

Discussion: “Analysis of a System of Linear Delay Differential Equations” (Asl, F. M., and Ulsoy, A. G., 2003, ASME J. Dyn. Syst., Meas., Control, 125, pp. 215–223)

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This work presents a commentary of the article published by Asl and Ulsoy (2003, ASME J. Dyn. Syst., Meas., Control, 125, pp. 215–223). We show by an example that their method leads to inaccurate results and is therefore erroneous.
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Introduction

Consider a system of first order delay differential equations of the form

$$\dot{\mathbf{y}}(t) + \mathbf{A}\mathbf{y}(t - T) + \mathbf{B}\mathbf{y}(t) = 0 \tag{1}$$

where **A** and **B** are $n \times n$ matrices, and **y** is an $n \times 1$ vector.

The exact solution of Eq. (1) cannot, in general, be obtained. Special cases of Eq. (1) were considered by Chen et al. [1] and exact analytical solutions were obtained. In [2], Asl and Ulsoy offer a closed form solution to the general case in matrix form and compute the stability lobes numerically. Both studies in [1,2] are based on a solution of a transcendental equation expressed in terms of the Lambert function, which was first derived by Briggs in [3]. Because the solution involves the Lambert function and it is given as a series, reaching an exact analytical stability bound for Eq. (1) is still a problem.

Method of Asl and Ulsoy

Employing the Laplace transform method, Eq. (1) leads to

$$(s\mathbf{I} + \mathbf{A}e^{-sT} + \mathbf{B})Y(s) = 0 \tag{2}$$

where $Y(s)$ is the Laplace transform of $y(t)$. The authors of [2] call the relation

$$s\mathbf{I} + \mathbf{A}e^{-sT} + \mathbf{B} = 0 \tag{3}$$

as the characteristic equation in matrix form of Eq. (1). Clearly, this equation can be written in the form

$$\mathbf{P}e^{\mathbf{P}} = \mathbf{Q}, \quad \mathbf{P} = (s\mathbf{I} + \mathbf{B})T, \quad \mathbf{Q} = -\mathbf{A}Te^{\mathbf{B}T} \tag{4}$$

From (4), the authors of [2] deduce that

$$\mathbf{P} = W(\mathbf{Q}) \tag{5}$$

where $W(\cdot)$ is the well-known Lambert function. For further properties of the Lambert function, see [3,4] and the references therein.

If the equality in (4) were a matrix identity, then passing from (4) to (5) would be trivial. However, the equality in (4) means that

$$\mathcal{M}Z(s) = 0, \quad \mathcal{M} = \mathbf{P}e^{\mathbf{P}} - \mathbf{Q} \tag{6}$$

which is derived from (2), with $Y(s) = Te^{\mathbf{P}}Z(s)$. In this case, one must have the rank of \mathcal{M} less than n in order to have a nontrivial solution, which is equivalent to $\det \mathcal{M} = 0$. In fact, this condition leads to a correct characteristic equation of (1) as

$$\det(s\mathbf{I} + \mathbf{A}e^{-sT} + \mathbf{B}) = 0 \tag{7}$$

Unfortunately, \mathcal{M} is set equal to zero as a condition in [2]. We note that the solution space of (6) becomes all of \mathbb{R}^n in this case, i.e., every $Y(s)$ is a solution. Therefore, the use of (5) may not give the correct answer in general, as illustrated numerically in the next section.

Example and Conclusion

Let us reconsider the case study in [2] with

$$A = \begin{bmatrix} 0 & 0 \\ -\frac{K_c}{K_m}\omega_n^2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ \left(1 + \frac{K_c}{K_m}\right)\omega_n^2 & 2\zeta\omega_n \end{bmatrix},$$

$$\zeta = 0.05, \quad \omega_n = 150$$

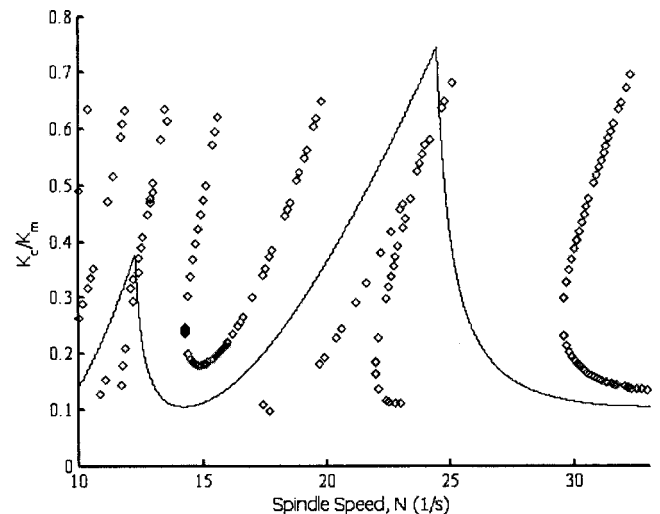


Fig. 1 Stability lobes diagram

As stated in [2], the s values are computed as eigenvalues of $(1/T)W(\mathbf{Q})-\mathbf{B}$, i.e.,

$$\det \left\{ s\mathbf{I} - \left[\frac{1}{T}W(-\mathbf{A}T\mathbf{e}^{\mathbf{B}T}) - \mathbf{B} \right] \right\} = 0 \quad (8)$$

In Fig. 1, the data points $N=1/T$, K_c/K_m , represented by a diamond sign, are obtained from (8) numerically for the principal branch of the Lambert function. An important observation is that these data points do not match with Fig. 9 of [2]. Figure 1 also displays, by the solid curve, the actual stability lobe diagram predicted analytically [5].

Using certain data points from Fig. 1, one can easily verify that (8) is satisfied but (7) is not. It is easy to check that $(19.9, 0.193993)$ at $s=180.520834i$ is such a point. On the other hand, it can also be shown that (7) holds but (8) does not for some data points from Fig. 1. For instance, this can be verified by using $(14.245976, 0.105)$ at $s=157.321327i$.

As a result, the method given by Asl and Ulsoy [2] results in s values which do not satisfy (7). Thus, the closed form solution of Eq. (1) given in [2] is not correct.

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