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Discussion

Discussion: "Analysis of a System of Linear Delay Differential Equations" (Asl, F. M., and Ulsoy, A. G., 2003, ASME J. Dyn. Syst., Meas., Control, 125, pp. 215–223)

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*This work presents a commentary of the article published by Asl and Ulsoy (2003, ASME J. Dyn. Syst., Meas., Control, **125**, pp. 215–223). We show by an example that their method leads to inaccurate results and is therefore erroneous.*
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Introduction

Consider a system of first order delay differential equations of the form

$$\dot{\mathbf{y}}(t) + \mathbf{A}\mathbf{y}(t-T) + \mathbf{B}\mathbf{y}(t) = 0 \tag{1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices, and $\mathbf{y}$ is an $n \times 1$ vector.

The exact solution of Eq. (1) cannot, in general, be obtained. Special cases of Eq. (1) were considered by Chen et al. [1] and exact analytical solutions were obtained. In [2], Asl and Ulsoy offer a closed form solution to the general case in matrix form and compute the stability lobes numerically. Both studies in [1,2] are based on a solution of a transcendental equation expressed in terms of the Lambert function, which was first derived by Briggs in [3]. Because the solution involves the Lambert function and it is given as a series, reaching an exact analytical stability bound for Eq. (1) is still a problem.

Method of Asl and Ulsoy

Employing the Laplace transform method, Eq. (1) leads to

$$(s\mathbf{I} + \mathbf{A}e^{-sT} + \mathbf{B})Y(s) = 0 \tag{2}$$

where $Y(s)$ is the Laplace transform of $y(t)$. The authors of [2] call the relation

$$s\mathbf{I} + \mathbf{A}e^{-sT} + \mathbf{B} = 0 \tag{3}$$

as the characteristic equation in matrix form of Eq. (1). Clearly, this equation can be written in the form

$$\mathbf{P}e^{\mathbf{P}} = \mathbf{Q}, \quad \mathbf{P} = (s\mathbf{I} + \mathbf{B})T, \quad \mathbf{Q} = -\mathbf{A}Te^{\mathbf{B}T} \tag{4}$$

From (4), the authors of [2] deduce that

$$\mathbf{P} = W(\mathbf{Q}) \tag{5}$$

where $W(\cdot)$ is the well-known Lambert function. For further properties of the Lambert function, see [3,4] and the references therein.

If the equality in (4) were a matrix identity, then passing from (4) to (5) would be trivial. However, the equality in (4) means that

$$\mathcal{M}Z(s) = 0, \quad \mathcal{M} = \mathbf{P}e^{\mathbf{P}} - \mathbf{Q} \tag{6}$$

which is derived from (2), with $Y(s) = Te^{\mathbf{P}}Z(s)$. In this case, one must have the rank of $\mathcal{M}$ less than n in order to have a nontrivial solution, which is equivalent to $\det\mathcal{M} = 0$. In fact, this condition leads to a correct characteristic equation of (1) as

$$\det(s\mathbf{I} + \mathbf{A}e^{-sT} + \mathbf{B}) = 0 \tag{7}$$

Unfortunately, $\mathcal{M}$ is set equal to zero as a condition in [2]. We note that the solution space of (6) becomes all of $\mathbb{R}^n$ in this case, i.e., every $Y(s)$ is a solution. Therefore, the use of (5) may not give the correct answer in general, as illustrated numerically in the next section.

Example and Conclusion

Let us reconsider the case study in [2] with

$$A = \begin{bmatrix} 0 & 0 \\ -\dfrac{K_c}{K_m}\omega_n^2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ \left(1 + \dfrac{K_c}{K_m}\right)\omega_n^2 & 2\zeta\omega_n \end{bmatrix},$$

$$\zeta = 0.05, \quad \omega_n = 150$$

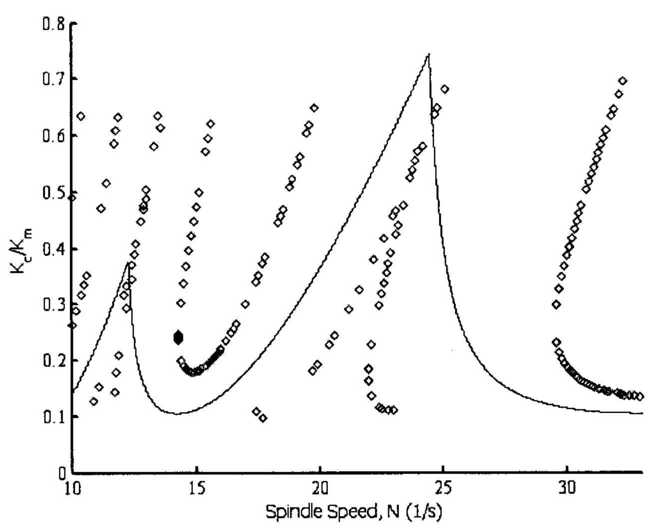

Fig. 1 Stability lobes diagram

As stated in [2], the s values are computed as eigenvalues of $(1/T)W(\mathbf{Q})-\mathbf{B}$, i.e.,

$$\det\left\{s\mathbf{I}-\left[\frac{1}{T}W(-\mathbf{A}Te^{\mathbf{B}T})-\mathbf{B}\right]\right\}=0 \tag{8}$$

In Fig. 1, the data points $N=1/T$, K_c/K_m, represented by a diamond sign, are obtained from (8) numerically for the principal branch of the Lambert function. An important observation is that these data points do not match with Fig. 9 of [2]. Figure 1 also displays, by the solid curve, the actual stability lobe diagram predicted analytically [5].

Using certain data points from Fig. 1, one can easily verify that (8) is satisfied but (7) is not. It is easy to check that $(19.9, 0.193993)$ at $s=180.520834i$ is such a point. On the other hand, it can also be shown that (7) holds but (8) does not for some data points from Fig. 1. For instance, this can be verified by using $(14.245976, 0.105)$ at $s=157.321327i$.

As a result, the method given by Asl and Ulsoy [2] results in s values which do not satisfy (7). Thus, the closed form solution of Eq. (1) given in [2] is not correct.

Acknowledgment

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