**Multifractal scaling properties of a growing fault population**

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**SUMMARY**

A numerical rupture model, introduced in Cowie, Vanneste & Sornette (1993), is used to simulate the growth of faults in a tectonic plate driven by a constant plate boundary velocity. We find that the plate initially deforms by uncorrelated nucleation of small faults reflecting the distribution of material properties. With increasing strain, growth and coalescence of existing faults dominate over nucleation, a power-law distribution of fault sizes appears, and the fault pattern is fractal. Furthermore, the combined effect of fault clustering and the correlation between fault displacement and fault size leads to a strongly multifractal deformation pattern. We show theoretically that the multifractal spectrum depends explicitly on the exponent \(c\), which defines the size distribution of the faults, as size and displacement are correlated. For different realizations of the numerical model, we calculate the exponent \(c\), and fractal structure of the deformation through time as strain accumulates. We explore in detail the time evolution of the capacity \(D_0\), information \(D_1\), and correlation \(D_2\) fractal dimensions. We relate these scaling parameters to the physical mechanisms of fault nucleation, growth and linkage during different phases of the deformation and discuss the factors that determine the values of the exponents. A consistently observed systematic decrease in the values of \(c\), \(D_1\) and \(D_2\) through time indicates that the relative strain contribution of the smallest faults decreases as the total strain increases, a signature of the localization of faulting.

**Key words:** fault scaling relationships, long-range elastic interactions, multifractal theory, network modelling, quenched disorder.

**INTRODUCTION**

The recognition that natural fault populations exhibit specific scaling properties has provided geologists with a powerful tool for interpreting and modelling fault development in the brittle crust. These so-called scaling relationships describe the size distributions of fault lengths (or displacements), the relationship between displacement and length, and the spatial pattern of faulting. The data that have been obtained from field studies indicate that faults, in continental regions at least, exhibit power-law size distributions (e.g. Kakimi 1980; Heffer & Bevan 1990; Walsh, Watterson & Yielding 1991; Jackson & Sanderson 1992; Peacock & Sanderson 1993; Villemin, Angelier & Sunwoo 1995), spatial clustering and branching configurations characteristic of fractal patterns (King 1983; Hirata 1989; Davy, Sornette & Sornette 1990; Vignes-Adler, Le Page & Adler 1991), and a displacement–length relationship of the form \(d \propto L^n\), where \(n = 1.0\) for faults in a single rock type and tectonic setting (Cowie & Scholz 1992a; Dawers, Anders & Scholz 1993. See Gillespie, Walsh & Watterson 1992; Cowie & Scholz 1992b; Hatton, Main & Meredith 1994 for a discussion of \(n\)). These scaling relationships have been used to predict (by extrapolation) the relative numbers, the spatial distribution, and the strain contribution of subresolution-scale faulting (Scholz & Cowie 1990; Walsh et al. 1991; Marrett & Allmendinger 1990, 1991, 1992; Gauthier & Lake 1993). The success of empirical extrapolation depends on the extent to which we understand the physical processes giving rise to these relationships and the instances in which the scaling changes or breaks down. Furthermore, determining what controls the value of the various scaling exponents will dictate just how far these extrapolations may be taken. These are the problems we...
address in this paper by presenting the results of a numerical rupture model which simulates fault growth.

An explanation for power-law size and spatial, i.e. fractal, distributions of faults was put forward by A. Sornette, Davy & Sornette (1990) and developed theoretically by D. Sornette, Davy & Sornette (1990). These workers proposed that fractal fault patterns are a consequence of the threshold nature of brittle failure and the redistribution of the resulting strain perturbations according to the equations of elasticity. Because the elastic strain field obeys Laplace's equation, stress perturbations decay algebraically with distance producing interactions and thus correlation at long range. At the same time, strong stress enhancements occur at fault tips and intersections. A combination of screening range. At the same time, strong stress enhancements occur at fault tips and intersections. A combination of screening and enhancement effects thus arise spontaneously and concentrate the deformation in some areas while leaving other areas relatively undeformed. It is this feature which is the hallmark of a fractal fault pattern. A further ingredient in the theoretical explanation is the presence of 'noise' in the deforming system. This 'noise' has been interpreted as either material heterogeneity (A. Sornette et al. 1990) or the rapid, and sometimes large, stress changes induced by earthquakes (Sornette & Virieux 1992).

Recently, we presented a numerical model which simulates seismic faulting in a thin plate (Cowie, Vanneste & Sornette 1993). The plate consists of a lattice of elements, each of which can support a finite elastic strain. This model contains the essential ingredients of the theory given by D. Sornette et al. (1990) in the simplest form: it is a scalar model but both short-range and long-range elastic effects are included, rupture occurs at a critical stress threshold and causes an instantaneous stress drop, and finally the material properties of the elements comprising the plate are randomly heterogeneous. Using this model, it was found that, even without preferential weakening of elements after rupture, the rupture pattern becomes localized in both time (earthquakes) and space (faults). Both the pattern of faulting and the distribution of earthquakes were found to exhibit scaling relationships similar to natural fault and earthquake populations. Using a different numerical approach, Poliakov & Herrmann (1994) observed the same localization behaviour in shear-band formation in elastoplastic materials. There are remarkable similarities between these numerically simulated fault patterns and those generated by the analogue models on which the theory, described above, was based (see A. Sornette et al. 1990, 1993). The consistency of these very different approaches (theory, numerical simulation, and analogue modelling) thus strengthens the central hypothesis put forward by D. Sornette et al. (1990). Furthermore, the simplicity of the numerical modelling suggests that the appearance of power-law scaling in fault populations is not conditional on special causes such as pre-existing structure, boundary conditions, fault kinematics or strain softening. The numerical approach has the advantage that the influence of these additional factors can now be assessed separately and quantitatively using a forward modelling philosophy. Because the model fault populations do not suffer from the sampling and resolution problems inherent in natural data sets, we are also able to develop and test techniques for quantifying fault patterns.

In this paper we use the model of Cowie et al. (1993) to explore in detail the initiation and evolution of the fault pattern. In particular, we analyse the fractal structure and the size–frequency distribution of the faults at different stages of the deformation and show how the scaling exponents relate to the mechanisms of fault nucleation, growth and linkage. In a theoretical section we demonstrate that a power-law distribution of fault displacements superimposed on a fractal fault pattern can explain the observed multifractal structure of the deformation field. This is the first time, to our knowledge, that a non-trivial multifractal spectrum is interpreted physically as stemming from a power-law measure. The spectrum provides a naturally intuitive representation of the large strain deviations that characterize such a long-tailed distribution of displacements.

THE MODEL

The model simulates antiplane shear deformation of an elastic plate using a 2-D square lattice. The lattice is made up of 180 $\times$ 180 elements which are oriented at 45° to the plate edges. Cyclic boundary conditions are applied in the x-direction to minimize lateral edge effects. A constant velocity is applied along one edge of the lattice ($y = 180$), while the other edge ($y = 0$) is kept fixed. Each element is assigned a stress threshold $\sigma$, which is drawn randomly from a probability distribution chosen in these computations to be uniform in the interval $(1 - \Delta/2, 1 + \Delta/2)$. We have considered a range of values of $\Delta$ between 0.2 and 1.9. These threshold values remain fixed throughout a particular simulation. The elastic properties of the elements are also fixed and the same for all elements. When an element ruptures, it undergoes an instantaneous stress drop by an amount given by $\sigma \beta/2$. The parameter $\beta$ simulates the effect of seismic 'undershoot' ($0 < \beta \leq 2.0$) or 'overshoot' ($2.0 < \beta \leq 4.0$). The stress field throughout the lattice is recalculated each time a rupture occurs such that the equation of static equilibrium is satisfied at each lattice node. The parameters $\Delta$ and $\beta$ represent two types of 'noise' in the model: $\Delta$ constitutes intrinsic heterogeneity of the material (quenched disorder) while $\beta$ determines the stress perturbations produced by rupture events which are constantly varying in space and time.

As this is a scalar model we are only concerned with stress and strain magnitudes. Furthermore, there is no physical lengthening or shortening of lattice elements, so there is no condition for mass conservation. Finally, because we use a 2-D lattice, this model applies when a thin plate approximation is valid. Consequently, for a crustal-scale analogy the length of an element is of the order of 10 km and the dimensions of the plate are of the order of 2000 km. The model faults are 'large' faults, i.e. they break through the entire thickness of the plate. For a plate boundary velocity of 10 cm yr$^{-1}$, one time unit in the model corresponds to between one month and one year, depending on the elastic limit assumed for crustal materials (see Cowie et al. 1993 for details).

CHARACTERISTICS OF FAULT EVOLUTION

Cowie et al. (1993), in their analysis of the rupture model, focused on the very long time-scale pattern of rupture and in
particular the properties of the earthquake distribution observed. In this paper we concentrate instead on the early phase of rupture development to investigate how the fault pattern emerges and evolves with increasing strain. Fig. 1 shows a set of fault maps generated at three different times during a representative simulation. In each map the elements that have ruptured are plotted in shades of grey. The grey scale is normalized in each map according to two different criteria, as explained in the caption to Fig. 1. In the left-hand panel of maps (Figs 1a, b and c), the shade of grey of an element depends on its accumulated displacement; black is the maximum displacement. In the right-hand panel (Figs 1d, e and f), the shade of grey refers instead to the size of the cluster to which that ruptured element belongs. A cluster is defined as a discontinuity in the strain field in the y-direction and a continuous set of ruptured elements in the x-direction. Using geological terminology, fault segments and splays that are structurally linked belong to the same cluster according to this definition. Note that, for the smaller faults, cluster size corresponds approximately to the tip-to-tip length of the fault. Following Cowie et al. (1993), we estimate that for Figs 1(a) and (d) the total accumulated strain is approximately 0.03 per cent, in Figs 1(b) and (e) the strain has increased to 0.07 per cent, and in Figs 1(c) and (f) the strain has reached 0.1 per cent.

The maps shown in Fig. 1 illustrate the characteristics of fault development which are common to all the simulations described in this paper. Initially the pattern of faulting consists of uncorrelated nucleation of small faults (Figs 1a and d). Some, but not all, of these faults continue to accumulate displacement and grow laterally, either by breaking previously unruptured elements or by linking with other ruptures (Figs 1b and e). Also, lateral fault splays and secondary strands are often formed during this growth phase. Eventually, in Figs 1(c) and (f) we see the formation of linkage structures, analogous to segment boundaries or relay ramps described in the geological literature, which connect adjacent faults together. The general progression through time is that the number of active faults decreases while the size and displacement of a few faults gradually increases. The final pattern consists of domains of inactive and randomly distributed short faults, inherited from the early deformation phase, bounded by major fault structures that remain intermittently active at long times. These major structures are highly complex, usually consisting of several anastomosing strands with varying displacements.

The presence of ‘noise’ in this model appears to be crucial to the generation of the fault pattern. In all the simulations we discuss here $\Delta \approx \beta$. The distribution of material properties (determined by $\Delta$) strongly influences the nucleation of faults for small amounts of deformation. When growth and linkage of faults start to develop, the stress field due to the faults becomes the more dominant effect. This occurs because the history of previous ruptures on a fault acts as a source or sink in the elastic strain field, thus trapping future slip events. As the largest ruptures may only occur on the largest faults, a positive feedback is created. This process leads to the growth of a few large faults at the expense of other faults which become inactive. However, rupture-induced stress changes (determined by $\beta$) may be sufficient to perturb this pattern at any instant. In this model, such a perturbation is manifested by intermittent periods of activity along different faults. In simulations where $\beta \geq \Delta$ we found that relatively few faults nucleate initially, the strain does not localize onto a well-defined fault trace and instead a wide zone of diffuse rupture activity persists. In this case the stress field in the plate is most sensitive to the temporal form of noise, which can change significantly in short intervals of space and time. Sornette, Miltenberger & Vanneste (1994) provide a detailed discussion of this behaviour.

**ANALYSIS OF THE FAULT PATTERNS**

**Size–frequency distribution**

We have analysed the statistics of the fault patterns produced by this model for several different realizations of the material properties in the plate but maintaining all other model parameters the same. First of all, we calculated the distribution of fault sizes in each simulation as a function of time to explore how the distribution varies as strain accumulates. The size of a fault is defined as the number of elements that belong to the same cluster (e.g. see Figs 1d, e, f and caption). In Fig. 2 we show, in a block diagram, the cumulative size–frequency distributions at seven different stages of the deformation for the model shown in Fig. 1. Due to the small size of even the largest clusters during the early stages of deformation, the distribution is exponential and plots as a curve on the log–log axes in Fig. 2. The exponential distribution evolves into a power-law distribution as the faults grow in size and as large faults form by coalescence of smaller faults. In this regime, we found that the cumulative distribution of fault sizes has the following form:

$$N_{\geq n} = \left( \frac{n}{n_{\text{max}}} \right)^{-c},$$

where $n$ is the number of elements belonging to a fault, and $n_{\text{max}}$ is the size of the largest fault. The top panels of Figs 3(a), (b), (c) and (d) show the variation in the value of the exponent $c$ as a function of time in four different simulations. The parameter $c$ was calculated in each case by fitting a least-squares line through the size distribution on a log–log plot. The error associated with each value is also shown in Fig. 3 and typically is of the order of 1 per cent. Fig. 3(c) corresponds to the simulation shown in Fig. 1. We found that when a power-law size distribution first appears, $c = 1.6–1.9$, but $c$ decreases rapidly with increasing strain, reaching an asymptotic value of 1.0–1.4.

This variation in $c$ is due to the competing effects of nucleation of new faults and growth of existing faults. Expression (1) is a cumulative distribution, so that increasing the number of faults will increase the value of $c$, whereas growth, particularly through coalescence, will result in a decrease. Thus, we have plotted, in the bottom panels in Fig. 3, the variation in the total number of faults and the size of the largest fault, $n_{\text{max}}$, over the same interval of time in each case. The number of faults increases rapidly at first while the size–frequency distribution is exponential. The appearance of a power-law distribution coincides with the onset of coalescence indicated by the increase in $n_{\text{max}}$. The vertical shaded bars in Fig. 3 highlight the time interval when $n_{\text{max}}$ increases most rapidly and the total number of
faults decreases slightly because coalescence has started to out-pace nucleation. This is usually the same time period over which \( c \) decreases most rapidly. Thus, the appearance of power-law size-scaling corresponds to the transition from a nucleation-dominated regime to a regime in which growth and coalescence dominate. In our model this transition and the subsequent decrease in \( c \) are part of a natural evolution in response to the accumulation of strain. A constant value of \( c \) is finally achieved when the deformation in the plate reaches saturation, i.e. the rates of growth and nucleation tend towards zero and \( c \) reaches an asymptotic value (see Fig. 3).

A similar evolution of \( c \) has been observed in analogue experiments of fault growth (A. Sornette et al. 1990, 1993). In these experiments \( c \) decreases from approximately 2.0 as deformation proceeds and reaches a value of 1.0 once a major fault runs right across the model and becomes the locus of virtually all the subsequent deformation. Sornette & Davy (1991) argued theoretically that \( c = 1.0 \) is the ‘attracting’ value for a fault population in a self-organized critical state in which deformation proceeds only by fault growth. However, the present model demonstrates that \( c \) will depend on the relative rates of both nucleation and growth. When deformation is accompanied by rotation, fault growth will be frustrated and we expect new faults to nucleate to accommodate the rotation (e.g. King 1983; Scotti, Nur & Estevez 1991). As tectonic rotations are ubiquitous, it seems unlikely that \( c = 1.0 \) should be a universal value. For example, if fault growth causes rotation, which in turn requires new faults to form, then the analysis of Sornette & Davy (1991) does not apply. However, in the absence of large rotations, we expect, from the results presented here, a strong correlation between the value of \( c \) and the total amount of deformation. If this is the case in nature, we may be in a position to predict fault populations from independent estimates of deformation, e.g. subsidence analyses in a sedimentary basin.

Another important point to be made here is that at any point in time the fault size distribution contains many faults that are no longer active, i.e. the value of \( c \) is controlled by both active and inactive fault populations. A measure of the size distribution of the active faults alone is given by the seismic \( b \) value of the earthquakes exhibited by this model, which we found to be approximately 0.45 at large times (Cowie et al. 1993). In general, we expect \( b < c \) because \( b \) is a measure of deformation increments, whereas \( c \) reflects many individual increments integrated over long time-scales.

The exact numerical values of \( c \) exhibited by this model are not directly equivalent to the size–frequency distribution exponent quoted for natural fault populations. In real data sets, fault length is typically the parameter measured; here we calculate cluster size which is the aggregate length of fault segments and splays that are structurally linked. In spite of this, we obtain \( c \) values in the range 1.0 \( \leq c \leq 2.0 \), which is approximately the range of real fault populations when sampled in two dimensions (e.g. Scholz & Cowie 1990; Marrett & Allmendinger 1991, 1992; Walsh et al. 1991; Scholz et al. 1993). This implies that size and length are approximately equivalent, which is probably the case for the smallest faults, or that these two parameters are correlated.

**Strain partitioning**

In these model simulations the strain across the plate is continually increasing through time as a result of the constant velocity boundary condition applied to the plate edge. However, this strain is being taken up by different faults at different stages during the evolution. Here we investigate the relative strain contribution of faults of differing size to the total accumulated strain as a function of time for the simulation shown in Fig. 1 and analysed in Fig. 3(c). At each time step considered in Fig. 3(c) we calculated the total strain and the proportion of this total accounted for by faults

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**Figure 1.** Evolution of the fault pattern as a function of time. Each plot is shown using shades of grey according to a normalized scale so that white is zero and black is the largest value in each case. The left-hand column of maps shows the accumulated displacement of each ruptured element at three points in model time, \((a) t^* = 50,000\), \((b) t^* = 100,000\), and \((c) t^* = 150,000\). Note that one model time unit equals between one month and one year. The right-hand column of maps \((d, e, f)\) shows the same three stages in the evolution but here the elements are shaded according to the size \((n)\) of the fault to which they belong. A fault is defined as a discontinuity in the strain field in the \( y \)-direction and a continuous set in the \( x \)-direction. Note that, for the shorter faults, fault size corresponds approximately to the tip-to-tip length \((L)\) whereas for the larger structures \( n > L \) because multiple branches and strands are included in our size definition.

**Figure 2.** 3-D block diagram showing how the cumulative size–frequency distribution of the faults changes through time for the simulation shown in Fig. 1. Initially the distribution is curved (data set 1). Once power-law size scaling develops (data sets 2–7) the slope of the distribution on log–log axes gives the value \( c \) in eq. (1). Data set 1 corresponds to Figs 1 (a) and (d), data set 3 corresponds to Figs 1 (b) and (c), and data set 4 corresponds to Figs 1 (c) and (f).
Figure 3. Variation of the size distribution exponent $c$ (top panels), the total number of faults (bottom panels, open diamonds), and the size of the largest fault, $n_{\text{max}}$ (bottom panels, filled diamonds), for four different simulations. The shaded areas indicate the regime where fault growth and coalescence start to dominate over nucleation in each example. Fig. 3(c) corresponds to the fault data shown in Figs 1 and 2.

less than or equal to a particular size, $n$. The results of this analysis are shown in Fig. 4; each curve represents a different time step plotted on normalized axes for comparison. The vertical dashed line indicates the strain accounted for by those faults that are less than half the size of the largest fault at each time considered. We found that, during the initial stages of the deformation (Fig. 1 a), at least 90 per cent of the strain is accounted for by the smallest half of the fault population. However, as time progresses and the total strain increases this proportion drops to only 40 per cent, the other 60 per cent being accommodated by the largest faults in the population. This changing behaviour is a corollary of the decreasing value of $c$ shown in Fig. 3(c) and indicates that the strain becomes more strongly convergent as the deformation proceeds. ‘Convergence’ in this context refers to the proportion of the total strain accounted for by faulting at different scales in a deformed region (see Scholz & Cowie 1990). Strongly convergent means that the largest faults account for virtually all of the strain, whereas weakly convergent means that the contribution from the smallest faults in a population is significant.
Scaling properties of a fault population

There has been considerable disagreement in the literature about the importance of small-scale faulting in regional strain. Scholz & Cowie (1990) argued that faults smaller than the thickness of the brittle crust may only contribute less than 10 per cent to the total fault strain budget. This conclusion was based on data from major plate boundary faults in Japan characterized by an exponent \( c = 1.0 \). In contrast, Walsh et al. (1991) and Marrett & Allmendinger (1992) analysed faults in extensional basins in the North Sea and obtained \( c = 1.5 \) so that as much as 40–50 per cent of the total in this case may be accounted for by faults with displacements of less than 100 m. These conflicting opinions could be explained by the results of the present model, i.e. the relative importance of small faults decreases as the total strain increases due to progressive localization and coalescence.

**Spatial distribution**

In order to analyse the spatial distribution and clustering properties of the model fault populations, we have performed a multifractal analysis of the displacement maps shown in Fig. 1 (a, b and c) using a box-counting algorithm. We consider each broken element as a point in the x–y plane and to each point we assign a weight equal to the amount of accumulated displacement on that element. This type of analysis therefore reflects not only the pattern of faulting but also the degree to which the displacement is concentrated in particular areas. The method involves calculating the ‘moments’ of the displacement distribution using

\[
M_q(r) = \sum_{i=1}^{N_q} P_i^q,
\]

where \( P_i \) is proportional to the sum of the displacements on all the elements in a box of size, \( r \), normalized by the total displacement of all the ruptured elements in the lattice. For this model, Cowie et al. (1993) verified that \( M_q(r) \sim r^{(q-1)D_q} \), which defines the fractal dimensions \( D_0 \) (capacity), \( D_1 \) (information) and \( D_2 \) (correlation), corresponding respectively to \( q = 0, 1, 2 \) (Grassberger & Procaccia 1983; see also Roux & Hansen 1990). Note that \( D_0 \) does not depend on the relative number or weighting of the elements in a particular box because \( P_i^0 \) always equals 1 unless a box is unoccupied. The information and correlation dimensions \( (D_1 \) and \( D_2 \)) depend on the clustering of ruptured elements as well as their displacements. As the value of \( q \) increases the regions where the faulting is most concentrated, and/or the displacements greatest, have increasing importance in eq. (2). This is illustrated schematically in Fig. 5 where we show a pattern of faults overlaid by a grid of boxes of size \( r \). The shading is used in Fig. 5 to indicate which boxes will contribute the most to the moment summation as \( q \) changes. Unoccupied boxes are shown white because they do not contribute to the moment sum. In Figs 5(b) and (c) the traces of the faults are shown with different line thicknesses to illustrate the observation that longer faults generally have greater displacements (e.g. Walsh & Watters 1988;
Figure 5. Sketch illustrating the method for calculating the multifractal structure of faults. See eq. (2) and text for explanation.

Marrett & Allmendinger 1991; Cowie & Scholz 1992b; Gillespie et al. 1992; Dawers et al. 1993). In Fig. 5(a), \( q = 0 \) and the grey shading is uniform, but for \( q > 0 \) the boxes that include the most or the largest displacement faults have the darkest shading.

Figure 6 shows the variation of \( D_0 \) and \( D_2 \) as a function of time during the evolution of faulting for the same four model simulations shown in Fig. 3. Note that Fig. 6(c) corresponds to the model results shown in Fig. 1. We found that \( D_0 \) always equals 2.0 because this parameter is insensitive to the distribution of faults and the pattern of displacements. Early in the simulations \( D_0 = D_1 = D_2 = 2.0 \), which indicates that the fault pattern approximates a homogeneous fractal and is space filling. However, a multifractal pattern gradually emerges, i.e. \( D_0 > D_1 > D_2 \).

The transition from a homogeneous pattern to multifractality is indicated by the vertical dashed lines. Note that the multifracture structure develops at approximately the same point as the appearance of the power-law size scaling shown in Fig. 3. The difference between \( D_1 \) and \( D_2 \) becomes more pronounced as time progresses or, in other words, the multifractality becomes stronger as the deformation proceeds. This is reflecting the fact that the deformation becomes localized onto a few major faults which accumulate large displacements. Eventually, \( D_1 \) and \( D_2 \) approach approximately constant values. But this is only achieved after a period of time considerably longer than that required for the evolution of the exponent \( c \).

In fact, eq. (2) can be calculated for a spectrum of both positive and negative values of \( q \), i.e. \( -\infty < q < \infty \). When \( q \) is positive, the moment summation is sensitive to the high strain regions as we have already shown; when \( q \) is negative, it is sensitive to regions of small or diffuse deformation. Thus, by considering a range of values of \( q \) we can obtain a measure of the heterogeneity of the deformation pattern. This is done by calculating \( \alpha \), using

\[
\alpha(q) = \frac{1}{\log(r)} \frac{\sum_{i=1}^{N(r)} P_i^{q} \log P_i}{M_q(r)},
\]

and \( M_q(r) \) is from eq. (2). The multifractal spectrum is then given by \( f(\alpha) \), where

\[
f(\alpha) = aq - D_q(q - 1).
\]

The correspondence between the moments of a distribution and \( f(\alpha) \) is that \( q \) is the slope of \( f(\alpha) \) at the point \((\alpha, f(\alpha))\). Fig. 7 shows the evolution of \( f(\alpha) \) as a function of time for the simulation shown in Fig. 1. For values of \( \alpha < 2.0 \) the slope of \( f(\alpha) \) is positive so that \( q > 0 \) and quantifies the large strains. Conversely, for \( \alpha > 2.0 \) the slope of \( f(\alpha) \) is negative so that \( q < 0 \), which quantifies the small strains. By definition, the maximum value of each curve corresponds to \( f(\alpha_{\alpha=0}) = D_0 \), the dimension of the support which in this case equals 2.0. Korvin (1992; p. 315) provides a useful and general explanation of multifractal theory in the context of earth science applications.

In spite of the rather large error bars, it is clear that the spectra \( f(\alpha) \) become broader with time. This is due to the fact that the distinction between large strain concentrations and relatively undeformed areas becomes stronger as the deformation localizes. For \( \alpha < 2.0 \) the spectral shape gradually converges indicating that strain localization is on structures that, once established, change less and less through time. Conversely, for \( \alpha > 2.0 \) the spectra diverge and the error bars are much larger, due to the fact that the low strain regions are incoherent both in the pattern of faulting and in the distribution of displacements. This incoherence reflects the random pattern of material properties, which controls the nucleation of the smaller faults, as well as the irregular temporal evolution of the small-scale faulting. For example, we have seen that the smaller faults are generally active for only short periods of time; they may become completely inactive in some areas, or remain intermittently active in the vicinity of one of the major faults zone.

Most published fractal analyses of natural fault and fracture populations quote only the capacity dimension, \( D_0 \) (e.g. Barton & Hsieh 1989; Villemin et al. 1995). However, several authors have argued that \( D_0 \) is sensitive to the shape of the faulted domain (Walsh & Watterson 1993) and the resolution threshold (Cowie et al. 1993), and thus is not a good measure of the true structure. Fig. 5 illustrates how multifractal analysis can give a great deal more information reflecting both the spatial distribution of faulting and
Scaling properties of a fault population

Theoretical explanation for multifractal scaling

The $f(\alpha)$ spectrum provides a mathematically precise and naturally intuitive description of the multifractal structure. For example, we can also define $\alpha$ in the following way:

$$P_i(r) = r^{\alpha}$$

where $P_i(r)$ is the normalized measure of the deformation in the $i$th box of size $r$. The number $N(\alpha)$ of boxes where $P_i$ has a singularity strength between $\alpha$ and $\alpha + \Delta \alpha$ is then given by

$$N(\alpha) = r^{-f(\alpha)}$$

where $f(\alpha)$ is the Hausdorff dimension of the spatial distribution of $\alpha$ values. We see from Fig. 7 that, at large times, the large strains are characterized by $\alpha_{q=0} = 1.0 - 1.5$, while the smallest strains are characterized by $\alpha_{q=0} = 3.0$. In order to understand the meaning of these $\alpha$ values it is easiest to compare them to $\alpha_{q=0}$, as $q = 0$ is insensitive to variations in the amount of displacement and degree of clustering. Compared to a uniform deformation, i.e. substituting $\alpha_{q=0} = 2.0$ in eq. (6), the largest displacements are fewer in number and are arranged along approximately linear zones, $P(r) \sim r^1$, whereas the lowest strain regions are characterized by many more small displacement faults which are anti-clustered over an area according to $P(r) \sim r^3$. (The term 'anti-clustering' is used here to describe faults that locally relax stress and inhibit magnitude of fault strain. A multifractal box-counting method similar to that presented here was used by Vignes-Adler et al. (1991) to analyse fracture patterns interpreted from satellite images and aerial photographs of West Africa, but the multifractality was found to be weak, such that $D_q(1 \leq q \leq 6) \approx 1.46$. Ouillon, Sornette & Castaing (in press) present a good discussion of multifractal techniques for analysing natural fault populations and the results of a study of the Tayma province of Arabia. They found a multifractal structure using only fault trace maps and obtained $D_{-1} = 2.0$, $D_1 = 1.94$, $D_2 = 1.85$, $D_{++} \approx 1.7$ for scales ranging from 12 km to 100 km. These values are comparable to our results shown in Fig. 6 at large times, in spite of the resolution limitations and lack of displacement information in the real data sets. Poliakov & Herrmann (1994) analysed their numerical results in essentially the same way as we do here and also found a strongly multifractal structure with $D_{-1} = 1.7$, $D_1 = 1.55$, $D_2 = 1.4$. However, analysing the time evolution of a multifractal fault structure is unique to this study as far as we know. The technique has been applied previously to earthquakes (e.g. Hirata, Satoh & Ito 1987; Henderson et al. 1992). A temporal analysis is obviously very difficult to do for natural data sets but it is interesting to note that Ouillon et al. (in press) found a difference between large-scale and small-scale fault maps; at scales of only 1 m to 5 km they found the fault pattern to be homogeneous and space filling, i.e. $D_0 \approx D_1 \approx 2.0$. Making an analogy with the model presented here, we might interpret these small-scale faults as representing an early phase of the deformation and preserved between major fault systems which take up most of the deformation.

Figure 6. Time evolution of the information dimension, $D_1$ (open circles), and correlation dimension, $D_2$ (points), for the four different realizations of the model shown in Fig. 3. Note that $D_0$ is always 2.0 for this model. All curves show $D_1 = D_2$ during the early stages of the deformation but with increasing time $D_1 > D_2$. The dashed line indicates where the multifractal structure starts to develop strongly; this coincides with the appearance of a power-law size distribution (see shaded areas in Fig. 3). At large times $D_1$ and $D_2$ reach approximately constant values.
the nucleation of adjacent faults.) Thus, the multifractal spectrum is reflecting both the relative number of faults having a given displacement and the degree of fault clustering as a function of displacement. In other words, the term multifractal derives from superimposing a power-law (i.e. fractal) distribution of fault displacements onto a fractal fault pattern.

In order to illustrate this point, we have taken the fault data shown in Fig. 1 and recomputed the multifractal spectrum making three different assumptions. The results are shown in Fig. 8. Curve A is exactly the same as that shown in Fig. 7 for time $t_4$ and is thus the unadulterated model fault data set. To calculate curve B we set all the displacements to exactly the same value in order to isolate variations in spatial clustering. Curve C was calculated using the actual values for the displacements but reassigning these values randomly to the ruptured elements so that the dependence of clustering on displacement was destroyed. Finally, curve D was obtained by assuming all the elements belonging to the same fault have the same displacement,

![Figure 7. Multifractal spectra calculated at four different times: $t_1 = \text{(triangles)}, t_2 = \text{(crosses)}, t_3 = \text{(squares)}, \text{ and } t_4 = \text{(circles)}$ for the simulation shown in Fig. 1. Each spectrum is obtained by first calculating $M_h(r)$ (eq. 2) and $L_s(r)$ (eq. 4) to obtain $D_h$ and $\alpha$ and then calculating $f(\alpha)$ using eq. (5). The error bars $\Delta(f(\alpha))$ were obtained using $\Delta(f(\alpha)) = |\alpha| \Delta(\alpha) + |\alpha - 1| \Delta(D_h)$. In model time units $t_1 = 50000, t_2 = 110000, t_3 = 180000, t_4 = 250000.\]

![Figure 8. Plot demonstrating the origin of the multifractal structure by comparing the multifractal spectrum shown in Fig. 7 for time $t_4$ (curve A: circles) with three other hypothetical fault patterns: curve B (triangles) assumes that all the faults formed at time $t_4$ have exactly the same displacement; curve C (points) was obtained using the observed displacements but they were randomly reassigned along the fault traces; curve D (squares) assumes that each fault has a constant value of displacement along its entire length such that $d(x) = d_{\text{max}}$, where $d_{\text{max}}$ is the observed maximum value on each fault. Note that curves A and D are very similar for $\alpha \approx 2.0$, whereas curves B and C differ greatly from curve A for all values of $\alpha$. See text for discussion.\]
Scaling properties of a fault population

probability that \( \epsilon = P(r) \), is given by

\[
\text{Prob}(r^2d = r^\alpha) = \text{Prob}(d = r^\alpha - 2),
\]

and substituting the right-hand side of eq. (9) into eq. (8) we obtain

\[
\text{Prob}(\epsilon = r^\alpha) = r^{-\phi(\alpha - 2)\alpha(\alpha - 2)}.
\]

Note that the width of the interval \( d(d) \) is proportional to \( d \) in a distribution that exhibits power-law scaling. To obtain an expression for \( f(\alpha) \) we use the definition of the Hausdorff dimension equal to the maximum value attained on that fault, i.e. \( d(x) = d_{\text{max}} \). As maximum displacement and fault size are correlated, this is similar to performing the fractal analysis on Figs 1(d)–(f) instead of Figs 1(a)–(c). We found that the pattern of faulting in the absence of displacement information forms an approximately homogeneous fractal with \( D_0 = D_1 = D_2 = \cdots = 2.0 \) (curve B), and suppressing the clustering but including the displacement distribution produces a spectrum that is only slightly broader (curve C).

Neither curve B nor curve C adequately explains the model data. However, the correspondence between curve D and curve A, at least for large displacements \( \alpha < 2.0 \), indicates that multifractality in the original fault data set is due to the fact that the largest displacements are concentrated or clustered along the largest faults, as is indeed the case in Fig. 1. This result implies that the function \( (\alpha, f(\alpha)) \) depends on the frequency distribution of fault sizes, characterized by the exponent \( c \), because a decrease in \( c \) with increasing strain means more large faults are forming which should produce a more pronounced multifractality. This is indeed what we observe (see Figs 2, 6 and 7).

The relationship between the function \( (\alpha, f(\alpha)) \) and \( c \) can be derived explicitly if we assume that \( d(x) = d_{\text{max}} \) for each fault and if we exploit the fact that \( d_{\text{max}} \) correlates with fault length, as previously discussed in relation to Fig. 5. In other words, the variability in displacement along the same fault is removed so that small displacements can only occur on small faults. Let us call \( d_j \) the displacement on the \( j \)th element of the lattice, and let us assume that the \( d_j \) are distributed according to a power law, i.e.

\[
\text{Prob}(d_j) = d_j^{\phi-d_j} \quad (8)
\]

where \( \phi \) is an exponent that we discuss below. In a box of size \( r \), there are \( r^2 \) elements, so that the cumulative strain in that box is proportional to \( r^2d \), which we denote \( \epsilon \). We have also shown that a spatial measure of the intensity of deformation is given by eq. (6), i.e. \( P(\epsilon) = r^\alpha \), and that \( \alpha \neq 2 \) when the fault displacements are included. The

\[
(\alpha, f(\alpha)) = 2 + (\phi - 1)(\alpha - 2) \quad (10)
\]

which gives

\[
f(\alpha) = 2 + (\phi - 1)(\alpha - 2). \quad (13)
\]

In Fig. 9 we have plotted eq. (13) using different values of \( \phi \) and compared it with the multifractal spectra obtained at two different times \( (t_1 \text{ and } t_2) \) during the simulation shown in Fig. 1, assuming that each fault has a uniform displacement equal to the maximum (as for curve D in Fig. 8). Fig. 10 shows the histogram distribution of fault displacements at the same two points in time, which we use for obtaining \( \phi \) (eq. 8). Comparing Figs 9 and 10, we see that the decrease from \( \phi = 3 \) to \( \phi = 2 \) for the largest displacements is consistent with the spectra broadening from time \( t_1 \) to time \( t_2 \) in the region \( \alpha = 2.0 \). The theory fits the data less well for \( \alpha > 2.0 \), although the large error bars in this region mean that we have less confidence in the spectral shape. In fact, small changes in slope from \( \phi = 0 \) to \( \phi = 0.5 \) produce a very large change in the predicted spectral shape.

Figure 9. Dashed lines show theoretically predicted spectral shape \( f(\alpha) = 2 + (\phi - 1)(\alpha - 2) \) (eq. 13) for \( \phi = 0, 0.5, 1, 2 \) and 3; curves show the multifractal spectra calculated at time \( t_1 \) (solid squares) and \( t_2 \) (open squares), assuming in each case that the faults have uniform displacement profiles with \( d(x) = d_{\text{max}} \).
As fault displacement and fault length (or size) are correlated (see above) then we can derive eq. (13) in terms of the exponent $c$ instead of $\phi$. However, $\phi$ defines the histogram distribution whereas $c$ defines a cumulative distribution, so that $\phi = c + 1$, i.e. $c(t_1) = 2$, $\phi(t_1) = 3$, $c(t_4) \approx 1.4$, $\phi(t_4) \approx 2$. In this case eq. (13) becomes $f(\alpha) = 2 + c(\alpha - 2)$.

Note that the exponent $\phi$ for small displacement faults differs from those with large displacements, reflecting their different physical origins. The interpretation of $\phi = 0$ is that $\text{Prob}(d)$ is constant, which is what we expect from a diffuse nucleation of small faults in the early regime where the heterogeneity of the material properties dominates; as the heterogeneity is uniform we expect the associated strain field to be uniform. (Note that $\phi = 1$ is a special case because $f(\alpha) = 2.0$ for all $\alpha$.) The fact that $\phi$ is not a constant means that the fault pattern is not scale-invariant even though it is fractal. If we had used a fractal rather than a uniform distribution of material properties, we would expect to find $\phi > 0$ at all scales because $\text{Prob}(d)$ will not be constant even for the smallest faults in this case. Therefore, if this model is a realistic analogue for crustal deformation we might conclude that strictly scale-invariant fault and fracture patterns can only develop if the material heterogeneity is fractal over the same range of scales as the structures that form therein. It could be argued that this will be the case if the crust inherits a texture from previous tectonic events. An obvious example where this will not be the case is in recently deposited sedimentary rocks where the material properties are determined by depositional processes.

**CONCLUSIONS**

Using a numerical rupture model which simulates the nucleation and growth of faults in a tectonic plate, we have been able to document the origin and evolution of fractal fault patterns. We found that three ingredients appear to be crucial to producing this phenomenon. First, the material in which the faults form deforms elastically below a critical threshold for rupture so that long-range and short-range elastic strains can be sustained. This leads to correlation in the fault pattern which is the hallmark of fractal patterns. The second ingredient is that rupture of the material occurs at a critical threshold and produces an abrupt stress drop. The resulting stress perturbation scales with the size of the rupture event and leads to a positive feedback which concentrates the deformation. Thirdly, the presence of 'noise' in the deforming system is very important. In this model a uniformly random material structure constitutes a quenched disorder while the stress perturbations produced by the rupture process introduce noise that varies in space and time. The heterogeneity in the material properties controls the nucleation of faults when deformation commences and leads to certain faults being the preferred loci of future growth and displacement accumulation. The time-varying noise source results in shifting activity between different faults in the population. Factors previously considered to be important in fault development, for example, pre-existing structure, kinematics, and strain softening, were not required to produce the realistic fault patterns obtained here.

We have analysed in detail the evolution of the size–frequency distribution and the diagnostic fractal dimensions of the faults produced during progressive deformation of the plate. The fractal analysis uses a box-counting algorithm which includes the displacements as well as the positions of the faults and thus quantifies the strain pattern produced by faulting. The earliest phase of the deformation is characterized by random nucleation of faults across the entire plate which have an exponential size distribution and form a homogeneous fractal pattern such that the capacity dimension ($D_0$), the information dimension ($D_1$) and the correlation dimension ($D_2$) are all approximately equal to 2.0. With increasing deformation, fault growth and coalescence begin to dominate over nucleation, a power-law distribution of sizes appears and $D_0 > D_1 > D_2$. In this regime we characterize the size distribution by the component $c$ and show that $c$, $D_1$ and $D_2$ all decrease as the deformation proceeds until an asymptotic regime is achieved, at which time the fault pattern is fully formed. At this stage the plate has become divided up into blocks that are relatively undeformed and tectonically quiescent, bounded by major fault zones along which the continuing rupture activity and the largest displacements are concentrated.

The decrease in $c$, $D_1$ and $D_2$ through time is indicative of progressive localization of strain in the plate. The amount of strain accounted for by the smallest faults in the population, when compared to the largest faults, also progressively decreases as $c$ decreases. Thus we have been able to show that the relative importance of small-scale faulting in the total strain budget actually depends on the amount of strain accommodated. Finally, we obtained the multifractal spectrum of the strain pattern through time by calculating the generalized dimensions $D_q$ for $q$-positive (sensitive to large strains) and $q$-negative (sensitive to small strains) and showed theoretically how the spectrum may be related to the exponent $c$. The relationship we obtain is indicating that the largest displacements occur along the longest faults (i.e. large strains concentrated in linear zones), whereas the smallest displacements primarily occur on the smallest faults which are randomly distributed (i.e. low strains extend over broad areas). This is the first time that the size distribution
of faults has been related explicitly to the spatial distribution
and, furthermore, it is the first time, to our knowledge, that a
non-trivial multifractal spectrum has been interpreted
physically as stemming from a power-law measure.

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REFERENCES

properties of fractures, in 28th International Geological
Congress Field Trip Guidebook T385, p. 36, American
Geophysical Union, Washington, DC.

displacement-length relationship of faults using a post-yield

relationship for faults: Data synthesis and discussion, J. struct.
Geol., 14, 1149–1156.

model for the spatio-temporal evolution of faults, J. geophys.

of a proposed fractal nature of continental faulting, Nature,
348, 56–58.

normal faults: Displacement-length scaling, Geology, 21,
1107–1110.

Gauthier, B.D.M. & Lake, S.D., 1993. Probabilistic modeling of
faults below the limit of seismic resolution in Pelican Field,
Bull., 77, 761–777.

Gillespie, P., Walsh, J.J. & Watterson, J., 1992. Limitations of
displacement and dimension data from single faults and the
correlations for data analysis and interpretation, J. struct.
Geol., 14, 1157–1172.


Hefter, K.J. & Bevan, T.G., 1990. Scaling relationships in natural

Henderson, J.R., Main, I.G., Meredith, P.G. & Sammonds, P.R.,
1992. The evolution of seismicity at Parkfield, California:
Observation, experiment, and a fracture-mechanical interpreta-
tion, J. struct. Geol., 14, 905–914.

structure in rock fracture geometry at various scales, Pageoph,
131, 157–170.

distribution of microfracturing in rock, Geophys. J. R. astr.

Jackson, P. & Sanderson, D.J., 1992. Scaling of fault displacements
from the Badajoz-Cordoba shear zone, SW Spain, Tectonophysics,
210, 179–190.

Kakimi, T., 1980. Magnitude-frequency relation for displacement of
minor faults and its significance in crustal deformation, Bull.

King, G., 1983. The accommodation of large strains in the upper
lithosphere and other solids by self-similar fault systems: The
geometrical origin of b-value, Pageoph, 124, 761–816.

Amsterdam.

Marrett, R. & Allmendinger, R.W., 1990. Kinematic analysis of

brittle faulting: sampling of fault populations, J. struct. Geol.,
13, 735–738.

Marrett, R. & Allmendinger, R.W., 1992. Amount of extension on
'small' faults: An example from the Viking Graben, Geology,
20, 47–50.

Ouillon, G., Sornette, D. & Castaing, C., 1995. Organisation of
joints and faults from 1 cm to 100 km scales revealed by
Optimization Anisotropic Wavelet Coefficient Method and
multifractal analysis, Nonlinear Processes in Geophysics, in
press.

Peacock, D.C.P. & Sanderson, D.J., 1993. Strain and scaling of
faults in the chalk at Falmborough Head, UK, J. struct. Geol.,
16, 97–108.

of plastic shear bands in rocks, Geophys. Res. Lett., 21,
2143–2146.

Roux, S. & Hansen, A., 1990. Introduction to multifractality, in
Disorder and Fracture, pp. 17–30, ed. Charmet, J.C., Roux, S.


and block rotation in three dimensions, J. geophys. Res.,
96, 12225–12243.


Sornette, A., Davy, Ph. & Sornette, D., 1993. Fault growth in
brittle-ductile experiments and the mechanics of continental

Sornette, D. & Davy, Ph., 1991. Fault growth model and the
universal fault length distribution, Geophys. Res. Lett., 18,
1097–1081.

Sornette, D. & Virieux, J., 1992. A theory linking large time
tectonics and short time deformations of the lithosphere,

Sornette, D., Davy, Ph. & Sornette, A., 1990. Structuration of the
lithosphere in plate tectonics as a self-organized critical

physics of fault patterns self-organised by repeated
earthquakes, Pageoph, 142, 491–527.

of fracturing in two African regions, from satellite imagery to
ground scale, Tectonophysics, 196, 69–86.

of faults extents and offsets: Implications for brittle
deforestation evaluation, the Lorraine coal basin (NE France),
in Fractals in the Earth Sciences, pp. 205–266, eds Barton, C.C.
& LaPointe, P.R., Plenum Press, New York.

between displacements and dimensions of faults, J. struct.
Geol., 10, 239–247.

using the standard box-counting technique: valid and invalid

Walsh, J., Watterson, J. & Yielding, G., 1991. The importance of