with the upper and lower limits of the thermostat, are plotted in the $\sigma^* - \hat{\lambda}$ plane. The actual upper and lower snap-through temperatures θ^u and θ^l are related to $(\tau^u\hat{\lambda})$ and $(\tau^l\hat{\lambda})$ by

$$\theta^u = \frac{4(\tau^u\hat{\lambda})}{F}\left(\frac{h}{l}\right)^2 + \theta_r \tag{15a}$$

$$\theta^l = \frac{4(\tau^l\hat{\lambda})}{F}\left(\frac{h}{l}\right)^2 + \theta_r \tag{15b}$$

where θ_r is the reference temperature. In order to give an indication of the numerical values for θ^u and θ^l consider the case $\hat{\lambda} = 2.9$, $\sigma^* = 0.75$, $k = 11.0$. From the discusser's Fig. 4, we have $(\tau^u\hat{\lambda}) \approx 1.0$, $(\tau^l\hat{\lambda}) \approx 0.5$. Then, if we take $\theta_r = 72$ deg F, $F = 0.0000146$/deg F, $h = 0.008$ in., and $l = 0.5$ in., we obtain $\theta^u = 142$ deg F and $\theta^l = 107$ deg F. Thus snap-through of prestressed tied arches can occur at temperatures which are of interest in the design of thermostats.

References

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Authors' Closure

The authors would like to thank Professor Clifton for his discussion and his interesting results on prestressed tied arches. A similar problem of tied arches has been discussed by A. M. Wahl[3] in which he also reports reasonable temperature gradients for critical snap-through temperature. However, our purpose was only to reexamine the thermal buckling of hinged arches as has been considered previously by Timoshenko.

Since the notation of the discusser is somewhat different from ours and, furthermore, his analysis concerns circular arches alone, it is of interest to indicate how our results can be modified to include the effect of prestraining.

If we consider H and M as the total horizontal thrust and bending moment of the prestrained arch subject to a temperature rise Θ, then our equations (1)–(9) and (11)–(12) still hold for Professor Clifton's problem. However, our equations (10) and (22) should be modified as

$$w(\pm a) = u(-a) = 0$$
$$u(a) = -\Delta$$

and

$$H = \frac{\gamma_3}{2a}\Delta - \frac{\gamma_3}{4a}\int_{-a}^{a}[(w')^2 - (w_0')^2]dx + \gamma_4\Theta - \frac{\gamma_5}{2a}\int_{-a}^{a}(w'' - w_0'')dx$$

where Δ is the total shortening of the distance between the ends of the arch. Let Δ_p be the prestrained shortening of the distance between the ends of the arch; then

$$\Delta = \Delta_p - \frac{2a}{A_tE_t}(H - H_1) - 2a\alpha_t\Theta$$

where A_t, E_t, and α_t are the cross-sectional area, Young's modulus, and coefficient of linear thermal expansion of the tie, respectively, and H_1 is the prestrained horizontal thrust which satisfies our modified equations (21) and (22), i.e.,

$$\left(\gamma_2 - \frac{\gamma_5^2}{\gamma_3}\right)(w_1'' - w_0'') + H_1w_1 = \frac{\gamma_5}{\gamma_3}H_1$$

and

$$H_1 = -\frac{\gamma_3}{4a}\int_{-a}^{a}[(w_1')^2 - (w_0')^2]dx - \frac{\gamma_5}{2a}\int_{-a}^{a}(w_1'' - w_0'')dx + \frac{\gamma_3}{2a}\Delta_p$$

where w_1 is the deformed shape of the arch due to prestrain.

[3] Wahl, A. M. "Analysis of the Valverde Thermostat," JOURNAL OF APPLIED MECHANICS, Vol. 11, TRANS. ASME, Vol. 66, Sept. 1944, pp. A-183–A-189.

Shock Waves in Mathematical Models of the Aorta[1]

B. R. SEYMOUR.[2] Any system of nonlinear hyperbolic equations which governs nondispersive waves (such as the author's equations (1)–(3)) have exact solutions which describe disturbances propagating into a uniform region. These *simple wave* solutions have been known for over a century [1, 2].[3] Also conditions at an acceleration front in a dissipative medium whose response is still governed by nonlinear hyperbolic equations are well documented in the literature [3, 4]. Such dissipative mechanisms are produced for example when the medium is rate-dependent or inhomogeneous. Although shock formation is possible in both dissipative and nondissipative systems there is a striking difference in the conditions under which a shock will form. Further, although there are two types of shock, *compressive* and *expansive*, for a particular system *only one type may form.* The author appears to be unaware of these well-known results. He computes the shock formation distance of a *compressive* shock for inviscid flow in a uniform tube by an unnecessary expansion in powers of Δt about the front of the pulse. In his experiment he is then unable to detect an *expansive* shock and concludes that "an improved experimental setup will therefore be needed." In fact it may be *impossible* to produce an expansive shock for the particular tube material he is using.

The author further states that if the vessel were tapered, when the system is dissipative, his results would not be substantially altered except that they could no longer be presented in "simple closed form but would have to be obtained numerically for each case." I will here briefly present the nondissipative simple wave solution and a closed-form solution for the propagation of an acceleration front into a tapered vessel. The essential differences between shock formation in dissipative and nondissipative systems and conditions for the formation of expansive and compressive shocks are discussed.

The author's system of equations for inviscid, hydraulic flow down a uniform flexible tube, (1)–(3), possesses an exact solution, the simple wave, which satisfies the boundary condition that at $x = 0$

$$p(0, t) = f(t) \tag{1}$$

and which describes a wave traveling into the uniform region $x > 0$. If the "wavelet" α = const is tagged by the time $t = \alpha$ when it passed $x = 0$, and if $T(x, \alpha)$ denotes the arrival time of the wavelet α at the position x, so that $T(0, \alpha) = \alpha$, then the simple wave solution can be written in the implicit form

[1] By George Rudinger, published in the March, 1970, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 37, TRANS. ASME, Vol. 92, Series E, pp. 34–37.

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[3] Numbers in brackets designate References at end of Discussion.

$$P = f(\alpha) \quad \text{and} \quad V = \frac{1}{\rho}\int^{f(\alpha)} \frac{dp}{C(p)}, \tag{2}$$

where

$$T(x, \alpha) = \alpha + x[V + C(P)]^{-1} \tag{3}$$

and

$$C(P) = \sqrt{\frac{A(P)}{\rho A'(P)}}. \tag{4}$$

In (2)–(4) the notation $g(x, t) = G(x, \alpha)$ is used. The waves described by (2)–(4), which have no restrictions on their amplitude or the shape of the applied signal, are *amplitude dispersed*. The form of this dispersion is particularly simple: constant values of f propagate with an invariant speed, $(V + C(P))$ which is determined by f. The condition that a shock form is that wavelets coalesce, that is, that $\partial T/\partial\alpha = 0$. Conditions (2)–(4) therefore imply that a shock will first form at $X = X_S$ where

$$X_S = \min_{\alpha} \left\{\frac{\rho C(V + C)^2}{f'(\alpha)(1 + \rho CC_{,p})}\right\} \tag{5}$$

The shock need not occur at the front ($\alpha = 0$), in fact *a shock need not occur at all.* A compressive (expansive) front, when $f'(\alpha) > 0(<0)$, will form a shock only if

$$1 + \rho C(P)C'(P) > 0(<0). \tag{6}$$

This condition of course depends on the particular equation of state of the wall material. A particular case when no shock may form is when condition (6) becomes an equality. Then (4) implies that the wall equation is of the form

$$p - p_0 = K(A^{-2} - A_0^{-2}). \tag{7}$$

For this particular wall material the nonlinearity introduced through the wall equation (7) balances the nonlinear convective terms in the equations of motion. For this extremely artificial case the wave travels undistorted (this is equivalent to choosing the ratio of specific heats, γ, equal to -1 for the flow of a polytropic gas—the basic equations, in Lagrangian form, are then linear).

If it is assumed that the front of the pulse travels into a region where there is no flow, and that the shock forms at the front, then *if a shock forms* it will form at

$$X_{S0} = \frac{\rho C_0}{f'(0)(1 + \rho C_0 C_{0,p})} \tag{8}$$

where

$$C_0 = C(p_0) \quad \text{(a constant)}$$

This is the result obtained by the author. Note that condition (6) must still hold for a shock to form at all.

For inviscid flow in a uniform tube the only condition on the applied pressure pulse for a shock to form is that $f'(\alpha)$ be positive (negative) for a compression (expansion) shock. It is physically unrealistic that for a particular tube material it is possible to apply a pulse of arbitrarily small amplitude and rate and still produce a shock if the pulse propagates far enough. However, if an attenuating mechanism is present, such as friction, wall properties varying with distance, or rate-dependence of the wall material, then the initial pressure rate must be greater than a *critical rate,* defined by the tube material, for a shock to form. The effect of all the foregoing dissipative mechanisms has been presented for an acceleration front and an arbitrary shaped, time periodic signal function [5, 6]. Simply to illustrate this result, I choose the mechanism mentioned by the author as having no substantial effect, that is a variable tube area, when

$$A = A(p, x). \tag{9}$$

Consider the variation in strength of an acceleration front propagating into the region $x > 0$ of a tube through which there is a steady flow given by

$$p_s + \tfrac{1}{2}v_s = \text{const} \tag{10}$$

and

$$A(p_s, x)v_s = \text{const.} \tag{11}$$

If $t = \psi(x)$ is the arrival time of the front $\alpha = 0$ at the point x then the variation in strength of the front is given in closed form by

$$\frac{\partial p}{\partial t}(x, \psi(x)) = f'(0)\left(\frac{w_0}{w_s}\right)\left(\frac{c_s^3 A_0}{c_0^3 A_s}\right)^{1/2}[1 - f'(0)M(x)]^{-1}, \tag{12}$$

where

$$A_s(x) = A(p_s, x), \qquad c_s(x) = c(p_s, x), \qquad w_s(x) = v_s + c_s$$

and

$$M(x) = \left(\frac{A_0}{c_0^3}\right)^{1/2} w_0 \int_0^x \left(\frac{c_s}{A_s}\right)^{1/2} (\rho w_s^3)^{-1}(1 + \rho c_s c_{s,p})ds. \tag{13}$$

(A_0, c_0, w_0 are these functions evaluated at $x = 0$?)

Conditions (12) and (13) show that a compressive (expansive) shock will form in a tapered tube *only if*

$$\underset{(<)}{f'(0) >} \min_x [M(x)]^{-1} = \text{critical rate, } C_r. \tag{14}$$

The shock will form at $x_s = \min_x (Y)$ where Y satisfies

$$f'(0)M(Y) = 1. \tag{15}$$

The result for a uniform tube is recovered by writing

$$M(x) = M'(0)x. \tag{16}$$

Then

$$C_r = \min_x \left(\frac{1}{M'(0)x}\right), \tag{17}$$

$$\to 0 \quad \text{as} \quad x \to \infty.$$

Thus a shock will form in the dissipative system only if the applied pressure rate is greater (or less) than a critical rate, C_r, defined by the tube. If the applied rate is less (greater) than C_r then the attenuating mechanism will dominate the amplitude dispersion due to nonlinearities and *no shock will form, no matter how far the front propagates.* We illustrate this further by taking a specific simple example, when the wall equation is linear and of the form

$$A = a(x)p/p_0. \tag{18}$$

Then $c_{,p} = 0$ and the only form of nonlinearity arises from the equations of motion. For a front moving into fluid at rest at pressure p_0,

$$M(x) = \frac{A_0^{1/2}}{c_0}\left(\frac{\rho}{p_0^5}\right)^{1/4}\int_0^x a(s)^{-1/2}ds. \tag{19}$$

If the area variation is of the form

$$a(x) = a(0)e^{2kx}, \tag{20}$$

then

$$M(x) = C_r^{-1}(1 - e^{-kx}), \tag{21}$$

where

$$C_r = \left(\frac{A_0}{a(0)}\right)^{1/2}\left(\frac{\rho}{p_0^5}\right)^{1/4}(c_0 k)^{-1} \tag{22}$$

Then for $k > 0$

$$\min_x [M(x)]^{-1} = C_r \text{ (a const)}. \quad (23)$$

Thus, for a vessel with varying area, a compressive (expansive) shock will form only if

$$\frac{\partial p}{\partial t}(0, 0) = f'(0) \underset{(<)}{>} C_r, \quad (24)$$

while for a uniform vessel the equivalent condition is that

$$f'(0) \underset{(<)}{>} 0. \quad (25)$$

There is obviously a striking difference between conditions for the formation of shocks in *dissipative* and *nondissipative* systems, and for the formation of *compressive* and *expansive* shocks.

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4 Varley, E., and Cumberbatch, E., *Journal of the Institute for Mathematical Applications*, Vol. 1, 1965, pp. 101–112.

5 Seymour, B. R., and Varley, E., "Hemorheology," *Proceedings of the First International Conference*, 1966; Oxford-New York, Pergamon Press, 1967.

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Author's Closure

The properties of *simple waves* are indeed well known, as emphasized by Dr. Seymour. However, there are still relatively few investigations of fluid-mechanical aspects of blood flow which are based on nonlinear partial differential equations. It seemed therefore preferable to derive some of these properties in footnote 4 of the paper instead of referring the reader to a purely mathematical discussion or to a derivation for a different physical system. The purpose of the paper was to show that shock discontinuities *may* arise during the numerical solution of specific problems and to illustrate this fact by a few simple examples. Since the shock-formation distance may be well within the range of interest, the important conclusion was that one should be aware of this possibility when preparing a program for machine computation.

Equation (7) of the discussion represents an "equation of state" for the wall for which the nonlinearity introduced through the wall equation balances the nonlinear convective terms in the equations of motion. This case is not as artificial as may appear at first sight. In fact, Olsen and Shapiro (reference [10] of the paper) pointed out that this equation of state leads to a good description of certain wave processes in elastic ducts and that the absence of a steepening tendency of the waves accounts for the satisfactory results that are often obtained on the basis of linearized equations. This case could have been included in the paper as an example of infinite shock-formation distance.

Particularly valuable is Dr. Seymour's comment that a minimum rate of pressure rise may be necessary for a shock to form if the flow is dissipative or if the wall properties vary with distance or are rate-dependent. In specific cases, it may be impractical to establish such a criterion beforehand. It would then still be necessary to consider the possibility of shock formation during numerical computations.

The comments on the exploratory experiment briefly discussed in the paper seem to be based on a misunderstanding. That it was *not* intended to produce an expansion shock should be clear from the description of the experimental setup which indicates that the flow was stopped *downstream* of the elastic tube. Consequently, the fluid was brought to rest by an upstream-traveling compression wave. The reason for the experiment being inconclusive was that the falling steel ball settled on the constriction too slowly and therefore did not stop the flow fast enough to produce a shock wave.

Thermal Stresses Near a Prolate Spheroidal Inclusion[1]

M. K. KASSIR.[2] The authors have presented a useful study concerning a technically important problem. There is much to be desired in the closed-form solution constructed in this work compared to the infinite series one discussed previously by Florence and Goodier (authors' reference [6]). The determination of the thermal stress-strain field around various degenerate cases of the prolate spheroidal inclusion (e.g., penny-shaped or needle-shaped cracks) merely amounts to elementary limiting procedure. In the infinite series solution, one has to face the sometimes troublesome question of convergence and this becomes more evident when the degenerate case is associated with singular stress. Indeed, as Florence and Goodier discovered, their solution for the title problem fails to converge when the spheroidal cavity degenerates into a penny-shaped surface of discontinuity or crack.

With reference to the particular case of a "needle-shaped" cavity and the bounded stresses at its ends, it may be useful to point out that such results immediately follow from the available solutions concerning an insulated elliptical crack disturbing an otherwise uniform flow of heat normal to the crack plane [1, 2].[3] One has merely to set $b \to 0$ ($2b$ is the minor axis of the ellipse) and apply the usual limiting rules to equations (60) of [1] or (54) of [2] to arrive at the results reached by the authors; namely, the shear stresses become bounded at the tips of the line crack. Of course, the shear stresses are singular at the rim of the elliptical surface of discontinuity. Similar results also follow, as noted by Williams [3], by shrinking the ellipsoidal cavity to a needle-shaped one whose geometrical axis is normal to the uniform tension at infinity.

References

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3 Williams, M. L., "Stress Singularities, Adhesion, and Fracture," *Proceedings of the 5th U. S. National Congress of Applied Mechanics*, 1966, pp. 451–464.

Authors' Closure

The authors would like to thank Professor Kassir for his kind comments on our paper. Are we to understand from his remark

[1] By M. A. Hussain and S. L. Pu, published in the June, 1970, issue of the JOURNAL OF APPLIED MECHANICS, Vol. 37, TRANS. ASME, Vol. 92, Series E, pp. 403–408.

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[3] Numbers in brackets designate References at end of Discussion.