(k = 2, 3)$ as a function $N$. The values of $M_N(E_0, 0^+_1 - 0^+_k)$ for $N = 4$ are the monopole matrix elements without ground state correlation which we discussed in detail in the above. They are $M_{N=4}(E_0, 0^+_1 - 0^+_2) = 1.39 \text{ fm}^2$ and $M_{N=4}(E_0, 0^+_1 - 0^+_3) = 2.36 \text{ fm}^2$. These values are very close to the values given in Eqs. (3.3) and (3.4) for which the normalization constants $N_k$ are not accounted. We clearly see that the values of $M_N(E_0, 0^+_1 - 0^+_k)$ grow almost monotonously as $N$ becomes larger. At $N = 30$, the values are already converged, and they are $M_{N=30}(E_0, 0^+_1 - 0^+_2) = 4.01 \text{ fm}^2$ and $M_{N=30}(E_0, 0^+_1 - 0^+_3) = 3.56 \text{ fm}^2$. These converged values are very close to the observed values, and the difference from the observed values is only within 13%. Thus we have shown that by taking into account the ground state correlation theoretical values are improved as to attain the reproduction of the data within a factor of 1.13.

An important reason why the ground state correlation enhances the monopole strengths is explained as follows. We study the deviation of the ground-state wave function $|0^+_1\rangle$ obtained by the $^{12}\text{C}+\alpha$ OCM from the doubly closed shell wave function. For this purpose we define a modified doubly closed shell model wave function $\Phi_0^+(\beta)$ (\beta denoting the size parameter of the $^{12}\text{C}+\alpha$ relative wave function) and calculate the squared overlap of it with the OCM ground state wave function $|0^+_1\rangle$ obtained with the full model space,

$$\Phi_0^+(\beta) = N_{g0}(\beta) \frac{1}{\sqrt{16C_4}} A\{[R_{40}(r, \beta)\phi_{L=0}(^{12}\text{C})]_{J=0}\phi(\alpha)\}, \quad (3.7)$$

$$P(\beta/(3\nu_N)) = |\langle \Phi_0^+(\beta)|0^+_1\rangle|^2, \quad (3.8)$$

where $N_{g0}(\beta)$ is the normalization constant. When $\beta$ is equal to $3\nu_N$, the wave function $\Phi_0^+(\beta = 3\nu_N)$ is equivalent to the doubly closed shell model wave function in Eq. (2.6), which originally has the $\alpha$ cluster degree of freedom or sort of like a seed of $\alpha$ clustering, as discussed in §2.1. For $\beta < 3\nu_N$, $\Phi_0^+(\beta)$ expresses a wave
Fig. 1. Dependence of the monopole strengths $M_N(E0)$ on the model space of the ground state wave function characterized as quanta $N$ [see Eq. (3.6)]. The square and circle points correspond to $M_N(E0; 0^+_1 \rightarrow 0^+_2)$ and $M_N(E0; 0^+_1 \rightarrow 0^+_3)$, respectively.

function in which the $^{12}$C$+\alpha$ relative motion or the seed of $\alpha$ clustering is swollen in comparison with those in the doubly closed shell model wave function. Thus, the study of the dependence of the squared overlap $P$ on the parameter $\beta/(3\nu_N)$ gives a rough indication on the degree of the deviation from the doubly closed wave function, i.e. the degree of $\alpha$ clustering activated in the ground state. We found that $P$ has the maximum value of 0.958 at $\beta/(3\nu_N) = 0.847$ [vs $P = 0.890$ at $\beta/(3\nu_N) = 1$]. The result of $\beta/(3\nu_N) = 0.847$ means that in the ground state wave function $|0^+_1\rangle$ the $^{12}$C$+\alpha$ relative motion or the seed of $\alpha$ clustering is definitely swollen in comparison with those in the doubly closed shell model wave function. Thus the ground state correlation makes the structure of the ground state closer to the $^{12}$C$+\alpha$ cluster structure in the $0^+_2$ and $0^+_3$ states, and then the monopole strengths become significantly larger in comparison with those with no ground state correlation.

3.2. Monopole transition matrix elements in $^{12}$C

The analytical expression of the monopole transition matrix element $M(E0, 0^+_1 - 0^+_2)$ is demonstrated in Eq. (2.64), which depends on the nucleon size parameter $\nu_N$ and width parameter of the Hoyle state $\gamma$. The expression of the monopole matrix element consists of three parts like the case of $^{16}$O, and the dominant part is the radial integral referring to the relative motions among three $\alpha$ clusters, $\langle R_{40}(r, \nu_N)|r^2|R_{60}(r, \nu_N)\rangle = \sqrt{21}/8/\nu_N$. The strength of the radial integral is a few times larger than the single particle monopole strengths, $\langle 0s|r^2|1s\rangle$ and $\langle 0p|r^2|1p\rangle$.

In the present study we use the value $\nu_N = 0.168$ fm$^{-2}$, which reproduces the observed rms radius of $^{12}$C with the $SU(3)$ shell model wave function in Eqs. (2.51)
Table II. $\xi_5$ and monopole matrix element $M(E_0, 0^+_2 - 0^+_1)$ in $^{12}\text{C}$ calculated at several values of $\gamma$. $M(E_0, 0^+_2 - 0^+_1)$ is given as $\xi_5 \times 0.882/\nu_N$ fm$^2$, where the value $\nu_N = 0.168$ fm$^{-2}$ is used. $R_{\text{rms}}(0^+_2)$ is the corresponding rms radius of the Hoyle state to the adopted values of $\nu_N$ and $\gamma$.

<table>
<thead>
<tr>
<th>$\gamma$ [fm$^{-2}$]</th>
<th>$\xi_5$</th>
<th>$M(E_0, 0^+_2 - 0^+_1)$ [fm$^2$]</th>
<th>$R_{\text{rms}}(0^+_2)$ [fm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0238</td>
<td>0.338</td>
<td>1.775</td>
<td>3.56</td>
</tr>
<tr>
<td>0.0182</td>
<td>0.252</td>
<td>1.326</td>
<td>3.79</td>
</tr>
<tr>
<td>0.0143</td>
<td>0.191</td>
<td>1.004</td>
<td>4.03</td>
</tr>
<tr>
<td>0.0115</td>
<td>0.147</td>
<td>0.773</td>
<td>4.28</td>
</tr>
</tbody>
</table>

$\sim (2.53)$. The monopole matrix element in Eq. (2.64) which is expressed as $M(E_0, 0^+_2 - 0^+_1) = \xi_5 \times 0.882/\nu_N$ fm$^2$ contains $\xi_5$ which depends on $\gamma$. In Table II, we display $\xi_5$ and $M(E_0, 0^+_2 - 0^+_1)$ calculated at several $\gamma$ values. According to Ref. 15), we should use the value of $\gamma \approx 0.018$ fm$^{-2}$. With this value of $\gamma$, the Bose-condensate wave function has a very large (almost 100 %) overlap with the full solution of the $3\alpha$ RGM for the Hoyle state and gives an rms radius 3.8 fm for the Hoyle state. For $\gamma = 0.018$ fm$^{-2}$, we obtain $\xi_5 = 0.25$, which is relatively small in comparison with $\eta_6 = 0.379$ and $\zeta_6 = 0.570$ in the case of $^{16}\text{O}$ in §3.1. This leads to

$$M(E_0, 0^+_2 - 0^+_1) = 0.25 \times \frac{0.882}{\nu_N} = 1.3 \text{ fm}^2,$$

which is of the same order of magnitude as the observed value $(5.4 \pm 0.2 \text{ fm}^2)$ but reproduces only about 25 % in comparison with that. This value of about 25 % is a little bit smaller in contrast to that of the $^{16}\text{O}$ case (see the previous subsection §3.1) in which our simple estimates are larger than about 40 % of the experimental data.

We should note that in more realistic situation the description of the ground state adopted here for $^{12}\text{C}$ using the $SU(3)$ shell model is not necessarily good and a deviation from the $SU(3)$ shell model representation should be taken into account. According to the structure study of $^{12}\text{C}$ with the $3\alpha$ orthogonality condition model (OCM), the $SU(3)(\lambda, \mu) = (0, 4)$ component with the lowest oscillator quantum ($N_{TOT} = 8$) is only about 60 % in the ground state. The smallness of the $SU(3)(\lambda, \mu) = (0, 4)$ component with the lowest oscillator quantum ($N_{TOT} = 8$) is in contrast to the $^{16}\text{O}$ case, in which the ground state is well described by the doubly closed shell model wave function $(0s)^4(0p)^{12}$: The $SU(3)(\lambda, \mu) = (0, 0)$ component with the lowest quantum ($N_{TOT} = 12$) of the $^{16}\text{O}$ ground state is as large as about 90 %.  

Here we demonstrate the effect of the ground state correlation to the monopole matrix element by adopting the following wave function for the ground state:  

$$\Psi_G(\tilde{\gamma}, \nu_N) = N_G \sqrt{\frac{4!14!}{12!}} A \left\{ \exp \left\{ - \tilde{\gamma} \left( 2s^2 + \frac{8}{3} t^2 \right) \right\} \phi(\alpha_1) \phi(\alpha_2) \phi(\alpha_3) \right\}, \tag{3.10}$$

where $N_G$ is normalization constant. This wave function is called THSR wave function and depends on two parameters $\tilde{\gamma}$ and $\nu_N$. We impose the condition that this
wave function reproduces the observed rms radius, 2.47 fm, of the ground state. Then, the ratio $\tilde{\gamma}/\nu_N$ is the only parameter which describes the property of the ground state. It is noted that $\Psi_G(\tilde{\gamma}, \nu_N)$ with $\tilde{\gamma}/\nu_N = 1$ agrees with the $SU(3)$ $(\lambda, \mu) = (0, 4)$ shell model wave function in Eqs. (2.51) – (2.53), demonstrating directly that the wave function originally has sort of like a seed of $3\alpha$ clustering. Taking the $\tilde{\gamma}$ value a little smaller than $\nu_N$, $\Psi_G(\tilde{\gamma}, \nu_N)$ deviates from the $SU(3)$ shell model wave function and the slightly more relaxed spatial clustering of $3\alpha$ clusters than the $SU(3)$ wave function is induced in the ground state, i.e. $\Psi_G(\tilde{\gamma}, \nu_N)$ expresses sort of like a generalized $SU(3)$ wave function in which the seed of $3\alpha$ clustering are slightly swollen in comparison with the original $SU(3)$ wave function. This is the ground state correlation taking into account here, which is similar to the case in $^{16}\text{O}$ as discussed in §3.1. In Ref. 15), it is reported that the ground state wave function of the $3\alpha$ RGM calculation of Refs. 8) and 9) can be well approximated by this kind of wave function. The amount of the $3\alpha$-like ground state correlation, thus, can be characterized by the ratio $\tilde{\gamma}/\nu_N$, which should be less than or equal to unity. In the $3\alpha$ cluster model, the nucleon size parameter $\nu_N$ is usually chosen to reproduce the rms radius of $\alpha$ cluster, $\nu_N = 0.275$ fm $^{-2}$ which is larger than that for the $SU(3)$ shell model wave function ($\nu_N = 0.168$ fm $^{-2}$) shown above. The estimation of $\tilde{\gamma}/\nu_N$ for the ground state wave function given by Ref. 15) is as small as $\tilde{\gamma}/\nu_N \sim 0.29$. This small value indicates that the ground state of $^{12}\text{C}$ has a significant amount of the $3\alpha$ correlation. Below we change the value of the parameter $\tilde{\gamma}/\nu_N$ from 1.0 down to 0.27.

The wave function of the Hoyle state is constructed so as to be orthogonal to the ground state wave function $\Psi_G$ in Eq. (3.10) and is given as follows:

$$\Psi_H(\gamma, \tilde{\gamma}, \nu_N) = N_H (1 - P) \sqrt{\frac{4!4!4!}{12!}} A \exp \left\{ - \gamma \left( 2 s^2 + \frac{8}{3} t^2 \right) \right\} \phi(\alpha_1) \phi(\alpha_2) \phi(\alpha_3),$$

(3.11)

$$P \equiv |\Psi_G(\tilde{\gamma}, \nu_N)\rangle \langle \Psi_G(\tilde{\gamma}, \nu_N)|,$$

(3.12)

where $N_H$ is normalization constant. The width parameter $\gamma$ is determined so as to reproduce the rms radius of the Hoyle state, 3.8 fm. This exotic structure of the Hoyle state was found to be described simply with a single $\alpha$-condensate wave function given in Eq. (3.11). The monopole matrix element is given by

$$M(E0, 0^+_1 - 0^+_2) = \langle \Psi_G(\tilde{\gamma}, \nu_N) \mid \frac{1}{2} \sum_{i=1}^{12} (\mathbf{r}_i - \mathbf{r}_G)^2 \mid \Psi_H(\gamma, \tilde{\gamma}, \nu_N) \rangle,$$

(3.13)

depending only on the parameter $\tilde{\gamma}/\nu_N$.

Table III shows the values of the monopole matrix elements [Eq. (3.13)] calculated at several $\tilde{\gamma}/\nu_N$ values. We see that the monopole matrix element increases as the ratio $\tilde{\gamma}/\nu_N$ decreases from unity, namely as the $3\alpha$-like correlation becomes stronger in the ground state. This can be reasonably understood from the fact that the ground state wave function $\Psi_G$ with stronger $3\alpha$-like correlation has larger $3\alpha$-cluster component which makes larger the overlap with the Hoyle state wave function.
Table III. Dependence of the monopole matrix element in $^{12}$C on the amount of 3α-like correlation involved in the ground state, which is characterized by $\tilde{\gamma}/\nu_N$. The monopole matrix element is given as $M(E_0,0^+_1-0^+_1)=|\Psi_G(\tilde{\gamma},\nu_N)|\sum_{i=1}^{12}(r_i-r_G)^2/2|\Psi_H(\gamma,\tilde{\gamma},\nu_N)|$, where $\Psi_G(\tilde{\gamma},\nu_N)$ and $\Psi_H(\gamma,\tilde{\gamma},\nu_N)$ are the ground state and Hoyle state wave functions, respectively. The rms radius of the ground-state wave function $\Psi_G$ is fixed to the experimental one (2.47 fm). Then, the ratio $\tilde{\gamma}/\nu_N$ is only the parameter to describe the property of the ground state. For a given value of $\tilde{\gamma}/\nu_N$, the value of $\gamma$ in $\Psi_H(\gamma,\tilde{\gamma},\nu_N)$ is chosen so as to reproduce the rms radius of the Hoyle state (3.8 fm). See the text for details.

<table>
<thead>
<tr>
<th>$\tilde{\gamma}/\nu_N$</th>
<th>$M(E_0,0^+_1-0^+_1)$ [fm$^2$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.000</td>
<td>1.326</td>
</tr>
<tr>
<td>0.705</td>
<td>1.810</td>
</tr>
<tr>
<td>0.498</td>
<td>2.473</td>
</tr>
<tr>
<td>0.309</td>
<td>3.597</td>
</tr>
<tr>
<td>0.274</td>
<td>4.035</td>
</tr>
</tbody>
</table>

$\Psi_H$ with the dilute 3α cluster structure, and then the monopole matrix element becomes larger. At the value of $\tilde{\gamma}/\nu_N \sim 0.27$, the monopole matrix element is about 4.0 fm$^2$, which is about three times larger than that for $\tilde{\gamma}/\nu_N = 1$, and is closer to the observed value 5.4 ± 0.2 fm$^2$. It is noted that $\tilde{\gamma}/\nu_N \sim 0.27$ gives the nucleon size parameter $\nu_N \sim 0.26$ fm$^{-2}$ which corresponds to the value used usually in the microscopic 3α cluster model calculations.$^{1),8),9),15),18}$ Without the ground state correlation the calculated monopole value is smaller than the observed value by a factor of 4.15 but now with inclusion of the ground state correlation the calculated monopole value changed to be smaller only by a factor of 1.35 than the observed value.

§4. Discussion and summary

The monopole transitions from cluster states to ground states in light nuclei are rather large which is comparable with the single particle strength. The single particle estimate of the monopole transition is based on the assumption that the excited state has a one-particle one-hole excitation from the ground state. However, the cluster structure is very different from the shell-model-like structure of the ground state, and its state is described as a superposition of many-particle many-hole configurations when it is expanded by shell model configurations. This means that in the excited state with a cluster structure, the component of a one-particle one-hole excitation from the ground state configuration is expected to be very small. Therefore the observation of rather large monopole strengths for cluster states which are comparable with single particle strength looks not to be easy to explain. Under this kind of understanding it has been often regarded that the monopole transition occurs through the mixing of shell model wave function $|\text{shell}\rangle$ and the cluster model wave function $|\text{cluster}\rangle$

\begin{align}
|\text{Ground}\rangle &= \alpha|\text{shell}\rangle + \beta|\text{cluster}\rangle, \\
|\text{Excited}\rangle &= -\beta|\text{shell}\rangle + \alpha|\text{cluster}\rangle.
\end{align}
Monopole Excitation to Cluster States

Since it is assumed that the monopole operator $O_M$ does not connect $|\text{cluster}\rangle$ and $|\text{shell}\rangle$, $\langle \text{cluster}| O_M |\text{shell}\rangle = 0$, the monopole matrix element is considered to come from the diagonal matrix elements (for example see Ref. 19)),

$$\langle \text{Excited}| O_M |\text{Ground}\rangle = \alpha \beta (\langle \text{cluster}| O_M |\text{cluster}\rangle - \langle \text{shell}| O_M |\text{shell}\rangle).$$  (4.3)

Our explanation of the strong monopole transition between ground state and excited cluster states is quite different from this explanation. We insist that the order of magnitude of the strong monopole transition is given by the matrix element $\langle \text{cluster}| O_M |\text{shell}\rangle \neq 0$,

$$\langle \text{Excited}| O_M |\text{Ground}\rangle \approx \langle \text{cluster}| O_M |\text{shell}\rangle, \text{ (for order of magnitude).}$$  (4.4)

Our argument is based on the Bayman-Bohr theorem which says that the $SU(3)$ shell-model wave function describing rather well the structure of the ground state of light nuclei is equivalent in most cases to cluster model wave function. The implication of this theorem is that the clustering degree of freedom is already embedded even in the shell model wave function. In the present study the monopole excitation of the ground state to cluster states is understood as just the excitation of the inter-cluster relative motion in the ground state to the inter-cluster relative motion in excited cluster states. This resembles the monopole excitation of the single nucleon motion. Our understanding was explicitly shown to be true by deriving the analytical expressions of the monopole matrix elements.

In this paper we analyzed the monopole transitions in $^{16}$O between the ground state and $^{12}$C $+$ $\alpha$ clusters states ($0^+_2$ and $0^+_3$) together with the one in $^{12}$C between the ground state and $3\alpha$ cluster state ($0^+_2$: Hoyle state). According to the Bayman-Bohr theorem, the doubly closed shell model wave function of $^{16}$O, which has the $SU(3)(\lambda \mu) = (00)$ symmetry and total quanta $N_{TOT} = 12$, is equivalent to the $^{12}$C $+$ $\alpha$ cluster wave function as well as $^{12}$C $2^+$ $+$ $\alpha$ with orbital angular momentum of the inter-cluster relative motion $L_r = 0$ and 2, respectively. The number of oscillator quanta of the inter-cluster relative motion $N_r$ is 4. Similarly, the ground state wave function of $^{12}$C with $SU(3) (\lambda \mu) = (04)$ and $N_{TOT} = 8$ is equivalent to the $3\alpha$ cluster wave function. The number of oscillator quanta of the inter-cluster relative motion with respect to each of two Jacobi coordinates is $N_r = 4$. On the other hand, the $0^+_2 [0^+_3]$ state of $^{16}$O has a $^{12}$C$0^+$ $+$ $\alpha$ cluster structure $[^{12}$C$2^+$ $+$ $\alpha]$ with the relative orbital angular momentum $L_r = 0 [L_r = 2]$. The $0^+_2$ state of $^{12}$C has a $3\alpha$ cluster structure with S-wave relative angular momenta referring to two Jacobi coordinates for $3\alpha$ clusters.

The analytical expressions of the monopole matrix elements we derived for the above transitions in $^{16}$O and $^{12}$C are composed of three factors. The most important factor is the radial integrals with harmonic oscillator wave functions $\langle N_r = 6, L_r |r^2|N_r = 4, L_r \rangle$ with $L_r = 0$ or 2. The values of these integrals are a few times larger than the single particle monopole transition matrix elements in $p$-shell nuclei, $\langle 1s|r^2|0s \rangle = \langle N_r = 2, L_r = 0 |r^2|N_r = 0, L_r = 0 \rangle$ and $\langle 1p|r^2|0p \rangle = \langle N_r = 3, L_r = 1 |r^2|N_r = 1, L_r = 1 \rangle$. The second factor is the amplitude (not squared amplitude) of the $2\hbar \omega$ - excited harmonic oscillator wave function in the cluster states.
which is denoted as $\eta_6$ for $M(E0, 0^+_1 - 0^+_2)$ and $\zeta_6$ for $M(E0, 0^+_1 - 0^+_3)$ in $^{16}\text{O}$ and $\xi_5$ for $M(E0, 0^+_1 - 0^+_2)$ in $^{12}\text{C}$. They are not so small; $\eta_6 = 0.38$, $\zeta_6 = 0.57$, and $\xi_5 = 0.19$.

The third factor is due to the antisymmetrization among nucleons, which is denoted as $\sqrt{\tau_{3,4}/\tau_{0,6}}$ or $\sqrt{\tau_{2,4}/\tau_{2,6}}$ in $^{16}\text{O}$ and $\sqrt{\langle F_4 \rangle / \langle F_5 \rangle}$ in $^{12}\text{C}$. Since the quantities with strong antisymmetrization effect are contained in the form of ratio, the third factor has magnitude close to unity. As is expected from the analytical expressions, the calculated numerical values of the monopole matrix elements were shown to have the same order of magnitude as the observed values which are comparable with the single nucleon strength.

Although the calculated values of the monopole matrix elements without ground state correlation surely reproduce the order of magnitude of the observed values, when compared with the data in detail, they are a few times smaller than the observed values. In the case of $^{16}\text{O}$, two kinds of theoretical values of $M(E0, 0^+_1 - 0^+_2)$ are 60 % and 38 % of the observed values, respectively, while those of $M(E0, 0^+_1 - 0^+_3)$ are 96 % and 63 %. In the case of $^{12}\text{C}$, theoretical value of $M(E0, 0^+_1 - 0^+_2)$ is 25 % of the observed value. Therefore we next investigated the effect of the ground state correlation to the monopole matrix elements. The ground state correlation we considered was the one caused by the activation of the clustering degree of freedom described by Bayman-Bohr theorem. In the calculation of the monopole strength without ground state correlation, we only considered the clustering degree of freedom rather in a static way. Namely we did not consider the dynamical effect of the clustering degree of freedom which excites the ground state configuration toward including higher quantum configurations. We know that the clustering degree of freedom described by Bayman-Bohr theorem has the physical reality because we observe many excited cluster states which are formed by exciting the clustering degree of freedom embedded in the ground state. Therefore taking into account the ground state correlation caused by the clustering degree of freedom described by Bayman-Bohr theorem is very natural and should be studied.

The investigation of the effect of the ground state correlation to the monopole strength in $^{16}\text{O}$ was made in the framework of the $^{12}\text{C} + \alpha$ OCM. It is because, in discussing the monopole strength without ground state correlation in §2, we used the results of the $^{12}\text{C} + \alpha$ OCM in Ref. 3). We repeated the same calculation as one in Ref. 3). We found that 1) increasing the amount of the ground state correlation, the monopole strengths are growing almost monotonously, and 2) at a full amount of the ground state correlation, the monopole strengths are reproduced within a factor of 1.13 in comparison with the observed values. The reason why the ground state correlation enhances the monopole strengths was discussed with a simple approach. In the case of $^{12}\text{C}$, the investigation of the effect of the ground state correlation to the monopole strength was performed by expressing the ground state with the so-called THSR wave function.$^{14}$ This wave function has two parameters $\tilde{\gamma}$ and $\nu_N$. When $\tilde{\gamma} = \nu_N$, the wave function is just equal to the $SU(3)$ wave function with $N_TOT = 8$ and $(\lambda, \mu) = (0, 4)$. As we make the ratio $\tilde{\gamma}/\nu_N$ smaller than unity, the wave function contains more amount of the ground state correlation. We found that at a full amount of the ground state correlation, the monopole strength is reproduced within a factor of 1.35 in comparison with the observed value.
The implication of the Bayman-Bohr theorem has been misunderstood such that the cluster model description is rather unnecessary, since a cluster model wave function is equivalent to a shell model wave function. The existence of cluster states especially as excited states is well established these days. Thus, the implication of the Bayman-Bohr theorem should be understood straightforwardly as follows. If the ground state is well described by an SU(3) shell model wave function equivalent to a cluster model wave function, the ground state possesses two different characters simultaneously, shell-model-state character and cluster-model-state character. This means that the ground state has mean-field degree of freedom and clustering degree of freedom simultaneously. Both of them can be excited, when the nucleus is stimulated by an external field. The monopole excitation to excited cluster states demonstrates us directly the evidence that the clustering degree of freedom is embedded in the ground state. In this paper we showed that the clustering degree of freedom embedded in the ground state can reproduce the order of magnitude of the monopole strength even without taking into account the ground state correlation. Moreover it was demonstrated that, if we take into account the ground state correlation activating the clustering degree of freedom described by the Bayman-Bohr theorem, the monopole strengths are reproduced within a factor of 1.13 in $^{16}$O and within a factor of 1.35 in $^{12}$C, in comparison with the observed values.

Our present study ascertains that the monopole transition between cluster and ground states in light nuclei is generally strong as to be comparable to the single particle strength. The measurement of strong monopole transitions or excitations, therefore, is in general very useful for the study of cluster states.

One of the authors (Y. F.) is grateful for the financial assistance from the Special Postdoctoral Researcher Program of RIKEN.

Appendix A

---

The Energy-Weighted Sum Rule of Monopole Transition by the Use of the Jacobi Coordinate---

We discuss here the energy-weighted sum rule of the monopole transition. The sum rule is written as follows:

$$\sum_k |\langle k | \frac{1}{2} \sum_{i=1}^A (r_i - r_G)^2 |g \rangle|^2 (E_k - E_g) = \frac{\hbar^2}{2m} A R_{\text{rms}}^2,$$

(A.1)

$$R_{\text{rms}}^2 = \frac{1}{A} \langle g | \sum_{i=1}^A (r_i - r_G)^2 |g \rangle,$$

(A.2)

where $|g \rangle$ and $E_g$ stand for the ground state and its energy, respectively, $|k \rangle$ and $E_k$ represent the $k$-th excited state and its energy, respectively, and $r_G$ stands for the center-of-mass coordinate.

In $^{16}$O, the observed value of $R_{\text{rms}}$ is 2.67 fm and then the energy-weighted sum rule value ($\hbar^2/2m$)\text{16}$R_{\text{rms}}^2$ is 2361 fm$^4\cdot$MeV. In the case of the $0^+_2$ state at 6.05 MeV which has $M(E0, 0^+_1 - 0^+_2)_{\text{exp}} = 3.55$ fm$^2$, the energy-weighted monopole transition strength is $(3.55)^2 \times 6.05 = 76.3$ fm$^4\cdot$MeV. This value is 3.2 % of the energy-weighted
sum rule value. In the case of the 0\(_{3}^{+}\) state at 12.05 MeV which has \(M(E0, 0_{1}^{+} - 0_{3}^{+})_{\text{exp}} = 4.03\) fm\(^2\), the energy-weighted monopole transition strength is \((4.03)^{2} \times 12.05 = 196\) fm\(^4\)·MeV. This value is 8.3 % of the energy-weighted sum rule value. The sum of the energy-weighted monopole transition strengths of 0\(_{2}^{+}\) and 0\(_{3}^{+}\) states is 11.5 % of the energy-weighted sum rule value. In \(^{12}\)C, the observed value of \(R_{\text{rms}}\) is 2.37 fm and then the energy-weighted sum rule value \((\hbar^{2}/2m)12R_{\text{rms}}^{2}\) is 1395 fm\(^4\)·MeV. In the case of the 0\(_{2}^{+}\) state at 7.66 MeV which has \(M(E0, 0_{1}^{+} - 0_{2}^{+})_{\text{exp}} = 5.4\) fm\(^2\), the energy-weighted monopole transition strength is \((5.4)^{2} \times 7.66 = 223\) fm\(^4\)·MeV. This value is 16 % of the energy-weighted sum rule value. These percentage values show that the strength of the monopole transition or excitation to cluster states shares an appreciable portion of the energy-weighted sum rule value.

The formula of the energy-weighted sum rule of monopole transition is obtained by calculating the double commutator of the monopole transition operator \(O_{M}\) and the system Hamiltonian \(H\)

\[
[O_{M}, [H, O_{M}]], \quad O_{M} = \frac{1}{2} \sum_{i=1}^{A} (r_{i} - r_{G})^{2}. \quad (A\text{-3})
\]

The calculation of the double commutator looks tedious due to the existence of the center-of-mass coordinate \(r_{G}\) but it can be made very easily by using the Jacobi coordinate. For the Hamiltonian \(H\) with momentum-independent interaction, \(H\) can be replaced by the kinetic energy operator \(K\)

\[
[O_{M}, [H, O_{M}]] = [O_{M}, [K, O_{M}]], \quad K = -\frac{\hbar^{2}}{2m} \sum_{i=1}^{A} \left( \frac{\partial}{\partial r_{i}} \right)^{2} - \frac{\hbar^{2}}{2Am} \left( \frac{\partial}{\partial r_{G}} \right)^{2}. \quad (A\text{-4})
\]

Now we introduce the normalized Jacobi coordinates as

\[
x_{j} = \sqrt{\frac{j}{j+1}} \left( \frac{1}{j} \sum_{i=1}^{j} r_{i} - r_{j+1} \right), \quad j = 1 \sim A - 1, \quad (A\text{-5})
\]

\[
x_{A} = \sqrt{\frac{1}{A}} \sum_{i=1}^{A} r_{i} = \sqrt{Ar_{G}}. \quad (A\text{-6})
\]

One can easily check that the linear transformation from \(\{r_{i}, i = 1 \sim A\}\) to \(\{x_{j}, j = 1 \sim A\}\) is unitary. Therefore we have

\[
\sum_{i=1}^{A} r_{i}^{2} = \sum_{j=1}^{A} x_{j}^{2}, \quad (A\text{-7})
\]

\[
\sum_{i=1}^{A} (r_{i} - r_{G})^{2} = \sum_{i=1}^{A} r_{i}^{2} - Ar_{G}^{2} = \sum_{j=1}^{A-1} x_{j}^{2}, \quad (A\text{-8})
\]

\[
\sum_{i=1}^{A} \left( \frac{\partial}{\partial r_{i}} \right)^{2} = \sum_{j=1}^{A} \left( \frac{\partial}{\partial x_{j}} \right)^{2}, \quad (A\text{-9})
\]
\[
\sum_{i=1}^{A} \left( \frac{\partial}{\partial r_i} \right)^2 - \frac{1}{A} \left( \frac{\partial}{\partial r_G} \right)^2 = \sum_{j=1}^{A-1} \left( \frac{\partial}{\partial x_j} \right)^2.
\]

(A.10)

Thus we have

\[
O_M = \frac{1}{2} \sum_{j=1}^{A-1} x_j^2,
\]

(A.11)

\[
K = -\frac{\hbar^2}{2m} \sum_{j=1}^{A-1} \left( \frac{\partial}{\partial x_j} \right)^2,
\]

(A.12)

When we use the above expressions of \(O_M\) and \(K\) by the normalized Jacobi coordinates, we can easily obtain the following result:

\[
[O_M, [H, O_M]] = [O_M, [K, O_M]] = \frac{\hbar^2}{m} \sum_{j=1}^{A-1} x_j^2 = \frac{\hbar^2}{m} \sum_{i=1}^{A} (r_i - r_G)^2.
\]

(A.13)

We thus have the formula of the energy-weighted sum rule of monopole transition as follows:

\[
\frac{\hbar^2}{2m} \langle g | \sum_{i=1}^{A} (r_i - r_G)^2 | g \rangle = \frac{1}{2} \langle g | [O_M, [H, O_M]] | g \rangle
\]

\[= \sum_k |\langle k | \frac{1}{2} \sum_{i=1}^{A} (r_i - r_G)^2 | g \rangle|^2 (E_k - E_g). \quad (A.15)
\]

Appendix B

Description of \(^{16}\text{O}\) and \(^{12}\text{C}\) Ground States with \(SU(3)\) Wave Function

The total number of the oscillator quanta \(N_{TOT}\) possessed by the doubly closed shell wave function of \(^{16}\text{O}\) is \(N_{TOT} = 12\). For \(N_{TOT} = 12\), it is possible to construct many \(^{12}\text{C}+\alpha\) cluster wave functions with various \(SU(3)\) symmetry \((\lambda, \mu)\), \(A\{[\mathcal{R}_4(r)\phi(12\text{C})]_{(\lambda,\mu)}\phi(\alpha)\}\). These wave functions, however, become all zero except for \((\lambda, \mu) = (0,0)\), because of the nature of the doubly closed wave function. Using the following relation

\[
[\mathcal{R}_{4L}(r)\phi_L(12\text{C})]_{J=0} = \sum_{(\lambda,\mu)} \langle (4,0)L, (0,4)L | (\lambda,\mu)0 | [\mathcal{R}_4(r)\phi(12\text{C})]_{(\lambda,\mu)} \rangle,
\]

(B.1)

we have

\[
A\{[\mathcal{R}_{4L}(r)\phi_L(12\text{C})]_{J=0}\phi(\alpha)\}
\]

\[= \sum_{(\lambda,\mu)} \langle (4,0)L, (0,4)L | (\lambda,\mu)0 | A\{[\mathcal{R}_4(r)\phi(12\text{C})]_{(\lambda,\mu)}\phi(\alpha)\} \rangle
\]

\[= \langle (4,0)L, (0,4)L | (0,0)0 | A\{[\mathcal{R}_4(r)\phi(12\text{C})]_{(\lambda,\mu)=(0,0)}\phi(\alpha) \rangle, \quad (B.2)
\]
for $L = 0, 2, \text{and } 4$. This relation is an explanation of the equalities of Eqs. (2-6) – (2-8).

Similar argument holds for the ground state of $^{12}$C. Although there can be constructed many $3\alpha$ cluster wave functions with various $SU(3)$ symmetry $(\lambda, \mu)$, $\mathcal{A}[[\mathcal{R}_{N_1}(s, 2\nu N)\mathcal{R}_{N_2}(t, (8/3)\nu N)]_{(\lambda, \mu)}\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)]$ for $N_{TOT} = N_1 + N_2 = 8$ which is the lowest number of the total oscillator quanta for $^{12}$C, only one wave function with $(\lambda, \mu) = (0, 4)$ is non-vanishing which is possible for $N_1 = N_2 = 4$.\textsuperscript{17,20} Therefore we have the following relations

\[
\mathcal{A}[[\mathcal{R}_{N_1,L}(s, 2\nu N)\mathcal{R}_{N_2,L}(t, (8/3)\nu N)]_{J=0}\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)] = \delta_{N_1,4}\delta_{N_2,4}\langle(4, 0)L, (4, 0)L||(0, 4)0\rangle \times \mathcal{A}[[\mathcal{R}_{N_1}(s, 2\nu N)\mathcal{R}_{N_2}(t, (8/3)\nu N)]_{(\lambda, \mu)=(0, 4)}\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)],
\]

(B.3)

for $L = 0, 2, \text{and } 4$. This relation is an explanation of the equalities of Eqs. (2-51) – (2-53).

\section*{Appendix C}

\textbf{Dependence of $\mathcal{M}(E_0, 0^+_1 - 0^+_2)$ on the Width Parameter $\gamma$ of the $3\alpha$ Condensed Wave Function}

First we note

\[
\mathcal{A}\{\hat{\chi}_H(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)\} = \mathcal{A}\{\hat{\chi}_{HG}(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)\}
- \langle\mathcal{R}_{4,4,L=0}^{8,J=0}(s, t)|\hat{\chi}_{HG}(s, t)\rangle \mathcal{A}\{\mathcal{R}_{4,4,L=0}^{8,J=0}(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)\}.
\]

(C.1)

Then we obtain

\[
\langle\hat{\chi}_H\rangle = \langle\hat{\chi}_{HG}\rangle - \left(\langle\mathcal{R}_{4,4,L=0}^{8,J=0}(s, t)|\hat{\chi}_{HG}(s, t)\rangle\right)^2 \langle\mathcal{R}_{4,4,L=0}^{8,J=0}(s, t)\rangle,
\]

(C.2)

with the notation

\[
\langle G \rangle = \langle G(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)\mathcal{A}\{G(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)\}\rangle.
\]

(C.3)

Next we note

\[
\mathcal{A}\{\hat{\chi}_{HG}(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)\} = \sum_{n \geq 4} \sum_{n_1+n_2=n} \langle\mathcal{R}_{2n_1,2n_2,0}^{2n,J=0}(s, t)|\hat{\chi}_{HG}(s, t)\rangle
\times \mathcal{A}\{\mathcal{R}_{2n_1,2n_2,0}^{2n,J=0}(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)\}
\]

(C.4)

\[
= \left(\frac{2\sqrt{\nu_N\gamma}}{\nu_N + \gamma}\right)^3 \sum_{n=4}^{\infty} \left(\frac{\nu_N - \gamma}{\nu_N + \gamma}\right)^n A\{F_n(s, t)\phi(\alpha_1)\phi(\alpha_2)\phi(\alpha_3)\},
\]

(C.5)

\[
F_n(s, t) = \sum_{n_1+n_2=n} \sqrt{\frac{(2n_1+1)!!(2n_2+1)!!}{(2n_1)!!(2n_2)!!}} R_{2n_1,2n_2,0}^{2n,J}(s, t),
\]

(C.6)

where the following formula is used

\[
\langle R_{2n,0}(r, \beta)\rangle \sqrt{\frac{(2\beta')^2}{\pi}} e^{-\beta' r^2} = \sqrt{\frac{(2n+1)!!}{(2n)!!}} \left(\frac{2\sqrt{\beta\beta'}}{\beta + \beta'}\right)^{\frac{n}{2}} \left(\beta - \beta'\right)^n.
\]

(C.7)
From Eqs. (C.3) and (C.5) we obtain

\[ \langle \hat{\chi}_{HG} \rangle = \left( \frac{2\sqrt{\nu_N \gamma}}{\nu_N + \gamma} \right)^6 \sum_{n=4}^{\infty} \left( \frac{\nu_N - \gamma}{\nu_N + \gamma} \right)^{2n} \langle F_n \rangle. \] (C.8)

By noticing

\[ \langle R_{4,4,0}^{S,J=0}(s,t) | \hat{\chi}_{HG}(s,t) \rangle = \frac{5!!}{4!!} \left( \frac{2\sqrt{\nu_N \gamma}}{\nu_N + \gamma} \right)^3 \left( \frac{\nu_N - \gamma}{\nu_N + \gamma} \right)^4, \] (C.9)

\[ \langle F_4 \rangle = \left( \frac{5!!}{4!!} \right)^2 \langle R_{4,4,0}^{S,J=0} \rangle = \left( \frac{5!!}{4!!} \right)^2 \frac{1}{(\hat{N}_g)^2}, \] (C.10)

we obtain

\[ \langle \hat{\chi}_H \rangle = \left( \frac{2\sqrt{\nu_N \gamma}}{\nu_N + \gamma} \right)^6 \sum_{n=5}^{\infty} \left( \frac{\nu_N - \gamma}{\nu_N + \gamma} \right)^{2n} \langle F_n \rangle \equiv \frac{1}{(\hat{N}_H)^2}, \] (C.11)

\[ D_{H,2n_1,2n_2} \equiv \sqrt{\frac{(2n_1 + 1)!!(2n_2 + 1)!!}{(2n_1)!!(2n_2)!!}} \left( \frac{2\sqrt{\nu_N \gamma}}{\nu_N + \gamma} \right)^3 \left( \frac{\nu_N - \gamma}{\nu_N + \gamma} \right)^{n_1+n_2}. \] (C.12)

By combining all these formulas we have

\[ M(E_0, 0^+_2 - 0^+_1) = \frac{1}{2} \frac{\hat{N}_H}{\hat{N}_g} (D_{H,6,4} + D_{H,4,6}) \langle R_{40}(r, \nu_N) | r^2 | R_{60}(r, \nu_N) \rangle \] (C.13)

\[ = \frac{1}{2} \sqrt{\left( \frac{2\sqrt{\nu_N \gamma}}{\nu_N + \gamma} \right)^6 \sum_{n=5}^{\infty} \left( \frac{\nu_N - \gamma}{\nu_N + \gamma} \right)^{2n} \langle F_n \rangle} \]

\[ \times 2 \sqrt{\frac{5!!}{4!!}} \left( \frac{2\sqrt{\nu_N \gamma}}{\nu_N + \gamma} \right)^3 \left( \frac{\nu_N - \gamma}{\nu_N + \gamma} \right)^5 \langle R_{40}(r, \nu_N) | r^2 | R_{60}(r, \nu_N) \rangle \] (C.14)

\[ = \sqrt{\frac{7}{6}} \frac{\langle F_4 \rangle}{\langle F_5 \rangle} \xi_5 \langle R_{40}(r, \nu_N) | r^2 | R_{60}(r, \nu_N) \rangle, \] (C.15)

\[ \xi_5 = \sqrt{\frac{\langle F_5 \rangle}{\langle F_5 \rangle + \sum_{n=6}^{\infty} \left( \frac{\nu_N - \gamma}{\nu_N + \gamma} \right)^{2(n-5)} \langle F_n \rangle}}. \] (C.16)

**Appendix D**

*Estimation of \( \eta_N \) and \( \zeta_N \) by Ref. 3*

In Ref. 3) the \(^{16}\text{O} \) states are expressed by microscopic \(^{12}\text{C} + \alpha \) cluster wave functions where \(^{12}\text{C} \) cluster can be excited to its first \( 2^+ \) and \( 4^+ \) states. This coupled channel problem is solved by using the coupled channel OCM (orthogonality condition model). The wave functions \( \Phi \) are expressed by the \( SU(3) \)-coupled basis
of the $^{12}\text{C} + \alpha$ cluster model space, which is in the case of $J = 0$

$$
\Phi = \sum_{Nq} \xi_{Nq} \Phi_{Nq}, \quad \text{(D.1)}
$$

$$
\Phi_{N,q=(\lambda,\mu)} = N_{Nq} \frac{1}{\sqrt{\mu_{Nq}}} \frac{1}{\sqrt{16C_4}} A\{[R_N(r, 3\nu_N)\phi(1^{2}\text{C})](\lambda,\mu)\phi(\alpha)\}, \quad \text{(D.2)}
$$

$$
\mu_{N,q=(\lambda,\mu)} = \left\langle [R_N(r, 3\nu_N)\phi(1^{2}\text{C})](\lambda,\mu)\phi(\alpha)\right| A\{[R_N(r, 3\nu_N)\phi(1^{2}\text{C})](\lambda,\mu)\phi(\alpha)\right\rangle. \quad \text{(D.3)}
$$

The $SU(3)$-coupled basis wave functions $\Phi_{N,q=(\lambda,\mu)}$ are ortho-normalized and are the eigenfunctions of normalization kernel with eigenvalues $\mu_{N,q=(\lambda,\mu)}$. The values of the expansion coefficients $\xi_{Nq}$ are given in Table I of the second paper of Ref. 3).

Since the wave function of the $0_{2}^{+}$ state shows predominantly a $^{12}\text{C}(0^{+}) + \alpha$ structure, we approximated in this paper the wave function of the $0_{2}^{+}$ state by a pure $^{12}\text{C}(0^{+}) + \alpha$ structure

$$
\Phi(0_{2}^{+}) = \sum_{N \geq 6} \eta_N \Phi_N, \quad \text{(D.4)}
$$

$$
\Phi_N = \frac{1}{\sqrt{\tau_{0,N}}} \frac{1}{\sqrt{16C_4}} A\{R_{N0}(r)Y_{00}(\hat{r})\phi_{L=0}(1^{2}\text{C})\phi(\alpha)\}. \quad \text{(D.5)}
$$

This means we adopted the following approximation for each $N$

$$
\sum_{q} \xi_{Nq} \Phi_{Nq} \approx \eta_N \Phi_N. \quad \text{(D.6)}
$$

Therefore we have

$$
\eta_N \approx \sqrt{\sum_{q} (\xi_{Nq})^2}. \quad \text{(D.7)}
$$

The values $\sum_{q} (\xi_{Nq})^2$ are tabulated in Table I of the first paper of Ref. 3). For $N = 6$, we have $\eta_6 \approx \sqrt{0.144} = 0.379$.

From Table I of the second paper of Ref. 3), we have in the case of $N = 6$ for $0_{2}^{+}$

$$
\sum_{q} \xi_{6q} \Phi_{6q} = 0.338\Phi_{6,(4,2)} + 0.171\Phi_{6,(2,0)}, \quad \text{(for } 0_{2}^{+}). \quad \text{(D.8)}
$$

We should note here that $\Phi_{6,(2,0)}$ is nothing but $\Phi_{6,(2,0)}^{d}$ which absorbs the total monopole strength of $\Phi_{CS}$ within the $^{12}\text{C} + \alpha$ cluster model space. On the other hand, there holds

$$
\Phi_6 = 0.750\Phi_{6,(4,2)} + 0.661\Phi_{6,(2,0)}. \quad \text{(D.9)}
$$

This relation is easily obtained by comparing $\langle \Phi_6 | O_M | \Phi_{CS} \rangle$ with $\langle \Phi_{6,(2,0)} | O_M | \Phi_{CS} \rangle$:

$$
\langle \Phi_6 | O_M | \Phi_{CS} \rangle = \frac{1}{2} \sqrt{\frac{\tau_{0.4}}{\tau_{0.6}}} \langle R_{40}(r, \nu_N) | r^2 | R_{60}(r, \nu_N) \rangle = \frac{0.784}{\nu_N}, \quad \text{(D.10)}
$$

$$
\langle \Phi_{6,(2,0)} | O_M | \Phi_{CS} \rangle = \frac{1.19}{\nu_N}. \quad \text{(D.11)}
$$
Since the monopole operator $O_M$ does not connect $\Phi_{CS}$ with $\Phi_{6, (4,2)}$, we have

$$\langle \Phi_{6, (2,0)} | \Phi_6 \rangle = \frac{0.784}{1.19} = 0.661.$$  \hfill (D-12)

Using $\eta_6 = 0.379$ we have

$$\eta_6 \Phi_6 = 0.284 \Phi_{6, (4,2)} + 0.250 \Phi_{6, (2,0)}.$$  \hfill (D-13)

The comparison of Eq. (D.8) with Eq. (D.13) tells us that the approximation of $\sum_q \xi_{6q} \Phi_{6q}$ with $\eta_6 \Phi_6$ is not very good and that the monopole strength of $\sum_q \xi_{6q} \Phi_{6q}$ is weaker than that of $\eta_6 \Phi_6$.

The arguments for the $0^{+}_3$ state can be made completely in the same way as the $0^{+}_2$ state. We obtain following relations:

$$\zeta_6 \approx \sqrt{0.325} = 0.570,$$  \hfill (D-14)

$$\sum_q \xi_{6q} \Phi_{6q} = 0.471 \Phi_{6, (4,2)} + 0.321 \Phi_{6, (2,0)}, \quad (\text{for } 0^+_3),$$  \hfill (D-15)

$$\Psi_6 = 0.493 \Phi_{6, (4,2)} + 0.870 \Phi_{6, (2,0)},$$  \hfill (D-16)

$$\zeta_6 \Psi_6 = 0.281 \Phi_{6, (4,2)} + 0.496 \Phi_{6, (2,0)}.$$  \hfill (D-17)

The approximation of $\sum_q \xi_{6q} \Phi_{6q}$ with $\zeta_6 \Psi_6$ is not also so good and the monopole strength of $\sum_q \xi_{6q} \Phi_{6q}$ is also weaker than that of $\zeta_6 \Psi_6$.

References

7) T. Yamada and Y. Funaki, to be prepared.