Schwinger-Dyson and Bethe-Salpeter Approach
to Strong Interaction Dynamics and Chiral Symmetry Breaking

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Maskawa and Nakajima initiated the approach based on the Schwinger-Dyson and Bethe-Salpeter equations to the strong interaction dynamics in QCD-like gauge theories, in particular, as a consistent framework to study the dynamical chiral symmetry breaking. We review here this approach including the later developments such as the improved ladder approximation by Higashijima and Miransky.

Subject Index: 101, 130, 136, 232

§1. Introduction

Maskawa had a strong interest in the spontaneous breaking of chiral symmetry in the strong interaction and in particular in the Nambu-Jona-Lasinio (NJL) paper.\(^1\) He calculated the pion decay constant \(f_\pi\) in the NJL model and persistently examined how \(f_\pi\) could be decreased by adjusting the free parameters in the NJL model: the coupling constant and the UV cutoff. But he says in his Nobel lecture; “I could get no definite answer since the contribution from the momentum region near the UV cutoff was most dominant.”

He thought from this experience that he must consider the problems in renormalizable theories to get definite answers in any case. This led him, on the one hand, to investigating the \(CP\) violation possibility in the framework of renormalizable Weinberg-Salam model together with Makoto Kobayashi in 1972.\(^2\) On the other hand, it made him reconsider the spontaneous chiral symmetry breaking in a renormalizable model of strong interaction, Abelian massive vector gluon model\(^3\),\(^4\). This was done in 1974 together with one of the present authors, H. Nakajima, his graduate student at that time. To the authors’ knowledge, it was the first work in which the dynamical chiral symmetry breaking was examined in a Lorentz invariant renormalizable quantum field theory.

Maskawa and Nakajima (MN) worked out in a specific approximation scheme which is now called ‘ladder approximation’. They first observed that their Schwinger-Dyson (SD) equation determining the quark propagator is actually consistent with the Bethe-Salpeter (BS) equation for the axial vertex in the same ladder approxi-
mation from the viewpoint of chiral symmetry; that is, the Ward-Takahashi (WT) identity for the axial vertex function and the quark propagator is satisfied if they are determined by the SD and BS equations in that approximation. This WT identity, therefore, guaranteed the presence of the massless Nambu-Goldstone pole in the axial-vertex function if the solution of the SD equation for quark propagator develops a non-vanishing mass function dynamically in the chiral symmetry case with vanishing bare quark mass. Since they worked in a renormalizable model, they could take the limit in which the UV cutoff $\Lambda$ goes to infinity. They noticed the importance of distinguishing the infinitesimal bare mass from the exactly zero mass. The infinitesimal bare mass $m_0 \propto \Lambda^{-\varepsilon}$ ($\varepsilon > 0$) itself becomes zero when $\Lambda \to \infty$, but it can generate a finite renormalized effect as an explicitly chiral symmetry breaking mass term. By this distinction they could distinguish the “super-conducting” solution from the “normal-state” solution to the SD equation and found the critical coupling for the spontaneous chiral symmetry breaking.

The primary purpose of this paper is to review the work of Maskawa-Nakajima from the modern viewpoint. In §2, however, we first present a systematic method based on the effective action for deriving the SD and BS equations which are mutually consistent with the flavor gauge symmetry including the chiral symmetry. This is based on the work of Bando et al.\textsuperscript{5}) which generalizes Maskawa-Nakajima’s consistency to arbitrary order beyond the ladder approximation. In §§3 and 4, we explain the set up of Maskawa-Nakajima and summarize their results in the papers I\textsuperscript{3}) and II\textsuperscript{4}) in the form of various theorems. We present them with making necessary corrections to the errors in the original MN papers, which were caused by a misreading of the maximum eigenvalue of the kernel in Landau gauge. Finally in §4, we shall describe several later developments after the work of Maskawa and Nakajima.

§2. Formalism and the Ward-Takahashi identity

We want a consistent set of approximate SD and BS equations such that the propagator and the vertex functions determined as the solution to them satisfy the Ward-Takahashi identities. The most systematic way to derive such a consistent set of SD and BS equations would be to use the effective action for the propagator $S_F(x, y) \equiv \langle T\bar{\psi}(x)\psi(y) \rangle_A$ in the presence of external background gauge field $A_\mu$, as was done in Ref. 5). Even without the device of the external gauge field, the effective action is useful to discuss the vacuum energy.\textsuperscript{6})

2.1. Effective action

Let us consider QCD system in the presence of external background gauge fields $A_\mu$ coupling to the flavor charges which are orthogonal to the color degrees of freedom:

$$\mathcal{L}_{\text{QCD}}(A) = -\frac{1}{4} F_{\mu\nu}^\alpha F^{\alpha \mu\nu} + \bar{\psi} i\gamma^\mu \left( \partial_\mu - ig_s T^\alpha G_\mu^\alpha - iA_\mu \right) \psi, \quad (2.1)$$

where $G_\mu^\alpha$ are the color gluon fields and $T^\alpha$ ($\alpha = 1, \cdots, N_c^2 - 1$) are the generator matrices of $SU(N_c)$ color group in the quark representation. Note that this flavor
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Fig. 1. Two particle irreducible (wrt fermion-line) diagrams contributing to $K_{2\text{PI}}^{(1)}$ and $K_{2\text{PI}}^{(2)}$ in QCD.

The double wavy line represents the gluon propagator $D_{\mu\nu}$ and the solid line represents the fermion propagator $S_F$.

gauge field

$$A_\mu(x) \equiv A_\mu^a(x)\lambda^a,$$

is just a non-dynamical background field which was introduced as a convenient devise to derive a consistent set of SD and BS equations and will eventually be set equal to zero. We have assumed vector coupling $L_{\text{int}} = \bar{\psi}\gamma^\mu A_\mu \psi$ for the external flavor gauge field just for notational simplicity. The axial vector case, in which we are actually interested for the present case of chiral symmetry, can be obtained simply by replacement $\gamma^\mu \to \gamma^\mu\gamma_5$.

The effective action for the two-point (or multi-point) functions was introduced by Dominicis and Martin and Cornwall, Jackiw and Tomboulis. The effective action $\Gamma[S_F, A]$ for the quark propagator $S_F(x, y) \equiv \langle T\psi(x)\bar{\psi}(y) \rangle_A$, in this case, is given by the formula

$$\Gamma[S_F, A] = i \text{Tr} \ln S_F - \text{Tr} (iD/S_F) + i^{-1} K_{2\text{PI}}[S_F],$$

where the external gauge field $A_\mu$ appears only in the covariant derivative

$$D_\mu = \partial_\mu - iA_\mu,$$

and $K_{2\text{PI}}$ stands for the two particle irreducible (wrt fermion-line) diagram contributions: in the present QCD theory, we can expand the $K_{2\text{PI}}$ into power series in the color gauge coupling $\alpha_s = g_s^2/4\pi$,

$$K_{2\text{PI}} = K_{2\text{PI}}^{(1)} + K_{2\text{PI}}^{(2)} + \cdots$$

and $K_{2\text{PI}}^{(1)}$ and $K_{2\text{PI}}^{(2)}$ are diagrammatically given by Fig. 1. More explicitly the first term $K_{2\text{PI}}^{(1)}$ is given by

$$K_{2\text{PI}}^{(1)} = -\frac{g_s^2}{2} \int d^4x d^4y \text{ tr} (S_F(x, y) i\gamma_\mu T^\alpha S_F(y, x) i\gamma_\nu T^\alpha) D^{\mu\nu}(x - y).$$

Here $D^{\mu\nu}$ is the tree level gluon propagator given by

$$D^{\mu\nu}(x) = \int \frac{d^4p}{i(2\pi)^4} e^{-ipx} \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \left( \frac{1}{p^2} - \frac{1}{p^2 - A^2} \right),$$

where $p^2 = (p^2 - A^2)^2 - 4a^2$ and $A^2$ is the light cone component of the momentum.

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where we have included an ultraviolet cutoff $\Lambda$ for definiteness. If we use the running coupling constant as was done in the improved ladder approximation by Higashijima$^9$ and Miransky,$^{10}$ the coupling constant $g_s$ should be replaced by the running coupling function $g_s(p^2)$ with gluon momentum $p_\mu$ in the argument.$^{11}$ It should be emphasized that the dependence must be only through the gluon momentum, but not the fermion momenta, in order to maintain the external flavor gauge invariance, as we explain below.

2.2. SD equation

The SD equation for the quark propagator follows from the stationarity condition of the effective action:

$$\frac{\delta \Gamma[S_F,A]}{\delta S_F(x,y)} = 0.$$  (2.6)

Using the above formula (2.3) for $\Gamma[S_F,A]$, this reads more explicitly

$$i S_F^{-1} = i \not{D} - i^{-1} \frac{\delta K_{2PI}}{\delta S_F}.$$  (2.7)

If we take only the lowest order term in $K_{2PI}$, $K^{(1)}_{2PI}$, then this SD equation further reduces to

$$i S_F^{-1} = i \not{D} + \not{A} + i^{-1} K * S_F,$$  (2.8)

with $K * S_F$ defined by

$$K * S_F \equiv g^2_s(i \gamma_\mu T^a)S_F(y,x)(i \gamma_\nu T^a)D^{\mu\nu}(x-y).$$  (2.9)

Diagrammatically, this is expressed as shown in Fig. 2.

Equation (2.7) is the SD equation determining a solution $S_F = S_F[A]$ for the fermion propagator, on an arbitrary external background gauge field $A_\mu$. The solution $S_F[A]$ is expanded into a power series in the external gauge field $A_\mu$:

$$S_F[A] = S_F + i A^a_\mu G^a_3 + \frac{i^2}{2} A^a_\mu A^b_\nu G^a_4 G^b_{\mu\nu} + \cdots,$$  (2.10)

where $a$, $b$ and $c$ denote the flavor indices. Here and henceforth the space-time coordinates and the integrations are suppressed, i.e., $A^a_\mu G^a_3 \equiv \int d^4z A^a_\mu(z)G^a_3(x, y; z)$, etc. The function $G^{a_1\mu_1, \cdots, a_n\mu_n}(x, y; z_1, \cdots, z_n)$ defines a fermion 2-point function with $n$ vector vertices inserted:

$$G^{a_\mu}(x, y; z) \equiv \frac{1}{i} \frac{\delta S_F(x, y; A)}{\delta A^a_\mu(z)} \bigg|_{A=0} = \langle 0 | T j^{a\mu}(z) \psi(x) \bar{\psi}(y) | 0 \rangle,$$  (2.11)

$$i S_F^{-1} = i \not{D} + \not{A} - i \not{g} \not{T}^a S_F \not{g} \not{T}^a.$$  

Fig. 2. Schwinger Dyson equation derived from the effective action $\Gamma$ using $K_{2PI} = K^{(1)}_{2PI}$. 

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$\not{D}$ represents the Dirac derivative.

$T$ denotes the trace over color indices.

$g$ represents the Dirac gamma matrices.

$\not{T}^a$ represents the color gamma matrices.

$\not{g} \not{T}^a$ represents the tensor product of the Dirac gamma matrices and color gamma matrices.

$\not{D}$ represents the Dirac derivative.

$\not{A}$ represents the gauge potential.

$\not{S}_F$ represents the fermion propagator.

$\not{j}$ represents the fermion source.

$\not{g}$ represents the Dirac gamma matrices.

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$\not{A}$ represents the gauge potential.

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$\not{j}$ represents the fermion source.
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\[ \Gamma_{3}^{\mu} = \gamma^{\mu} + \tilde{\Gamma}_{3}^{\mu}, \]

Fig. 3. BS equation for \( \Gamma_{3} \).

and so on. This is because \( \delta/\delta A_{\mu}^{\alpha} \) yields an insertion of the vector current operator \( j_{a\mu}^{\alpha} = \bar{\psi} \gamma^{\mu} \lambda^{a} \psi \) to which the external gauge boson \( A_{\mu}^{\alpha} \) couples. Hereafter we suppress the flavor indices to denote \( G_{n+2}^{\mu_{1} \cdots \mu_{n}} \) simply as \( G_{n+2}^{\mu_{1} \cdots \mu_{n}} \), and write only \( \gamma^{\mu} \) in place of \( \gamma^{\mu} \lambda^{a} \) as vertex factors in the figures, accordingly.

2.3. BS equations for the vertices

Therefore the SD equation (2.7) for \( S_{F}[A] \) in fact gives not only the SD equation for the propagator \( S_{F} = S_{F}[A = 0] \) but also the Bethe-Salpeter (BS) equations for the \( (n+2) \)-point Green functions \( G_{n+2}^{\mu_{1} \cdots \mu_{n}} \). That is, the functional differentiation wrt. \( A_{\mu}(and \ then \ setting \ A = 0) \) of the SD equation (2.7) successively generates the BS equations for the \( G_{n+2}^{\mu_{1} \cdots \mu_{n}} \) functions. It is convenient to define the following vertex function by amputating the fermion legs:

\[ \Gamma_{n+2}^{\mu_{1} \cdots \mu_{n}} \equiv S_{F}^{-1} G_{n+2}^{\mu_{1} \cdots \mu_{n}} S_{F}^{-1}. \]

Let us demonstrate this more explicitly only for the simplest case in which we use the lowest order kernel \( (\mathcal{K}_{2PI} = \mathcal{K}_{2PI}^{(1)}) \). First differentiation \( \delta/\delta A_{\mu}|_{A=0} \) of Eq. (2.8) gives (see Fig. 3)

\[ \Gamma_{3}^{\mu} = \gamma^{\mu} + \tilde{K} \ast \Gamma_{3}^{\mu}, \]

where \( \tilde{K} \ast \Gamma_{3}^{\mu} \equiv K \ast (S_{F} \Gamma_{3}^{\mu} S_{F}) = K \ast G_{3}^{\mu} \) is defined in the same way as in Eq. (2.9).

2.4. External gauge invariance

By our assumption that the flavor freedom is orthogonal to the color, the flavor matrices \( \lambda^{a} \) commute with the color matrices \( T^{\alpha} \). Then we have the following lemma. **Lemma:** For any approximation for \( K_{2PI} \) by an arbitrary subset of diagrams contributing to \( K_{2PI} \), the effective action Eq. (2.3) is (external) gauge invariant:

\[ \Gamma[S_{F}, A] = \Gamma[S_{U}^{U}, A^{U}], \]

where the gauge transformation with \( U(x) = \exp (i\theta^{a}(x)\lambda^{a}) \) is given explicitly by

\[ A_{\mu} \rightarrow A_{\mu}^{U} = U A_{\mu} U^{-1} + \frac{1}{i} \partial_{\mu} U \cdot U^{-1}, \]

\[ S_{F}(x, y) \rightarrow S_{F}^{U}(x, y) = U(x) S_{F}(x, y) U^{-1}(y). \]

The proof is easy as shown in Ref. 5) in detail, and each term in Eq. (2.3) is separately gauge-invariant. Moreover, each diagram contributing to \( K_{2PI} \) is also separately invariant. Indeed, in any diagram, all the fermion lines are connected. Although they are separated by the interaction vertex factor \( g_{s} \gamma^{\mu} T^{\alpha} \) at each vertex, the gauge transformation matrices \( U(x) \) and \( U^{-1}(x) \) which appear from the two
propagators of both sides of the vertex point $x$ cancel each other since $U$ acts only in the the flavor space and is commutative with the color matrix $T^a$ at the vertex. This also explains the reason why the argument of the running coupling function must be the momentum of the gluon, since otherwise the vertex becomes non-local for the fermion lines.\footnote{A refinement of the proof and the generalization to the running coupling case was given by Kugo and Mitchard.\footnote{T. Kugo and H. Nakajima}}

Thus the gauge invariance of $\Gamma[S_F, A]$ holds at any order of approximation for $\mathcal{K}_{2PI}$.

2.5. Ward-Takahashi identity

We now show that the external gauge invariance of the effective action implies that the vertex functions determined by those BS equations satisfy the Ward-Takahashi identities.

It immediately follows from the external gauge invariance relation (2.13) that the solution of the SD equation (2.6) on the gauge transformed background $A^U_\mu$ is given by the gauge transformation $US_F[A]U^{-1}$ of the solution $S_F[A]$ on the original background $A_\mu$: that is,

$$S_F[A^U] = US_F[A]U^{-1}. \tag{2.15}$$

Substituting the expansion (2.10) into both sides of Eq. (2.15), we have

$$\text{LHS} = S_F + iA^U_\mu G^3_3 + \frac{i^2}{2} A^U_\mu A^U_\nu G^\mu\nu + \cdots,$n

$$\text{RHS} = US_F U^{-1} + iA_\mu U G^\mu_3 U^{-1} + \frac{i^2}{2} A_\mu A_\nu U G^{\mu\nu} U^{-1} + \cdots. \tag{2.16}$$

Considering, in particular, an infinitesimal gauge transformation $U = 1 + i\theta$ ($\theta = \theta^a \lambda^a$) and $A^U_\mu = A_\mu + D_\mu \theta$, and equating the same power terms in $A_\mu$ on both sides, we find

$$-i\partial^\mu G^{a\mu}_3 (x, y; z) = i\delta^4(z - x)\lambda^a S_F(x - y) - i\delta^4(z - y)S_F(x - y)\lambda^a, \tag{2.17}$$

and so on. These are just the Ward-Takahashi identities required by the external gauge invariance. Thus this proves that the fermion propagator $S_F$ and the vertices $\Gamma^\mu_{n+2\cdots n}$ determined by our SD and BS equations satisfy the Ward-Takahashi identities giving relations among them; namely, our approximations for the SD and BS equations are mutually consistent and gauge invariant.

We emphasize again that the WT identities are satisfied if we use SD and BS equations in the same order of approximation, that is, if they are both derived from the same effective action $\mathcal{K}_{2PI}$, irrespectively of the order of the approximation for it.

If we take the the lowest order approximation with $\mathcal{K}_{2PI} = \mathcal{K}_{2PI}^{(1)}$, we have the ladder SD equation in Fig. 2 with $A_\mu = 0$ and the ladder BS equation for the 3-point vertex $\Gamma^\mu_3$ in Fig. 3. This gauge invariance for $\Gamma^\mu_3$ in the simplest ladder approximation has been known for a long time to Maskawa and Nakajima.\footnote{\textsuperscript{3),\textsuperscript{*}}}
§3. Maskawa-Nakajima paper I

The QCD was not yet generally accepted as a true theory describing the strong interaction when Maskawa and Nakajima started to examine the dynamical chiral symmetry breaking. So they adopted a massive Abelian vector gluon model for the strong interaction in place of our QCD theory (2.1); therefore, their model is obtained by setting the color generator matrices $T^a$ equal to 1 and replacing the gluon propagator (2.5) by

$$D^{\mu\nu}(x) = \int \frac{d^4p}{i(2\pi)^4} e^{-ipx} D^{\mu\nu}(p),$$

$$D^{\mu\nu}(p) = \left( g^{\mu\nu} - \eta(p^2) \frac{p^{\mu} p^{\nu}}{p^2} \right) \left( \frac{1}{p^2 - \mu^2} - \frac{1}{p^2 - \Lambda^2} \right),$$

where $\eta(p^2)$ is the gauge-fixing function and $\mu^2$ the gluon mass square. They also put a bare mass term $-m_0 \bar{\psi} \gamma_5 \psi$ of the quark field when necessary to discuss the explicit chiral symmetry breaking. So the SD equation (2.8) in the absence of the external gauge field $A_\mu = 0$ for the fermion propagator

$$S_F(x) = \int \frac{d^4p}{i(2\pi)^4} e^{-ipx} \frac{-1}{\alpha(p^2) \dot{\beta} - \beta(p^2)}$$

now reads in their case as

$$\alpha(p^2) \dot{\beta} - \beta(p^2) = \dot{\beta} - m_0 + \int \frac{d^4k}{i(2\pi)^4} g_s^2 D^{\mu\nu}(p-k) \gamma_\mu \gamma_5 \psi \frac{1}{(k^2 - \alpha(k^2))^{\beta(k^2) - \alpha(k^2)}} \gamma_\nu.$$  (3.3)

To the authors’ knowledge, Maskawa and Nakajima are the first to notice the mutual consistency of the ladder SD equation of the form Fig. 2 (with the external gauge field $A_\mu$ set equal to zero) and the ladder BS equation for the axial vector vertex of the form Fig. 3 (with $\gamma_\mu$ replaced by $\gamma_5\gamma_5$). The important point is that the fermion propagator contained in the BS equation in Fig. 3 is that determined by the ladder SD equation.

The Ward-Takahashi identity (2.17) reads for the axial vertex $\Gamma_{5\mu}$ for the axial vector current $j_{5\mu} = \bar{\psi} \gamma_\mu \gamma_5 \psi$ in the presence of bare mass $m_0$

$$(p-k)^\mu \Gamma_{5\mu}(p,k) = iS_F^{-1}(p) \gamma_5 + \gamma_5 iS_F^{-1}(k) + 2m_0 \Gamma_5(p,k),$$

where $\Gamma_5(p,k)$ is the vertex for the pseudo-scalar operator $j_5 = \bar{\psi} \gamma_5 \psi$. If there is a solution $iS_F^{-1}(p) = \alpha(p^2) \dot{\beta} - \beta(p^2)$ for the SD equation with nonvanishing mass function $\beta(p^2)$, then the limit $q = p - k$ going to zero of the WT identity (3.4) gives

$$\lim_{q \to 0} q^\mu \Gamma_{5\mu}(p,p-q) = -2\beta(p^2) \gamma_5 + 2m_0 \Gamma_5(p,p).$$

*) This WT identity in the presence of explicit chiral symmetry breaking bare mass term can also be derived in the formalism in the previous section most conveniently by generalizing the WT identity by introducing chiral symmetric Yukawa interaction of scalar $S$ and pseudoscalar $P$ fields, $-\bar{\psi}(S + i\gamma_5 P)\psi$, and finally taking the “vacuum” with $\langle S \rangle = m_0$ and $\langle P \rangle = 0$. 
If the bare mass term $m_0$ is zero, this implies that there exist a massless pole term

$$\Gamma_{5\mu}(p, p - q) \sim -2\beta(p^2)\gamma_5 \frac{q^\mu}{q^2} \tag{3.6}$$

in the axial vector vertex $\Gamma_{5\mu}$. This is the Nambu-Goldstone boson pole for the spontaneous chiral symmetry breaking. MN have emphasized, however, that the massless NG pole does not appear even when a non-vanishing $\beta$ solution exists for $m_0 = 0$ for the case of no ultraviolet cutoff. The chiral symmetry is only in appearance in such a case, although $m_0 = 0$; the bare mass $m_0$ is not exactly zero but an infinitesimal quantity which is multiplied by an infinity of $\Gamma_{5\mu}$ such that the renormalized $m_0\Gamma_{5\mu}(p, p)$ gives exactly the finite $\beta(p^2)\gamma_5$.

Moreover they proved various properties of the SD equation for the first time in a rather mathematical and rigorous way. Here we summarize the various results obtained in the papers I and II in the form of theorems. For the proofs, we refer to the original papers.

### 3.1. Solutions without cutoff in the “Landau-like” gauge

Let us first consider the case without ultraviolet cutoff, i.e., taking $\Lambda \to \infty$ in Eq. (3.1), and adopt a gauge which we call “Landau-like” gauge, i.e.,

$$\eta(p^2) \equiv \frac{p^2}{p^2 - \mu^2} \tag{3.7}$$

By performing the angular integrations after Wick rotation $d^4k \to id^4k_E$ (suffix $E$ denoting the Euclidean momentum), the SD equation (3.3) is reduced to the following set of the equations for $\alpha(x)$ and $\beta(x)$ ($x \equiv p_E^2 \equiv -p_0^2 + p^2$):

$$\alpha(x) = 1 - g^2 \int_0^\infty dy L_{\mu^2}(x, y) \frac{y\alpha(y)}{y\alpha^2(y) + \beta^2(y)}, \tag{3.7a}$$

$$\beta(x) = m_0 + g^2 \int_0^\infty dy K_{\mu^2}(x, y) \frac{y\beta(y)}{y\alpha^2(y) + \beta^2(y)}, \tag{3.7b}$$

where

$$g^2 \equiv \left(\frac{g_s}{2\pi}\right)^2,$$

$$L_{\mu^2}(x, y) \equiv \frac{y\mu^2}{(x + y + \mu^2 + \sqrt{(x + y + \mu^2)^2 - 4xy})^2 \sqrt{(x + y + \mu^2)^2 - 4xy}},$$

$$K_{\mu^2}(x, y) \equiv \frac{2}{x + y + \mu^2 + \sqrt{(x + y + \mu^2)^2 - 4xy}} \frac{3}{4} \left(1 + \frac{\mu^2}{3\sqrt{(x + y + \mu^2)^2 - 4xy}}\right). \tag{3.8}$$

* This choice is made only in order that the integral kernel $L_{\mu^2}(x, y)$ should be a positive and less singular function.
Theorem 1-a: When $m_0 \neq 0$, there exists no solution for Eqs. (3.7a) and (3.7b). When $m_0 = 0$ and $0 < \overline{g}^2 < \overline{g}_c^2$, $(\overline{g}_c^2 \equiv (16/33)^2)^{*}$ there exist a continuously infinite number of independent solutions $(\alpha, \beta)$ for Eqs. (3.7a) and (3.7b). The solutions are specified by an ‘asymptotic parameter’ $\nu$ ($0 < \nu < \infty$) which determines the magnitude of the asymptotically dominant part of $\beta$, as

$$\beta_\nu(x) = \nu x^{\lambda_2(\overline{g})} + o(x^{-1/2}),$$

where

$$\lambda_2(\overline{g}) \equiv -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 3\overline{g}^2}.$$  

The $\beta_\nu(0)$ can be arbitrarily fixed by choosing a suitable $\nu$.

Theorem 1-b: Under the condition that $m_0 = 0$ and $0 < \overline{g}^2 < \overline{g}_c^2$, $(1/5 < \overline{g}_c^2 < \overline{g}_c^2 = (16/33)^2)$, and for an asymptotic parameter $\nu$, there exists only one solution $(\alpha, \beta_\nu)$ in the space $(D_\alpha, D_\beta)$, where**

$$D_\alpha \equiv \{ \alpha \mid \alpha(x) \geq \alpha_m(\overline{g}^2) \text{ for } \forall x, \alpha \in C \},$$

$$D_\beta \equiv \{ \beta \mid \beta \in C \},$$

with

$$\alpha_m(\overline{g}^2) \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - \overline{g}^2}.$$  

Theorem 1-c: For $m_0 = 0$, the identically vanishing $\beta$ is a solution for Eq. (3.7b). Under the condition $0 \leq \overline{g}^2 \leq 2$, there exists a solution $\alpha$ for Eq. (3.7a) with $\beta$ set identically vanishing. For $\overline{g}^2 > 8$, if it existed, the $\alpha$ should have zeros in the space-like momentum region.

The reason why the arbitrariness of $\nu$, consequently of the physical mass $m$, appears in the theorem 1-a is understood as follows. If we introduce a cutoff $\Lambda$ and let it go to infinity with a physical mass $m$ fixed, the bare mass $m_0(m, \Lambda)$ tends to zero; $\lim_{\Lambda \to \infty} m_0(m, \Lambda) = 0$. But this does not imply that the system recovers the chiral symmetry since the 4-divergence of the axial current vertex $\Gamma^5_{\mu\nu}$, $\partial^\mu \Gamma^5_{\mu\nu} = 2im_0 \Gamma_5$, does not vanish at the renormalization point but to a finite value determined by $m^{12}$ since the renormalization constant $Z^{-1}_p$ tends to infinity. That is, the solutions ($\beta \neq 0$) obtained for $m_0 = 0$ in Theorem 1-a, are not “superconducting” solutions (i.e., spontaneously chiral symmetry breaking solutions), but explicitly breaking solutions which appear as a result that an infinitesimal bare mass slips into the theory.

Theorem 1-b guarantees the uniqueness of such an explicitly chiral symmetry breaking solution for a given infinitesimal bare mass specified by the asymptotic parameter $\nu$.

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* Note that this critical value $\overline{g}_c^2$ is fixed by $\overline{g}_c^2/(\alpha_m(\overline{g}^2))^2 = 1/4$ where $\alpha_m(\overline{g}^2)$ is defined in Eq. (3.11).

** $C$ is a set of all continuous functions. It can be proved that for any solution $(\alpha, \beta)$ in which there exists a positive lower bound of $\alpha$, the $\alpha$ and the $\beta$ are continuous functions.
The solution with $\beta(x) \equiv 0$ treated in Theorem 1-c is the “normal-state” solution in the truly chiral symmetric system. It also says that such “normal-state solution” becomes pathological when the coupling constant becomes very large.

3.2. Solutions with cutoff in the Feynman gauge

Maskawa and Nakajima investigated the SD equation with a sufficiently large but finite cutoff $\Lambda$ in the Feynman gauge with vanishing gauge-function $\eta(k^2) \equiv 0$, in order to make the above argument more definite. In this case, the vanishing bare mass, $m_0 = 0$, implies chiral invariance because no infinity appears there.

With $\eta(k^2) \equiv 0$ in Eq. (3.1), the SD equation (3.3) yields the following equations to be considered in this case:

$$
\alpha(x) = 1 + g^2 \int_0^\infty dy L_\Lambda(x, y) \frac{y\alpha(y)}{y\alpha^2(y) + \beta^2(y)},
$$

$$
\beta(x) = m_0 + g^2 \int_0^\infty dy K_\Lambda(x, y) \frac{y\beta(y)}{y\alpha^2(y) + \beta^2(y)},
$$

where

$$
L_\Lambda(x, y) \equiv \left[ \frac{y}{\{x + y + \mu^2 + \sqrt{(x + y + \mu^2)^2 - 4xy}\}^2} - (\mu^2 \leftrightarrow \Lambda^2) \right],
$$

$$
K_\Lambda(x, y) \equiv \left[ \frac{2}{x + y + \mu^2 + \sqrt{(x + y + \mu^2)^2 - 4xy}} - (\mu^2 \leftrightarrow \Lambda^2) \right] .
$$

Theorem 2-a: When $0 < g^2 < 1/4$, there exists at least a solution $(\alpha, \beta)$ for the simultaneous equations (3.12a) and (3.12b) in the space $(D'_\alpha, D_\beta)$, and then for any fixed $\alpha$ in the space $D'_\alpha$, there exists a unique solution $\beta$ of Eq. (3.12b) in the space $D_\beta$, where

$$
D'_\alpha \equiv \{ \alpha \mid \alpha(x) > 1 \text{ for } \forall x, \alpha \in \mathcal{C} \} , \quad D_\beta \equiv \{ \beta \mid \beta \in \mathcal{C} \} .
$$

When $m_0 = 0$, we find that the identically vanishing $\beta$ is apparently a solution for (3.12b), and therefore from Theorem 2-a, we immediately conclude that the identically vanishing $\beta(x) \equiv 0$ is the unique solution for the simultaneous equations (3.12a) and (3.12b). This means that no “super-conducting” solution exists for such a small coupling $g^2 < 1/4$ irrespective of the value of $\Lambda$, and no Nambu-Goldstone boson appears.

*) The question about the occurrence of dynamical breaking of chiral symmetry is centered at that of the nonuniqueness of the solution $\beta$. Nonuniqueness of $\alpha$ is irrelevant to consideration of the existence of the “super-conducting” solution. We have been discussing only the positive definite solution $\alpha$, which is in fact positive if there are no negative norm states. We do not know much about the non-positive solution $\alpha$. Under the assumption $\alpha > 0$, one immediately has $\alpha > 1$ from Eq. (3.12a).
Here adding to the above theorem, we have the following one valid even for
\( m_0 \neq 0 \) by the argument of the final part of §2-a in the paper II:

Theorem 2-b: When \( 0 \leq \bar{g}^2 < 1/5 \), there exists a unique solution \((\alpha, \beta)\) in the
space \( (D'_\alpha, D_\beta) \) defined in (3.14) for Eqs. (3.12a) and (3.12b).

3.3. Model case analysis for rather strong coupling

It becomes impossible to verify the uniqueness of the solution for the strong
coupling case. To examine the situation for such a strong coupling case, Maskawa
and Nakajima considered a model case in which \( \alpha(x) \equiv 1 \), since
\( \alpha(x) \equiv 1 \) is in any case seen to be near to the true solution. They analyzed Eq. (3.12b) with
\( \alpha(x) \equiv 1 \) for \( m_0 = 0 \) and for rather strong coupling, and proved that there exists
another solution besides the identically vanishing solution \( \beta(x) \equiv 0 \) and, therefore,
the solution \( \beta \) is no longer unique for \( \bar{g}^2 > 1/4 \).

Equation (3.12b) for \( m_0 = 0 \) with with \( \alpha(x) \equiv 1 \) is given by

\[
\beta(x) = \bar{g}^2 \int_{0}^{\infty} dy K_A(x,y) \frac{y\beta(y)}{y + \beta^2(y)},
\]

where \( K_A \) is defined in (3.13). Maskawa and Nakajima gave a theorem for this form
of nonlinear integral equation but possessing a more general class of integral kernels
\( K(x,y) \), i.e.,

\[
\beta(x) = \psi \cdot \beta(x) \equiv \bar{g}^2 \int_{0}^{\infty} dy K(x,y) \frac{y\beta(y)}{y + \beta^2(y)}.
\]

Here the kernel \( K \) is assumed to belong to a set \( K \) of non-negative definite symmetric
\( L^2 \) kernels. The more detailed definition is omitted here and referred to the paper II.

Theorem 3: There exists a nonvanishing solution \( \beta \) (also \( -\beta \)) along with an
identically vanishing one, \( \beta \equiv 0 \), for Eq. (3.16), if \( K \in K \) and \( \bar{g}^2 \lambda_m > 1 \) for the
maximum eigenvalue \( \lambda_m \) of \( K \).

Since the kernel \( K_A \) in Eq. (3.15) is shown to belong to \( K \), there exists a non
vanishing solution \( \beta \) (“super-conducting” solution) along with the solution \( \beta \equiv 0 \)
(“normal-state” solution), when \( \bar{g}^2 \lambda_{\text{max}} > 1 \), where \( \lambda_{\text{max}} \) is the maximum eigenvalue
of \( K_A \). The \( \lambda_{\text{max}} \) tends to 4 as \( \Lambda \) tends to infinity and therefore \( \bar{g}^2 \lambda_{\text{max}} > 1 \) is
satisfied for the sufficiently large \( \Lambda \) if \( g^2 > 1/4 \), i.e., \( g^2 / 4\pi > \pi / 4 \).

It should also be noted from Theorem 1-c, that in the region \( \bar{g}^2 > 8 \), if the
“normal-state” solution existed for the equation without cutoff, the corresponding
propagator should necessarily have an unphysical singularity, i.e., a singularity in
the space-like momentum region. This fact implies that the “normal-state” solution
becomes unstable for a sufficiently large value of \( g^2 \).

To the question whether the dynamical breaking of chiral symmetry occurs in
the vector-gluon model, or not, Maskawa and Nakajima thus answered in the paper
I, “yes, for strong coupling, and no, for weak coupling even for the cutoff \( \Lambda \) going
to infinity”. The solution $(\alpha, \beta)$ obtained in the case of no cutoff for weak coupling, $0 < \sqrt{g^2} \leq 1/5$, is not a chiral symmetric solution (no Nambu-Goldstone boson!).

§4. Maskawa-Nakajima paper II

This answer of theirs is, however, the joined result of Theorems 1, 2 and 3. There they took a “Landau-like” gauge in Theorems 1 (3-a in the paper I), and the Feynman gauge in Theorems 2 (3-b in the paper I) and Theorem 3 (3-c in the paper I). We consequently dealt with slightly different equations and further approximated them as $\alpha \equiv 1$ in Theorem 3. Thus in the paper II, they gave the theorems for both the weak coupling and strong coupling cases persistently in the “Landau-like” gauge with a finite cutoff $\Lambda$ but without the approximation $\alpha \equiv 1$, and reinforced the results obtained in the paper I. The proofs of them are given in §2 of the paper II. Here we cite the theorems.

In the “Landau-like” gauge with cutoff, the integral equations to be considered are as follows:

\[
\begin{align*}
\alpha(x) &= 1 - \sqrt{g^2} \int_0^\infty dy L_c(x, y) \frac{y \alpha(y)}{y \alpha^2(y) + \beta^2(y)}, \\
\beta(x) &= m_0 + \frac{1}{\sqrt{g^2}} \int_0^\infty dy K_c(x, y) \frac{y \beta(y)}{y \alpha^2(y) + \beta^2(y)},
\end{align*}
\]

(4.1a)

(4.1b)

where

\[
\begin{align*}
L_c(x, y) &\equiv L_{\mu^2}(x, y) - (\mu^2 \leftrightarrow \Lambda^2), \\
K_c(x, y) &\equiv K_{\mu^2}(x, y) - (\mu^2 \leftrightarrow \Lambda^2).
\end{align*}
\]

(4.2)

with the kernels $L_{\mu^2}(x, y)$ and $K_{\mu^2}(x, y)$ given in Eq. (3.8) before.

4.1. Solutions with cutoff in the “Landau-like” gauge: weak coupling case

Theorem 4: For $0 \leq \sqrt{g^2} \leq 0.222$, there exists a unique solution $(\alpha, \beta)$ in the space $(D_\alpha, D_\beta)$ defined in Eq. (3.10) for Eqs. (4.1a) and (4.1b). (Proof in §2 in the paper II.)

Especially, for $m_0 = 0$, $(\alpha, \beta)_{\beta=0}$ is a unique solution. Thus we find that dynamical breaking of chiral symmetry does not occur, irrespective of the magnitude of the cutoff $\Lambda$ for weak coupling such that

\[
0 \leq g^2/4 \pi \leq g_1^2/4 \pi \equiv 0.222 \pi.
\]

(4.3)

4.2. Solutions with cutoff in the “Landau-like” gauge: rather strong coupling case

We consider Eqs. (4.1a) and (4.1b) for rather strong coupling case, firstly in the case $m_0 = 0$. We have
Theorem 5: For $2 \geq \bar{g}^2 \geq \bar{g}^2(\equiv 0.367)^*$ and for finite but sufficiently large $\Lambda$, there exists a solution $(\alpha, \beta)$ with nonvanishing $\beta$, along with a solution with identically vanishing $\beta$, for Eqs. (4.1a) and (4.1b) with $m_0 = 0$. (Proof in §2 in the paper II.)

It is thus found that the dynamical breaking of chiral symmetry does occur for a sufficiently large cutoff $\Lambda$ and for rather strong coupling such that

$$2\pi \geq g^2/4\pi \geq g_2^2/4\pi \equiv 0.367\pi.$$  \hspace{1cm} (4.4)

However, if there exists a sufficiently large bare mass $m_0$ for a finite $\Lambda$ in (4.1b), the solution $(\alpha, \beta)$ for Eqs. (4.1a) and (4.1b) turns out to become unique even for rather strong coupling. The following theorem shows this:

Theorem 6: If $m_0$ is sufficiently large for a finite $\Lambda$, and if $0 \leq \bar{g}^2 \leq 2$, there exists a unique solution $(\alpha, \beta)$ for Eqs. (4.1a) and (4.1b), in the space $(D_\alpha, D_\beta)$ in (3.10). (Proof in §2 in the paper II.)

§5. Later developments

After the MN paper, there appeared many works and developments in this field of research in which the strong interaction dynamics is analyzed by using SD and BS equations.\textsuperscript{13) This technique became important generally: In the context of QCD, it gives a viable technique situated in between the NJL 4-fermion effective approach and the full lattice gauge theory simulation. It gives a better approximation to QCD than NJL model and is more concise than the lattice approach. From this property of this method, it has been widely used, in particular, in technicolor theory\textsuperscript{14) and top-quark condensation model.\textsuperscript{15) There we had to know the strong interaction dynamics of new gauge theories beyond the standard model.

It is, however, beyond the scope of this paper to review all these developments and applications. Here we only mention two basic works performed after MN.

5.1. Massless gluon model

Although MN considered the massive gluon model, the situation of course becomes much easier and more explicit in the massless gluon case. Moreover, nowadays, the massless gluon is regarded as an even better approximation to QCD. This was done by Fukuda and Kugo (FK).\textsuperscript{16)

FK first observed that the SD equation (3.3), or Eq. (3.7) for the case of massless

\textsuperscript{*}) This critical value is determined by the Eq. (2.35) in the paper II, which reads $\lambda_{\text{max}} \bar{g}^2 > \alpha_M^2$ with $\alpha_M = 1 + g^2/8\alpha_m(\bar{g}^2) = (3 - \sqrt{1 - \bar{g}^2}/2)/2$. This condition can be solved to give $\bar{g}^2 > 16(10\lambda_{\text{max}} - 1 - 6\sqrt{\lambda_{\text{max}}(\lambda_{\text{max}} - 1)})/(8\lambda_{\text{max}} + 1)^2$. MN, however, misread the maximum eigenvalue $\lambda_{\text{max}}$ of the kernel $K_\epsilon$ to take $\lambda_{\text{max}} = 4$ in the $\Lambda \to \infty$ limit. The correct maximum eigenvalue is in fact given by $\lambda_{\text{max}} = 3$, which yields the condition $\bar{g}^2 > 0.366...$ as given here. We will explain the reason for this point later.
gluon propagator in the Landau gauge

\[ D^{\mu\nu}(p) = \frac{g^{\mu\nu} - p^\mu p^\nu/p^2}{p^2} \]  

(5.1)

becomes very simple; setting \( \mu^2 = 0 \) in Eq. (3.8) shows that the wave-function renormalization factor \( \alpha(x) \) remains identically to be 1, \( \alpha(x) \equiv 1 \), and the integral equation for the mass function \( \Sigma(x) \equiv \beta(x)/\alpha(x) \) becomes

\[
\Sigma(x) = m_0 + \frac{\lambda}{4} \left[ \int_0^x dy \frac{y}{x} + \int_x^{A^2} dy \frac{\Sigma(y)}{y + \Sigma^2(y)} \right]. \quad (\lambda \equiv 3\pi^2)
\]  

(5.2)

Here \( A^2 \) is a UV cutoff introduced on the fermion loop momentum integral.\(^*\) It is remarkable that this equation can be transformed into the following differential equation:

\[
\left[ x^2 \Sigma'(x) \right]' + \frac{\lambda}{4} \frac{x \Sigma(x)}{x + \Sigma^2(x)} = 0,
\]  

(5.3)

where the prime \( ' \) denotes \( d/dx \). This differential equation becomes equivalent to the original integral equation (5.2) if it is supplemented with the boundary conditions at UV and IR:

\[
\left. \frac{d}{dx} [x \Sigma(x)] \right|_{x = A^2} = m_0, \quad (5.4)
\]

\[
\left. x^2 \Sigma'(x) \right|_{x = \epsilon^2} = 0. \quad (5.5)
\]

(The \( \epsilon \) is an IR cutoff.) The asymptotic behavior is determined by the linearized equation:

\[
\left( x^2 \frac{d^2}{dx^2} + 2x \frac{d}{dx} + \frac{\lambda}{4} \right) \Sigma(x) = 0. \quad (5.6)
\]

By trying power form \( \Sigma(x) = x^a \), we find that the power \( a \) is

\[
a(a - 1) + 2a + \frac{\lambda}{4} = 0 \quad \Rightarrow \quad a = \frac{1}{2} \left( -1 \pm \sqrt{1 - \lambda} \right). \quad (5.7)
\]

When the coupling constant is weak such that \( \lambda \equiv 3\pi^2 < 1 \), the two powers are real and the asymptotic behavior is determined by the larger (i.e., dominant) one as

\[
\Sigma(x) \sim \text{const} \times x^{(-1 + \sqrt{1 - \lambda})/2} \quad (5.8)
\]

in conformity with MN’s result Eq. (3.9) for the massive gluon case. This monotonically decreasing power behavior clearly shows that the \( [x \Sigma(x)]' \) does not cross zero and the UV boundary condition (5.4) cannot be satisfied for the zero bare mass

\(^*\) Unlike the UV cutoff introduced before as the regulator mass for the gluon propagator, this cutoff for the fermion loop momentum in fact violates the consistency with the WT identity discussed in §2. But, if it goes to infinity, the cutoff becomes equivalent to the cutoff on the gluon momentum so that it will recover the consistency with the WT identity.
\[ m_0 = 0; \text{ that is, for the chiral symmetric system, only the trivial solution } \Sigma(x) \equiv 0 \text{ exists. If the non-zero bare mass } m_0 \text{ were present in the system, then the UV boundary condition would tell us that} \]
\[ m_0 \propto \Lambda^{-1+\sqrt{1-\lambda}}. \quad (5.9) \]

That is, the bare mass should be vanishing as this equation shows as \( \Lambda \to \infty \), to keep the low energy behavior intact. This also, of course, coincides with the MN result.

If the coupling constant is strong, i.e., \( \lambda > 1 \), then the powers (5.7) become complex so that the asymptotic behavior of \( \Sigma(x) \), and \([x \Sigma(x)]'\) as well, become oscillatory, so that the UV boundary condition (5.4) can be satisfied even for the zero bare mass. Therefore the system can have “super-conducting” solution aside from the trivial \( \Sigma(x) \equiv 0 \) solution in this strong coupling regime. The critical coupling for this “super-conducting” solution to exist for \( m_0 = 0 \) is thus given by
\[ \lambda_{cr} = 1 \quad \text{or} \quad \frac{g_{cr}^2}{\Lambda} = \frac{1}{3}. \quad (5.10) \]
in the \( \Lambda \to \infty \) limit. This critical value is the same as that in the massive gluon case considered by Maskawa-Nakajima,\(^*\) and is indeed consistent with the MN result given in Theorems 4 and 5.

The main point of Fukuda-Kugo’s paper was, however, not the behavior of the mass function \( \Sigma(x) \) in the spacelike region, but that in the timelike region. They indeed showed that the pole of the quark propagator \( \mathcal{S}_F(p) = i(\not{p} - \Sigma(p^2))^{-1} \) does not exit. We do not enter this subject here.

5.2. Improved ladder approach to QCD-like theories

Higashijima\(^9\) and Miransky,\(^10\) independently, proposed the improved ladder approximation in which the running coupling constant is used:
\[ \lambda(x) = \frac{3}{4\pi^2} C_2(F) g_s^2(x) = \frac{1}{x_0 + B \max(t, t_{IR})}, \quad (5.11) \]
where \( x = p_E^2 \), \( t = \ln x/x^2 \) and \( t_{IR} \) is an infrared cutoff above which \( \lambda(x) \) runs according to the leading logarithmic renormalization group but below which \( \lambda(x) \)

---
*\(^*\) The reason is the following: The critical coupling is given by the inverse of the maximum eigenvalue of the kernel \( K_c = K_{\mu_2} - K_{\Lambda^2} \) in Eq. (4.2) in the \( \Lambda \to \infty \) limit. Since the eigenfunction belonging to the maximum eigenvalue, \( \beta(x/m^2) \) \( (0 \leq x/m^2 \leq \Lambda^2/m^2) \), approaches the eigenfunction of the massless gluon theory except the small region, \( 0 \leq x/m^2 \leq O(1) \). But since this eigenfunction asymptotically damps only as \( \sim 1/\sqrt{x} \), the \( L^2 \)-norm of the eigenfunction is proportional to \( \ln(\Lambda^2/m^2) \). Due to this fact, the finite differences of the eigenfunctions and the kernels in the region \( 0 \leq x/m^2 \leq O(1) \) between the massive and massless gluon cases do not contribute to the estimate of the maximum eigenvalue \( \lambda_{max} \) so that the critical coupling constants coincide in both theories.

The situation will be completely different in the case of the improved ladder approximation where the running coupling constant is used. There the theory has an intrinsic mass parameter called \( \Lambda_{QCD} \), beyond which the coupling monotonically decreases, so that the eigenfunction \( \beta(x/\Lambda_{QCD}) \) will rapidly decrease as \( \sim 1/x \) asymptotically. In such a case, the gluon mass will largely affect the criticality of the theory.
is kept constant to avoid the divergent pole. $C_2(F)$ is the second Casimir of the fermion color representation, defined by

$$
\sum_a T^a T^a = C_2(F) \mathbf{1},
$$

(5.12)

which is $(N_c^2 - 1)/2N_c$ for $SU(N_c)$ fundamental representation. The parameter $B$ in (5.11) is the unique parameter characterizing the QCD-like theory. It is given, for instance, for the $SU(N_c)$ QCD with $N_f$ flavor quarks in the fundamental representation, by

$$
B \equiv \frac{b_0}{2} \cdot \frac{4\pi^2}{3C_2(F)} = \frac{1}{12C_2(F)} \left( \frac{12N_c - 2N_f}{3} \right),
$$

(5.13)

where $b_0$ is the lowest order coefficient of the $\beta$-function of renormalization group equation.

It is, however, not unique which momentum square $x \equiv p_E^2$ is used as the argument of the running coupling constant $\lambda(x)$ in the SD equation (cf. (3.3)); e.g., fermion momentum $k$ of internal line, or $p$ of external line, or $p - k$ of gluon momentum? Higashijima and Miransky proposed to use the larger one of the fermion momentum squares:

$$
\theta(x - y)\lambda(x) + \theta(y - x)\lambda(y). \quad (x \equiv p_E^2, \ y \equiv k_E^2)
$$

(5.14)

Then, the angle integration of $d^4k_E$ still can be performed so that $\alpha(x) \equiv 1$ remains to hold and the equation for the mass function $\Sigma(x)$ now reads in place of Eq. (5.2)

$$
\Sigma(x) = m_0 + \frac{1}{4} \left[ \lambda(x) \int_0^x dy \frac{y}{x} + \int_x^{A^2} dy \lambda(y) \right] \frac{\Sigma(y)}{y + \Sigma^2(y)}.
$$

(5.15)

This can still be rewritten into a differential equation

$$
\left[ \frac{\Sigma'(x)}{(\lambda(x)/4x)^*} \right]' = \frac{x\Sigma(x)}{x + \Sigma^2(x)}
$$

(5.16)

supplemented with the boundary conditions:

$$
\frac{d}{dx} \left[ \frac{x}{\lambda(x)} (\Sigma(x) - m_0) \right] \bigg|_{x = A^2} = 0,
$$

(5.17)

$$
\frac{\Sigma'(x)}{(\lambda(x)/x)^*} \bigg|_{x = \epsilon^2} = 0.
$$

(5.18)

Higashijima and Miransky showed that the asymptotic behavior of the mass function of spontaneously broken solution (for $m_0 = 0$) just coincides with Politzer’s exact QCD result derived from the operator product expansion and the renormalization group equation.\textsuperscript{17} Higashijima has also proved that the criticality condition for the spontaneous chiral symmetry breaking to occur is that the (maximum) value $\lambda(x = e^{t_{IR}})$ of the running coupling constant at the IR cutoff exceeds the critical coupling constant

$$
\lambda(x = e^{t_{IR}}) \geq \lambda_{cr} = 1.
$$

(5.19)
This Higashijima-Miransky approximation was also applied to the BS equations for the vector, axial vector, scalar and pseudo-scalar vertices, and not only the massless Nambu-Goldstone pole in the pseudo-scalar channel but also the vector, axial vector and scalar meson bound states were found. The computed result for their masses and decay constants (defined by equations like \( \langle 0 | \bar{\psi} \gamma^\mu d | \rho(\epsilon, k) \rangle \equiv f_\rho M_\rho \epsilon^\mu \)) is summarized in the following table:

<table>
<thead>
<tr>
<th></th>
<th>Our calculation</th>
<th>Experimental value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vector mass, ( M_\rho )</td>
<td>710 - 830</td>
<td>770 ± 3 (19)</td>
</tr>
<tr>
<td>Axial vector mass, ( M_{a_1} )</td>
<td>1110 - 1420</td>
<td>1260 ± 30 (19)</td>
</tr>
<tr>
<td>Scalar mass, ( M_{a_0} )</td>
<td>770 - 860</td>
<td>983 ± 3 (19)</td>
</tr>
<tr>
<td>( \rho ) decay constant, ( f_\rho )</td>
<td>200 - 400</td>
<td>204 ± 11 (20)</td>
</tr>
<tr>
<td>( a_1 ) decay constant, ( f_{a_1} )</td>
<td>270 - 720</td>
<td>200 ± 20 (21)</td>
</tr>
</tbody>
</table>

Here all the quantities are in MeV, and the scale was fixed by taking the pion decay constant to be \( f_\pi = 94 \) MeV. This result is really remarkable and impressive. The computation was done in the chiral symmetry limit so that it has no free parameters at all! This gives a strong support for the belief that the improved ladder SD and BS equation approach can give a reliable and concise method for analyzing the strong dynamics in QCD-like theories.

Despite this success, however, it was later observed that the original form of Higashijima-Miransky approximation is actually violating the Ward-Takahashi identity.11,22] This was surprising because the exactly massless Nambu-Goldstone pole was found in the axial vector vertex as well as directly in pseudo-scalar vertices. The detailed examination of the Ward-Takahashi identity (3.4) in the chiral limit \((m_0 = 0)\) revealed a slight deviation of the residue of the massless pole on the LHS from the prediction from the RHS.

The reason was, however, clear from the beginning in a sense. The chiral symmetry of the effective action \( \Gamma[S_\Sigma, A] \) can be kept only when the quark-gluon vertex is local wrt quark fields. Therefore, the vertex modification can be done only on the gluon propagator, so that the argument of the running coupling constant must be gluon momentum:

\[
\text{(I)} \quad g_\sigma((p_E - k_E)^2). \quad (5.20)
\]

There is no other choice for keeping the chiral symmetry. If one does not mind a slight violation, there are good approximations to this: noting that \((p_E - k_E)^2 = p_E^2 + k_E^2 - 2p_E k_E \cos \theta\) and the effect of the angle dependent part \(2p_E k_E \cos \theta\) is naturally expected to be small on average, we can expect that the choice

\[
\text{(II)} \quad g_\sigma(p_E^2 + k_E^2) \quad (5.21)
\]

should be a good approximation to the first choice (5.20). A further bold approximation to this is to replace \(p_E^2 + k_E^2\) by the larger one \(\max(p_E^2, k_E^2)\):

\[
\text{(III)} \quad g_\sigma(\max(p_E^2, k_E^2)) \quad (5.22)
\]

This is just Higashijima and Miransky’s original choice.
In Higashijima and Miransky’s choice (III), and in (II) as well, one can carry out the angle integration analytically and obtain $\alpha(x) \equiv 1$ in the Landau gauge $\eta(x) = 1$. We should, however, note that to have $\alpha(x) \equiv 1$ is not just simple but also a must in the improved ladder approximation. The very use of the running coupling constant requires it! Recall that $g_s(\mu^2)$ is the effective coupling constant between the fields renormalized at $\mu^2$. So, when using $g_s(\mu^2)$, we have to use the renormalized quark and gluon field satisfying $Z_2(\mu^2) = Z_3(\mu^2) = 1$ for the consistency. Since we are using the running coupling constant $g_s(p_E^2)$ with moving argument $p_E^2$, this requirement of unit wave-function renormalization factor has to be satisfied for any momentum: $Z_2(x) = Z_3(x) = 1$. The condition $Z_3(x) = 1$ for the gluon is automatically satisfied since the tree gluon propagator is used in the improved ladder approximation. The condition $Z_2(x) = 1$ for the quark field demands $\alpha(x) = 1$.

Thus, for the consistency, we should take the Landau gauge $\eta(x) = 1$ for the choices (II) and (III) in the improved ladder approximation, as was done above. In the case of choice (I), however, this condition $\alpha(x) \equiv 1$ is no longer satisfied simply by taking the Landau gauge. To have $\alpha(x) \equiv 1$ there, we have to use a non-local (i.e., $p^2$-dependent) gauge given by\(^^{11,23}\)

$$\eta(x) = \frac{2}{x^2 \lambda(x)} \int_0^x y dy \left(1 - y \frac{d}{dy}\right) \lambda(y). \quad (5.23)$$

Interestingly, this gauge function $\eta(x)$ approaches to the Landau gauge value $\eta = 1$ in both IR and UV limits, although it deviates from 1 in the important region $x \sim \Lambda_{QCD}^2$. We do not enter into the detail any more here.

The VEV $\langle \bar{\psi} \psi \rangle$ and the pion decay constant $f_\pi$ computed before\(^^{24}\) by using the original form of Higashijima and Miransky approximation for the SD and BS equations were recalculated in this consistent improved ladder approximation scheme.\(^{11}\) The obtained values do not deviate from the previous ones and agree within several percents. Therefore, the original Higashijima-Miransky approximation does not violates chiral symmetry so much and gives a sufficiently good and concise approximation for the study of QCD-like theories.

§6. Conclusions

In this paper we have quickly reviewed developments in the SD and BS approach to the QCD-like gauge theories which was pioneered by Maskawa and Nakajima and succeeded by Higashijima and Miransky. We think that this approach gives a powerful and still concise computational method for revealing the strong dynamics in QCD-like theories in various new circumstances.

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