Electric Multipole Transitions between High-Spin States with Different Shapes

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Electric multipole transitions between multi-quasiparticle states with different shapes are formulated within Nilsson-BCS model. As a specific case, we investigate the transition from the oblate \( I = 49/2^+ \) to the spherical \( I = 45/2^+ \) observed in \(^{147}\)Gd. The transition rate is shown to be very sensitive to the difference in deformations between the initial and final states. To explain the observed transition rate, it is necessary to mix a well-deformed oblate-shape state into the spherical \( I = 45/2^+ \) state.

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§1. Introduction

Electromagnetic transitions between quasiparticle states are very useful tools for studying nuclear structure in a wide range of nuclei including deformed nuclei. It is particularly interesting to study the electromagnetic transition between the states with different shapes. Recently, many properties have been studied in high-spin isomers (HSIs) in rare-earth nuclei.\(^1\) The level structures have been analyzed in various models, among which, the deformed independent particle model (DIPM)\(^2\) shows reasonable success in describing the observed excitation energies of these isomers, assigning deformation and configuration to each level. Most analyses so far have been based on energy levels and electromagnetic moments. The electromagnetic transitions between the states, however, have not been quantitatively studied using microscopic models. One of the main difficulties is that the study involves the evaluation of matrix elements between the states of different deformations.

Odahara et al.\(^1\) pointed out that the mixture of states of different shapes is important in explaining the observed transition rates in some transitions in HSIs. In particular, in the \( E2 \) transition from \( I = 49/2^+ \) to \( 45/2^+ \) in \(^{147}\)Gd, they concluded that spherical-oblate shape mixture can account for the observed rate of transition.

In this study, we obtain the formulas for the calculation of matrix elements of electric multipole operators between states with different deformations. The formulation is based on the expression of the BCS vacua in the Thouless form (see, e.g., Ref. 3)). We extend the formula to the matrix elements of multipole matrix elements between many-quasiparticle states with deformed and spherical bases. We
then apply the formulas to calculate electric quadrupole transitions from HSIs to neighboring states. The results are compared with experimental data in $^{147}\text{Gd}$. Although some results were already shown in Ref. 1), here we perform a detailed study including the effect of change in deformation.

This paper is organized as follows. Section 2 is devoted to the formulation of electromagnetic transitions in deformed nuclei in different shapes. We apply these formulas to the calculation of electric quadrupole strength from $^{49}\text{Gd}^+\text{isomer}$ in $^{147}\text{Gd}$ in §3. Discussion on spherical and oblate configuration mixing is presented in §4. Summary and conclusions are given in §5. Detailed mathematical formulas of Nilsson+BCS model are derived in the Appendix.

§2. Formulas for electromagnetic transitions

2.1. Matrix elements in the intrinsic frame

We calculate the matrix element of the electric transition with multipolarity $\lambda$:

$$\langle \epsilon'K' | M(E\lambda, \mu) | \epsilon K \rangle$$

in the intrinsic frame, where $\epsilon$ and $\epsilon'$ are nuclear potential deformation parameters, while $K$ and $K'$ are angular-momentum projections to the third axis of the intrinsic frame. The electric multipole ($E\lambda$) transition operator is written as

$$M(E\lambda, \mu) = e_n^{\text{eff}} Q_{n,\mu} + e_p^{\text{eff}} Q_{p,\mu},$$

where

$$Q_{\mu} = \sum_{n'l'j'm'nljm} \langle n'l'j'm'|r^{\lambda}Y_{\lambda\mu}|nljm\rangle \times c_{n'l'j'm'}^{\dagger}c_{nljm},$$

where $c_{n'ljm}^{\dagger}$ and $c_{nljm}$ are particle creation and annihilation operators of the single-particle state of node $n$, orbital angular momentum $l$, spin $j$, and projection $m$.

The states $|K\rangle$ and $|K'\rangle$ are written as the direct products of neutron and proton states:

$$|K\rangle = |\nu\rangle|\pi\rangle$$

$$|K'\rangle = |\nu'\rangle|\pi'\rangle.$$  

The neutron and proton states are assumed to have definite quasiparticle numbers:

$$|\rho\rangle = a_{\rho_1 \lambda_1}^{\dagger} a_{\rho_2 \lambda_2}^{\dagger} \cdots |f(\epsilon)\rangle$$

for $\rho = \nu, \pi$, where $|f(\epsilon)\rangle$ is the quasiparticle vacuum of the Nilsson Hamiltonian$^4$ with the potential deformation parameter $\epsilon$. The label $\nu_i$ ($\pi_i$) stands for a neutron (proton) single-quasiparticle state. The creation operator $a_{\lambda\mu}^{\dagger}(\epsilon)$ for the quasiparticle
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is connected to the creation $c_{\lambda\mu}^\dagger(\epsilon)$ and annihilation $c_{\lambda\mu}(\epsilon)$ operators of the real particle in the Nilsson basis labelled by asymptotic quantum numbers $[\lambda\mu] \equiv [Nn_3A\mu]$ as

$$a_{\lambda\mu}^\dagger(\epsilon) = u_{\lambda\mu}(\epsilon)c_{\lambda\mu}^\dagger(\epsilon) - v_{\lambda\mu}(\epsilon)c_{\lambda\mu}(\epsilon),$$  \hspace{1cm} (2.7)

where $u_{\lambda\mu}(\epsilon)$ and $v_{\lambda\mu}(\epsilon) = \sqrt{1 - u_{\lambda\mu}(\epsilon)^2}$ are BCS unoccupation and occupation amplitudes, respectively. The operators of the real particle in the Nilsson state can be expanded by spherical basis as

$$c_{\lambda\mu}^\dagger(\epsilon) = \sum_a d_{\lambda\mu a}(\epsilon)c_{a\mu}^\dagger,$$  \hspace{1cm} (2.8)

$$c_{\lambda\mu}(\epsilon) = \sum_a (-1)^{j-\mu}d_{\lambda\mu a}(\epsilon)c_{a,-\mu}.$$  \hspace{1cm} (2.9)

The amplitudes $d_{\lambda\mu a}(\epsilon)$ are obtained by the diagonalization of the Nilsson Hamiltonian.

The total matrix element of (2.2) becomes

$$\langle \nu' \pi' K' | (e_{\text{eff}}^n Q_n + e_{\text{eff}}^p Q_p) | \nu \pi K \rangle = e_{\text{eff}}^n \langle \nu' | Q_n | \nu \rangle \langle \pi' | \pi \rangle + e_{\text{eff}}^p \langle \pi' | Q_p | \pi \rangle \langle \nu' | \nu \rangle.$$  \hspace{1cm} (2.10)

The matrix elements between multi-quasiparticle states with different deformations are given in the Appendix.

2.2. $B(E\lambda)$ in the laboratory frame

The reduced matrix element between the states with the angular momenta $I$ and $I'$ in the laboratory frame is calculated as

$$\langle K' || M(E\lambda) || K \rangle = (2I + 1)^{1/2} \left[ (IK\lambda K' - K | I'K') \langle K' | M(E\lambda, K' - K) | K \rangle 
+ (-1)^{I+K} (I - K\lambda K + K' | I'K') \langle K' | M(E\lambda, K' + K) | K \rangle \right],$$  \hspace{1cm} (2.11)

where $|K\rangle$ is the time-reversed state defined as

$$|K\rangle = \exp(i\pi J_2) |K\rangle = \sum_J (-1)^{J+K} \Phi_{J,-K}.$$  \hspace{1cm} (2.12)

When $K + K' > \lambda$ as is the case in the present study, $|K\rangle$ has no contribution to (2.11). Thus, we have

$$B(E\lambda; KI \rightarrow K'I') = \frac{1}{2I + 1} \left| \langle K' || M(E\lambda) || K \rangle \right|^2$$
$$= (IK\lambda K' - K | I'K')^2 \langle K' | M(E\lambda, K' - K) | K \rangle^2.$$  \hspace{1cm} (2.13)

The above formalism can also be used for magnetic multipole transitions.6)
§3. Comparison with experiments

Let us now discuss the experimental results of $N = 83$ isotopes. Figure 1 shows a HSI of $I = \frac{49}{2}^+$ and two states of $I = \frac{45}{2}^+$ in $^{147}\text{Gd}$ discussed in Ref. 1. The experimental value is $B(E2; \frac{49}{2}^+ \rightarrow \frac{45}{2}^+) = 0.0188 \pm 0.0008 \approx 1.9 \times 10^{-2}$ in Weisskopf unit.\textsuperscript{7}

In the DIPM, the initial state with $I = \frac{49}{2}^+$ is interpreted as a stretched five-quasiparticle state with a large oblate deformation $\beta = -0.20$:

$$I = \frac{49}{2}^+ \rightarrow \left| (\beta = -0.20) \nu(f_{7/2} h_{9/2} i_{13/2}) \pi(h_{11/2}^2); K = \frac{49}{2} I = \frac{49}{2}^+ \right>,$$

where the label $l_j$ denotes the single-particle state at the spherical limit. The final state ($I' = \frac{45}{2}^+$), on the other hand, is assigned as a five-quasiparticle state with a spherical core:

$$K' = \frac{45}{2} I' = \frac{45}{2}^+.$$ (3.1)

The matrix element of $\mathcal{M}(E2)$ between the two states, however, vanishes because there is no transition matrix element connected by a one-body operator. To obtain finite transition matrix, an oblate state with $I' = \frac{45}{2}^+$ should be mixed with a coefficient $\alpha$:\textsuperscript{1)

$$I' = \frac{45}{2}^+ \rightarrow \sqrt{1 - \alpha^2} \left| \text{(spherical)} \nu(i_{13/2}) \pi(g_{7/2} d_{5/2} h_{11/2}^2); K' = \frac{45}{2} I' = \frac{45}{2}^+ \right>$$

$$+ \alpha \left| (\beta') \nu(f_{7/2} i_{13/2}) \pi(h_{11/2}^2); K' = \frac{45}{2} I' = \frac{45}{2}^+ \right>.$$ (3.3)

To evaluate the transition matrix element quantitatively, we perform the Nilsson+BCS calculation in the model space with the principal oscillator quantum numbers $N = 5–6$ ($N = 4–5$) for neutron (proton). The potential deformation parameter is defined as\textsuperscript{4)

$$h\omega_1 = h\omega_2 = h\omega_0 \left(1 + \frac{1}{3}\epsilon\right),$$
$$h\omega_3 = h\omega_0 \left(1 - \frac{2}{3}\epsilon\right).$$ (3.4-5)

The parameters for the Nilsson model are taken from Ref. 8): $(\kappa, \mu) = (0.062, 0.43)$ and $(0.062, 0.34)$ for $N = 5$ and 6 for neutron while $(\kappa, \mu) = (0.065, 0.57)$ and $(0.060, 0.65)$ for $N = 4$ and 5 for proton. We do not consider the coupling to $N \pm 2$ shells in this study since, in the stretched coordinate of the Nilsson model, the coupling to $N \pm 2$ shells by $l \cdot s$ and $l \cdot l$ interactions is very small.\textsuperscript{5) The oscillator

$$\frac{45/2^+}{[(\text{oblate})]} \quad \frac{49/2^+}{[(\text{oblate})]}$$

$$\frac{45/2^+}{[(\text{spherical})]} + \alpha[(\text{oblate})]?$$

Fig. 1. HSI with $I = \frac{49}{2}^+$ and possible decaying states in $^{147}\text{Gd}$.
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frequencies in the potential are determined to be \( \hbar \omega_0 = \hbar \omega_0^0 \left((1+\epsilon/3)^2(1-2\epsilon/3)\right)^{-1/3} \) with \( \hbar \omega_0^0 |_{p/n} = 41/A^{1/3}(1 \pm (N-Z)/(3A)) \) MeV. The BCS pairing strengths are determined using the formula from Ref. 9: 

\[
G_{p/n} = (g_0 \pm g_1 (N-Z)/A)A^{-1} \text{ MeV}
\]

with \( g_0 = 19.2, g_1 = 7.4 \). The deformation parameter \( \beta \) is related to the deformation of the harmonic oscillator potential by \( \epsilon = 0.95beta \) in the leading order. We will adopt this formula throughout this paper ignoring small deviations in higher order. Thus, the deformation parameter \( \beta = -0.20 \) corresponds to \( \epsilon = -0.19 \). The matrix elements of \( \langle n' j' m'|r^2 Y_{2\mu}|n lj m \rangle \) are calculated using the harmonic oscillator wave functions with \( \hbar/(M\omega_0) = 0.010 \times A^{1/3} \) barn. We adopt the effective charges \( e_{n}^{\text{eff}} = \epsilon \) and \( e_{p}^{\text{eff}} = 2\epsilon \). Note that the second term of the right-hand side of Eq. (3.3) can be connected to the initial \( I = \frac{49}{2} \) state by the quadrupole operator.

Figure 2 shows the calculated \( B(E2) \) values from \( |(\beta = -0.20) \nu(f_{7/2} h_{9/2} i_{13/2}) \pi(h_{11/2}^2); K = \frac{49}{2} I = \frac{49}{2}^+ \rangle \) to \( |(\beta') \nu(f_{7/2}^2 i_{13/2}) \pi(h_{11/2}^2); K' = \frac{45}{2} I' = \frac{45}{2}^+ \rangle \) as a function of the deformation \( \beta' \) of the final state. The value \( \beta' = -0.20 \) gives the same deformation as the initial state. The \( B(E2) \) value is appreciably large for \( \beta' = -0.2 \), as expected from the same deformation and shape as those of the initial state. As the oblate deformation decreases (\( \beta' \to 0 \)), the \( B(E2) \) value decreases quickly. Around \( \beta' \approx -0.12 \), the calculated \( B(E2) \) value is close to the experimental data. This case, however, does not correspond to the real situation because this means \( \alpha = 1 \) in Eq. (3.3). This is inconsistent with the result of DIPM. When the deformation is prolate \( \beta' \gtrsim 0 \), \( B(E2) \) becomes negligibly small even compared with the single-particle estimation (W. u.).

Fig. 2. \( B(E2) \) value from the initial state (3.1) to the second configuration of the right-hand side of Eq. (3.3), plotted as a function of \( \beta' \). To help the comparison, we indicate the experimental value (0.019) with a dashed line.
For deformation $\beta' = -0.20$ for the coupled configuration in Eq. (3.3), the $B(E2)$ value becomes 0.158 W. u. In this case, the mixing probability in Eq. (3.3) becomes

$$|\alpha|^2 \approx 0.019/0.158 \approx 0.12,$$

i.e., $|\alpha| \approx 0.35$. More precise analysis would be possible if the experimental value: $B(E2; \frac{45}{2}^+ \rightarrow \frac{49}{2}^+)$ between the higher oblate $\frac{45}{2}^+$ level to the isomer $\frac{49}{2}^+$ state were known. However, it would be difficult to identify experimentally those higher $\frac{45}{2}^+$ levels unless they are isomers.

The figure shows the behavior of $B(E2)$ with $\beta'$ down to $-0.4$. When $\beta'$ is further decreased (i.e., increased in magnitude) from $-0.2$, $B(E2)$ continues to increase until $\beta' \approx -0.26$ where $B(E2) \approx 0.26$ (W. u.). The large $B(E2)$ arises because the increase in the matrix $\langle \nu(f_{7/2}); \mu' = 5/2 | Q_n | \nu(h_{9/2}); \mu = 9/2 \rangle$ overcomes the decrease in the overlap of the initial and final states. After $\beta' \approx -0.26$, the $B(E2)$ value decreases with decreasing $\beta'$. Note that around $\beta' = -0.4$, the Nilsson states have the spherical ($j$) components that are quite different from those with $\beta' = -0.2$.

§4. Discussion

We have seen that the mixture of a deformed state in Eq. (3.3) can explain the large $B(E2)$ value observed experimentally in $^{147}$Gd. To be consistent with the DIPM analysis, the deformation of the mixed component must be largely oblate ($\beta' \sim -0.2$). Otherwise, the mixed amplitude would be too large.

As the the final state mixture, from the argument of DIPM, we have taken

$$|\langle \beta' \rangle \nu(f_{7/2}^2 i_{13/2}) \pi(h_{7/2}^2); K' = \frac{45}{2} I' = \frac{45}{2} \rangle,$$

which locates 0.54 MeV above the yrast $\frac{45}{2}^+$ state as the most promising candidate. There is another non-stretched state:

$$|\nu(f_{7/2} h_{9/2} i_{13/2}) \pi(h_{11/2}^2); K' = \frac{45}{2} I' = \frac{45}{2} \rangle,$$

which may have a large matrix element, because the structure is the same as the initial state in the spherical limit. This state (4.2) is located about 1.3 MeV above the yrast state. Therefore, the mixing amplitude is expected to be smaller from the denominator of the perturbation theory alone. The mixture of this component will also contribute to large $B(E2)$ in a way similar to the state (4.1). Among other states with $I = \frac{49}{2}^+$ nearby, it is difficult to find one that has a large $E2$ matrix element with the initial $I = \frac{49}{2}^+$ because both proton and neutron configurations are different.

Thus far, we have been considering the mixture of states in the yrast $|\frac{45}{2}^+\rangle$ state and have not discussed the mixture in the initial $|\frac{49}{2}^+\rangle$. About the states with $I = \frac{49}{2}^+$, mixture is more difficult to occur because the level density is sparser. In fact, the lowest non-yrast state that may be connected with the main component of $|\frac{45}{2}^+\rangle$ by $E2$ is about 1.3 MeV above the yrast.
§5. Summary and conclusions

We present the formulas to calculate electric transition strength between many-quasiparticle states with different shapes. In general, this formalism could be useful in analyzing electromagnetic transitions between the states with different deformations. These formulas were applied to calculate $B(E2)$ strength for the high-spin isomeric state $I = \frac{49}{2}^+$ in $^{149}$Gd. In DIPM, the isomer state $I = \frac{49}{2}^+$ and the decay state $\frac{45}{2}^+$ have different deformations $\beta = -0.2$ and $\beta = 0.0$, respectively. We showed that the experimental $B(E2)$ value from $I = \frac{49}{2}^+$ to $I = \frac{45}{2}^+$ in $^{149}$Gd can be explained using a mixture of the oblate states in the spherical yrast $I = \frac{45}{2}^+$ state. The analysis has been carried out quantitatively by calculating the matrix elements between the states with various values of the quadrupole deformation parameter. It turns out that the transition matrix element depends very much on the deformation and almost vanishes between a well-deformed oblate state and a spherical state. To explain the observed large $E2$ transition, the mixed component must have a large oblate deformation as much as the isomer state. The yrare $\frac{45}{2}^+$ with large oblate deformation has not been observed as an independent level, because it is very difficult to identify it experimentally. It would be interesting if such a level is found and provides a more precise test of the model in the future.

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Appendix

This appendix gives the formulas used in evaluating the matrix elements in Eq. (2.10). The quasiparticle vacuum $|f\rangle$ of the initial state (proton or neutron) with deformation $\epsilon$ is denoted as

$$|f(\epsilon)\rangle = N_f |f_0(\epsilon)\rangle,$$

(A-1)

where the normalization factor $N_f$ and the unnormalized state vector $|f_0\rangle$ in the Thouless form are given as

$$N_f = \prod_{\lambda\mu > 0} u_{\lambda\mu}, \quad (A.2)$$

$$|f_0\rangle = \exp\left(\frac{1}{2} \sum_{\alpha\beta} f_{\alpha\beta} c_{\alpha}^\dagger c_{\beta}^\dagger\right) |\text{vac}\rangle,$$

(A-3)

where $f_{\alpha\beta}$ with $\alpha = (nlj, \mu) = (a, \mu)$, $\beta = (b, \nu)$ is given as

$$f_{\alpha\beta} = \delta_{\mu, -\nu} (-1)^{l+\nu} \sum_{\lambda} \frac{u_{\lambda\mu}}{v_{\lambda\nu}} d_{\lambda a}^\mu d_{\lambda b}^\nu,$$

(A.4)
and $|\text{vac}\rangle$ is the closed shell. The coefficient $d^\mu_{\lambda, nlj}$ with negative $\mu < 0$ is defined as $d^\mu_{\lambda, nlj} = (-1)^{j_2 - \frac{1}{2}} (d^\mu_{\lambda, nlj})^*$. The quasiparticle vacuum $|g\rangle$ for the final state with deformation $\epsilon'$ is also expanded similarly with the matrix $g_{\alpha\beta}$.

The matrix element of any operator $\mathcal{O}$ between $|f\rangle$ and $|g\rangle$ is calculated as
\begin{equation}
\langle g|\mathcal{O}|f\rangle = \langle g|f\rangle \frac{\langle g|\mathcal{O}|f_0\rangle}{\langle g|f_0\rangle},
\end{equation}
where
\begin{equation}
\langle g|f\rangle = N_f N_g \langle g_0|f_0\rangle = N_f N_g \left[\det(1 + M)\right]^{1/2}
\end{equation}
with $M = fg^\dagger$. We define the following matrices for later calculations:
\begin{equation}
D = (1 + M)^{-1}, \quad \mathcal{F} = Df, \quad \mathcal{G} = g^\dagger D.
\end{equation}

In the following formulas including Eqs. (A.10), (A.14), (A.20), and (A.23), $\mu, \nu > 0$ is assumed for the third component of angular momentum of all creation and annihilation operators $a^\dagger_\mu$ and $a_\nu$. Otherwise, additional terms arising from commutation relations need to be taken into account.

A.1. Overlaps of quasiparticle states with different deformations

The one-quasiparticle-state overlap is given as
\begin{equation}
\langle g|a^\dagger_\nu a^\dagger_\mu|f\rangle = \frac{1}{u_\nu(\epsilon')u_\mu(\epsilon)} \langle g|c^\dagger_\nu c^\dagger_\mu|f\rangle
= \frac{1}{u_\nu(\epsilon')u_\mu(\epsilon)} \sum_{nlj'\nu'j} d^\mu_{\nu, nlj'}(\epsilon')d^\nu_{\mu, nlj} (\epsilon) \langle g|c_\nu c^\dagger_{\nu'} c^\dagger_\mu|f\rangle.
\end{equation}

Equation (A.8) gives the transformation from the quasiparticle operator $a^\dagger_\rho$ in the deformed basis to the real particle one $c^\dagger_{\lambda\mu}$ in the spherical basis through the real-particle $c^\dagger_{\lambda\mu}$ in the deformed basis. The matrix elements on the right-hand side are evaluated using the matrices defined as (A.7):
\begin{equation}
\langle g|c_\alpha c^\dagger_\alpha|f\rangle = \langle g|f\rangle \frac{\langle g_0|c^\dagger_\alpha c_\alpha|f_0\rangle}{\langle g_0|f_0\rangle} = \langle g|f\rangle D_{\alpha'\alpha},
\end{equation}
where $\langle f|g\rangle$ is given in (A.6) while $D$ is defined in (A.7).

The two-quasiparticle-state overlap is calculated in a similar manner:
\begin{equation}
\langle g|a^\dagger_\nu a^\dagger_\mu a^\dagger_\rho a^\dagger_\sigma|f\rangle = \frac{1}{u^\nu_\rho(\epsilon')u^\mu_\sigma(\epsilon')u^\rho_\mu(\epsilon)u^\sigma_\nu(\epsilon)} \langle g|c^\dagger_\nu c^\dagger_\mu c^\dagger_\rho c^\dagger_\sigma|f\rangle
= \frac{1}{u^\nu_\rho(\epsilon')u^\mu_\sigma(\epsilon')u^\rho_\mu(\epsilon)u^\sigma_\nu(\epsilon)} \sum_{nlj_1j_2j_1'j_2'j_3j_4} d^\mu_{\nu, nlj_1j_2}(\epsilon')d^\sigma_{\mu, nlj_3j_4}(\epsilon')d^\nu_{\rho, nlj_1j_3}(\epsilon)d^\rho_{\sigma, nlj_2j_4}(\epsilon)
\times \langle g|c_\nu c_\sigma c^\dagger_\rho c^\dagger_\mu c^\dagger_\sigma c^\dagger_\rho c^\dagger_\mu|f\rangle.
\end{equation}
where

\[ \langle g | c_{\beta'}^\dagger c_{\alpha'}^\dagger c_\alpha^\dagger | f \rangle = \langle g | f \rangle \frac{\langle g_0 | c_{\beta'} c_{\alpha'} c_\alpha^\dagger | f_0 \rangle}{\langle g_0 | f_0 \rangle}, \]  
(A-11)

and the second factor on the right-hand side is given as

\[ \frac{\langle g_0 | c_{\beta'} c_{\alpha'} c_\alpha^\dagger | f_0 \rangle}{\langle g_0 | f_0 \rangle} = P^{\alpha \beta} \left[ D_{\alpha' \alpha} D_{\beta' \beta} - \frac{1}{2} G_{\alpha \beta} F_{\alpha' \beta'} \right], \]  
(A-12)

where the unnormalized antisymmetrizer is defined as

\[ P^{\alpha \beta} [A(\alpha \beta)] = A(\alpha \beta) - A(\beta \alpha). \]  
(A-13)

Three-quasiparticle overlap is also given as

\[ \langle g | a'_p a'_p a_p^\dagger a_{p1} a_{p2} a_{p3} | f \rangle = \frac{1}{u_{p3}(\epsilon')u_{p3}(\epsilon')u_{p1}(\epsilon)u_{p2}(\epsilon)u_{p3}(\epsilon)} \langle g | c_{\alpha'}^\dagger c_{\alpha'}^\dagger c_{\alpha'}^\dagger | f \rangle \]  
\[ = \frac{1}{u_{p3}(\epsilon')u_{p3}(\epsilon')u_{p1}(\epsilon)u_{p2}(\epsilon)u_{p3}(\epsilon)} \sum_{n_1 n_2 n_3 n_4 n_5 n_6} \]  
\[ \times \langle g | c_{n_{3}} c_{n_{3}} c_{n_{3}} c_{n_{3}} c_{n_{3}} c_{n_{3}} | f \rangle \]  
\[ \times \langle g | c_{\gamma'} c_{\beta'} c_{\alpha'} c_{\alpha'} c_{\alpha'} | f \rangle = \frac{1}{4} P^{\alpha \beta} P^{\alpha' \beta'} \left[ 2 D_{\alpha' \alpha} D_{\beta' \beta} D_{\gamma' \gamma} - 4 D_{\alpha' \alpha} D_{\gamma' \beta} D_{\beta' \gamma} - 4 D_{\alpha' \alpha} F_{\beta' \gamma} G_{\beta \gamma} - 2 D_{\gamma' \beta} F_{\alpha' \beta} G_{\alpha \beta} - 2 D_{\gamma' \beta} F_{\alpha' \beta} G_{\alpha \beta} \right]. \]  
(A-14)

\[ \langle g | c_{\gamma'} c_{\beta'} c_{\alpha'} c_{\alpha'} c_{\alpha'} c_{\alpha'} | f \rangle = \frac{1}{4} P^{\alpha \beta} P^{\alpha' \beta'} \left[ 2 D_{\alpha' \alpha} D_{\beta' \beta} D_{\gamma' \gamma} - 4 D_{\alpha' \alpha} D_{\gamma' \beta} D_{\beta' \gamma} - 4 D_{\alpha' \alpha} F_{\beta' \gamma} G_{\beta \gamma} - 2 D_{\gamma' \beta} F_{\alpha' \beta} G_{\alpha \beta} - 2 D_{\gamma' \beta} F_{\alpha' \beta} G_{\alpha \beta} \right]. \]  
(A-15)

A.2. Matrix elements of the one-body operator between quasiparticle states

The matrix element of one-body transition between one-quasiparticle state is given as

\[ \langle g | a_p^\dagger Q a_{p1} | f \rangle = \frac{1}{u_{p'}(\epsilon')u_{p}(\epsilon)} \sum_{n l j m} d_{n l j}^\mu (\epsilon')d_{n l j}^\mu (\epsilon) \langle g | c_{n l j m} c_{\alpha}^\dagger Q c_{\alpha}^\dagger | f \rangle, \]  
(A-16)

where

\[ \langle g | c_{\alpha} Q c_{\alpha}^\dagger | f \rangle = \langle g | f \rangle \frac{\langle g_0 | c_{\alpha} Q c_{\alpha}^\dagger | f_0 \rangle}{\langle g_0 | f_0 \rangle} \]  
(A-17)
is obtained using
\[
\frac{(g_0|c^{\dagger}_\alpha Qc^{\dagger}_\alpha|f_0)}{(g_0|f_0)} = \text{Tr}[MDQ]D_{\alpha\alpha} + (DQD)_{\alpha\alpha} - (g^\dagger DQDf)_{\alpha\alpha}. \quad (A.19)
\]

The matrix elements of the one-body operator between two-quasiparticle states are calculated as
\[
\langle g|a'_\rho_2 a'_\rho_1 Qa_\rho_1 a_\rho_2|f \rangle = \frac{1}{u_{\rho_2}(\epsilon')u_{\rho_1}(\epsilon')u_{\rho_1}(\epsilon)u_{\rho_2}(\epsilon)} \sum_{n_1l_1j_1n_2l_2j_2n'_1l'_1j'_1n'_2l'_2j'_2} d_{\rho_2}^{l_2} n_2 l_2 j_2 (\epsilon') d_{\rho_1}^{l_1} n_1 l_1 j_1 (\epsilon') d_{\rho_1}^{l_1} n_1 l_1 j_1 (\epsilon) d_{\rho_2}^{l_2} n_2 l_2 j_2 (\epsilon) \\
\times \langle g|c_{n_2 l_2 j_2} c_{n'_1 l'_1 j'_1} c_{n'_1 l'_1 j'_1} c_{n_2 l_2 j_2} |f \rangle, \quad (A.20)
\]
where
\[
\langle g|c_{\beta'} c^{\dagger}_\alpha Qc^{\dagger}_\alpha c^{\dagger}_\beta |f \rangle = (f|g) \frac{(g_0|c^{\dagger}_\beta c^{\dagger}_\alpha Qc^{\dagger}_\alpha c^{\dagger}_\beta |f_0)}{(g_0|f_0)}, \quad (A.21)
\]
where the second factor of the right-hand side is given as
\[
\frac{(g_0|c_{\beta'} c^{\dagger}_\alpha Qc^{\dagger}_\alpha c^{\dagger}_\beta |f_0)}{(g_0|f_0)} = -\frac{1}{4} \rho^{\alpha \beta} \rho^{\alpha' \beta'} \left[ (2D_{\alpha' \alpha} D_{\beta' \beta} - F_{\alpha' \beta} G_{\alpha \beta}) \text{Tr}[DQ] \\
+ 4D_{\alpha' \alpha}(GQF)_{\beta' \beta} - (DQD)_{\beta' \beta} \right] \\
- 2F_{\alpha' \beta}(GQD)_{\beta' \beta} - 2G_{\alpha \beta}(DQF)_{\alpha' \beta}' \right], \quad (A.22)
\]
where \(Q_{\alpha \beta} \equiv \langle \alpha|\tau^\lambda Y_{\lambda \mu} |\beta \rangle \) for electric \(\lambda\)-pole transition.

The matrix elements of the one-body operator between three-quasiparticle states are also calculated as
\[
\langle g|a'_\rho_3 a'_\rho_2 a'_\rho_1 Qa_\rho_1 a_\rho_2 a_\rho_3 |f \rangle
\]
\[
= \frac{1}{u_{\rho_3}(\epsilon')u_{\rho_2}(\epsilon')u_{\rho_1}(\epsilon')u_{\rho_1}(\epsilon)u_{\rho_2}(\epsilon)u_{\rho_3}(\epsilon)} \sum_{n_1l_1j_1n_2l_2j_2n_3l_3j_3} \sum_{n'_1l'_1j'_1n'_2l'_2j'_2n'_3l'_3}
\times \langle g|c_{n_3 l_3 j_3} c_{n'_2 l'_2 j'_2} c_{n'_1 l'_1 j'_1} c_{n'_1 l'_1 j'_1} c_{n'_2 l'_2 j'_2} c_{n_3 l_3 j_3} |f \rangle. \quad (A.23)
\]
The last line of (A.23) becomes
\[
\langle g|c_{n_3 l_3 j_3} c_{n'_1 l'_1 j'_1} c_{n'_2 l'_2 j'_2} c_{n'_1 l'_1 j'_1} c^{\dagger}_{n'_2 l'_2 j'_2} c^{\dagger}_{n_3 l_3 j_3} |f \rangle
\]
\[
= \delta_{n'n} \delta_{l'l} \delta_{j'j} \delta_{\alpha'\alpha} \langle g|c_{n_3 l_3 j_3} c_{n'_1 l'_1 j'_1} c_{n'_2 l'_2 j'_2} c_{n'_1 l'_1 j'_1} c^{\dagger}_{n'_2 l'_2 j'_2} c^{\dagger}_{n_3 l_3 j_3} |f \rangle \\
- \langle g|c_{n_3 l_3 j_3} c_{n'_1 l'_1 j'_1} c_{n'_2 l'_2 j'_2} c_{n'_1 l'_1 j'_1} c_{n_3 l_3 j_3} c^{\dagger}_{n_3 l_3 j_3} c^{\dagger}_{n_2 l_2 j_2} c^{\dagger}_{n_3 l_3 j_3} |f \rangle. \quad (A.24)
\]
The three-particle matrix elements on the right-hand side are shown in (A.15). The four-particle matrix elements are calculated as

\[
\langle g | c_{\delta'} c_{\gamma'} c_{\alpha'} c_{\beta'} c_{\alpha} c_{\beta} c_{\gamma} c_{\delta} | f \rangle = \frac{\langle g | f \rangle (g_0 | c_{\delta'} c_{\gamma'} c_{\alpha'} c_{\beta'} c_{\alpha} c_{\beta} c_{\gamma} c_{\delta} | f_0 \rangle}{(g_0 | f_0 \rangle}, \tag{A.25}
\]

where

\[
\frac{\langle g_0 | c_{\delta'} c_{\gamma'} c_{\alpha'} c_{\beta'} c_{\alpha} c_{\beta} c_{\gamma} c_{\delta} | f_0 \rangle}{(g_0 | f_0 \rangle} = P^{\alpha \beta} P^{\gamma \delta} P^{\alpha' \beta'} P^{\gamma' \delta'} (I_1 + I_2 + I_3), \tag{A.26}
\]

where

\[
I_1 = \frac{1}{4} \left[ D_{\alpha' \alpha} D_{\beta' \beta} D_{\gamma' \gamma} D_{\delta' \delta} + D_{\alpha' \alpha} D_{\beta' \beta} D_{\gamma' \gamma} D_{\delta' \delta} + 4 D_{\alpha' \alpha} D_{\beta' \beta} D_{\gamma' \gamma} D_{\delta' \delta} \right], \tag{A.27a}
\]

\[
I_2 = \frac{1}{16} \left[ \mathcal{F}_{\alpha' \beta} \mathcal{F}_{\gamma' \delta} \mathcal{G}_{\alpha \beta} \mathcal{G}_{\gamma \delta} - 2 \mathcal{F}_{\alpha' \beta} \mathcal{F}_{\gamma' \delta} \mathcal{G}_{\alpha \gamma} \mathcal{G}_{\beta \delta} \right. \\
- 2 \mathcal{F}_{\alpha' \gamma} \mathcal{F}_{\beta' \delta} \mathcal{G}_{\alpha \beta} \mathcal{G}_{\gamma \delta} + 4 \mathcal{F}_{\alpha' \gamma} \mathcal{F}_{\beta' \delta} \mathcal{G}_{\alpha \gamma} \mathcal{G}_{\beta \delta} \right], \tag{A.27b}
\]

\[
I_3 = -\frac{1}{8} \left[ D_{\alpha' \alpha} D_{\beta' \beta} \mathcal{F}_{\gamma' \delta} \mathcal{G}_{\alpha \beta} + D_{\alpha' \alpha} D_{\beta' \beta} \mathcal{F}_{\gamma' \delta} \mathcal{G}_{\alpha \beta} \right. \\
+ D_{\gamma' \gamma} D_{\delta' \delta} \mathcal{G}_{\alpha \beta} + 4 \left( D_{\alpha' \alpha} D_{\gamma' \gamma} D_{\delta' \delta} \mathcal{F}_{\beta' \delta} \mathcal{G}_{\alpha \beta} + D_{\gamma' \gamma} D_{\delta' \delta} \mathcal{F}_{\beta' \delta} \mathcal{G}_{\alpha \beta} \right. \\
+ 4 \left( D_{\alpha' \alpha} D_{\gamma' \gamma} D_{\delta' \delta} \mathcal{F}_{\beta' \delta} \mathcal{G}_{\alpha \beta} + D_{\gamma' \gamma} D_{\delta' \delta} \mathcal{F}_{\beta' \delta} \mathcal{G}_{\alpha \beta} \right) \\
\left. + 8 \left( D_{\alpha' \alpha} D_{\gamma' \gamma} - D_{\alpha' \gamma} D_{\gamma' \alpha} \right) \mathcal{F}_{\beta' \delta} \mathcal{G}_{\beta \delta} \right]. \tag{A.27c}
\]

References