4 For constant wall temperature the total heat transfer for laminar flow is not appreciably changed by the oscillations. This agrees with the findings in [4, 6, and 7].

In this analysis the wall heat capacity and the axial heat conduction in the fluid have been neglected. These factors would be expected to moderate the responses given here.

The results obtained here are restricted to laminar flow. For oscillating flow this may require a lower critical Reynolds number than the value of about 2100 which applies for steady conditions. In [4] the critical Reynolds number based on the mean velocity for flow in a tube was found to be reduced to about 1500 when flow pulsations were present.

In the cases considered here the channel was unheated up to $X = 0$ and then heated thereafter. Hence there is a step function in the thermal boundary condition at $X = 0$. With the solution now known for this step function it would be possible, by superposition of results for steps at various X-values, to build up solutions for arbitrary variations in the boundary conditions with X.

The analysis given here can be carried out in the same manner for the circular tube geometry.

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DISCUSSION

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If we are interested only in the heat transfer far away from the thermal inlet length, the problem treated in this paper can be largely generalized: The slug-flow assumption can be relaxed; a wall temperature distribution varying linearly with x (with slope A) is also admissible; and most important of all, free-convection effects due to a body force per unit mass g in the negative x-direction, with pressure gradient term f, heat generation Q, and wall temperature all oscillating, can be investigated. A brief account follows with a discussion of suitable methods of solution, including one using the variational principle, for various shapes of cross section of the duct.

Similar to the steady case,[3] we have[4]

$$\rho \frac{\partial u}{\partial \tau} = \rho \beta g v - f(\tau) + \mu \nabla^2 u \tag{35}$$

$$\rho c \left(\frac{\partial v}{\partial \tau} + A u \right) = k \nabla^2 v + Q(\xi, \eta, \tau) \tag{36}$$

where ∇^2 is the two-dimensional Laplacian in the general coordinates ξ and η perpendicular to x, $v(\xi, \eta, \tau)$ is the temperature difference referred to the temperature on a certain element of the wall, f is $\partial p/\partial x$ plus g times ρ evaluated on this element, c is the specific heat capacity. On the boundary Γ of the cross section, v is prescribed generally by

$$v_\Gamma(s, \tau) = \sum_{n=0}^{\infty} [v_{\Gamma nc}(s) \cos n\omega\tau + v_{\Gamma ns}(s) \sin n\omega\tau]$$

$$= \text{Re} \sum_{n=0}^{\infty} v_{\Gamma n}(s) \exp (in\omega\tau)$$

where $v_{\Gamma n} = v_{\Gamma nc} - i v_{\Gamma ns}$, s is the arc length parameter along Γ, and Re denotes the real value. We assume similarly that

$$f = \text{Re} \sum_{n=0}^{\infty} f_n \exp (in\omega\tau)$$

and

$$Q = \text{Re} \sum_{n=0}^{\infty} Q_n(\xi, \eta) \exp (in\omega\tau)$$

Then, disregarding transients, we have u and v in the forms of

$$u = \text{Re} \sum_{n=0}^{\infty} u_n(\xi, \eta) \exp (in\omega\tau)$$

and

$$v = \text{Re} \sum_{n=0}^{\infty} v_n(\xi, \eta) \exp (in\omega\tau)$$

which, on substituting into equations (35) and (36), yield

$$\mu \nabla^2 u_n - i\rho n\omega u_n + \rho\beta v_n = f_n \tag{37}$$

and

$$k\nabla^2 v_n - i\rho c n\omega v_n - \rho c A u_n = -Q_n \tag{38}$$

with

$$\left. \begin{array}{l} u_n = 0 \\ v_n = v_{\Gamma n} \end{array} \right\} \text{ on } \Gamma \tag{39}$$

The steady part ($n = 0$) of this system has been solved for various shapes of Γ.[3] For $n \neq 0$, similar methods can also be used. The simpler problems that depend only on one space

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[3] See, e.g., P-C. Lu, "Combined Free and Forced-Convection Heat-Generating Laminar Flow Inside Vertical Pipes With Circular Sector Cross Section," Journal of Heat Transfer, Trans. ASME, Series C, vol. 82, 1960, pp. 227–232.

[4] The two equations become uncoupled when g or A vanishes.

variable (i.e., parallel plate channel, circular pipe or annulus, with compatible v_Γ and Q) have (37) and (38) as a system of ordinary differential equations and can be solved by standard methods. A more general method would be that of finite integral transforms[5] if Γ is a rectangle, a circular sector, a circle, two parallel lines, or concentric circles. By introducing suitable integral transforms, (37) and (38) can be reduced to two ordinary differential equations or even two algebraic equations, the solutions of which give, via suitable inverse formulas, u_n and v_n. However, the results, being in the forms of infinite series, may not be convenient to use. Approximate solutions in closed forms can be obtained by the following variational principle.

Setting

$$\begin{aligned} \bar{\nabla}^2 &= a^2\nabla^2 \quad (\bar{\xi} = \xi/a, \quad \bar{\eta} = \eta/a) \\ \bar{u}_n &= u_n/u_{0m} \\ \bar{v}_n &= kv_n/\rho c A u_{0m} a^2 \\ \bar{Q}_n &= Q_n/\rho c A u_{0m} \\ \bar{f}_n &= f_n a^2/\mu u_{0m} \end{aligned}$$

where a is a characteristic length and u_{0m} is the mean of u_0, we can nondimensionalize[5] equations (37) and (38) as

$$\bar{\nabla}^2\bar{u}_n - 2\,in\,M^2\,\bar{u}_n + Ra\bar{v}_n = \bar{f}_n$$

$$\bar{\nabla}^2\bar{v}_n - 2\,in\,M^2\mathrm{Pr}\,\bar{v}_n - \bar{u}_n = -\bar{Q}_n$$

where $Ra = \epsilon^4 = \rho^2 c\beta A g a^4/k\mu$.

This can be further reduced by introducing

$$\bar{V}_n = \epsilon^2\bar{v}_n$$

to

$$\bar{\nabla}^2\bar{u}_n - 2\,in\,M^2\bar{u}_n + \epsilon^2\bar{V}_n = \bar{f}_n \tag{40}$$

$$-\bar{\nabla}^2\bar{V}_n + 2\,in\,M^2\,\mathrm{Pr}\,\bar{V}_n + \epsilon^2\bar{u}_n = \epsilon^2\bar{Q}_n \tag{41}$$

with

$$\left.\begin{aligned} \bar{u}_n &= 0 \\ \bar{V}_n &= \epsilon^2\bar{v}_{\Gamma n} \end{aligned}\right\} \quad \text{on} \quad \Gamma \tag{42}$$

This system can be shown to be equivalent to stating that either the integral

$$I_C = \iint_S \left[\left(\frac{\partial\bar{u}_n}{\partial\bar{y}}\right)^2 + \left(\frac{\partial\bar{u}_n}{\partial\bar{z}}\right)^2 - \left(\frac{\partial\bar{V}_n}{\partial\bar{y}}\right)^2 - \left(\frac{\partial\bar{V}_n}{\partial\bar{z}}\right)^2 - 2\epsilon^2\bar{u}_n\bar{V}_n + 2\bar{f}_n\bar{u}_n + 2\epsilon^2\bar{Q}_n\bar{V}_n + 2\,in\,M^2\bar{u}_n^2 - 2\,in\,M^2\,\mathrm{Pr}\,\bar{V}_n^2\right] d\bar{y}d\bar{z} \tag{43}$$

or the integral

$$I_P = \iint_S \left[\left(\frac{\partial\bar{u}_n}{\partial\bar{r}}\right)^2 + \frac{1}{\bar{r}^2}\left(\frac{\partial\bar{u}_n}{\partial\phi}\right)^2 - \left(\frac{\partial\bar{V}_n}{\partial\bar{r}}\right)^2 - \frac{1}{\bar{r}^2}\left(\frac{\partial\bar{V}_n}{\partial\phi}\right)^2 - 2\epsilon^2\bar{u}_n\bar{V}_n + 2\bar{f}_n\bar{u}_n + 2\epsilon^2\bar{Q}_n\bar{V}_n + 2\,in\,M^2\bar{u}_n^2 - 2\,in\,M^2\,\mathrm{Pr}\,\bar{V}_n^2\right] \bar{r}d\bar{r}d\phi \tag{44}$$

is a minimum, where S is the region bounded by Γ. I_C is used for rectangular Cartesian co-ordinates; and I_P, polar.

This minimization can be carried out by the direct Ritz-Galerkin method. It is advisable, however, to use the semidirect method[6] whenever applicable.

[5] If A and u_{0m} are zero, other characteristic quantities can be used. What follows is still valid.

Authors' Closure

The authors would like to thank Mr. Lu for his interest in this problem and for his comments on the region far from the thermal inlet. The variational technique he presents is of interest as it points out a method using a single variational integral which has more than one Euler equation and hence can be used for solving a simultaneous set of partial differential equations. Readers who are unfamiliar with this can refer to Hildebrand[7] for the derivation. Some solutions like those proposed here for the fully developed region with arbitrary variations in heat generation and forced convection pressure gradient have been obtained by Zeiberg and Mueller[8] using transform methods.

An important point in this discussion is to decide where the fully developed solutions would be applicable when the flow is pulsating. The initial premise is that they apply sufficiently far down the channel (past the thermal entrance region) so that, for uniform heat input along the channel, the temperature gradient in the axial direction would be constant, $\partial T/\partial x = A$. The question then arises as to what is the length of the thermal entrance region for a pulsating flow. In Figs. 9 and 10 in the paper it is shown that the wall temperatures oscillate about the steady flow results, and these oscillations would persist throughout the duct length. Hence within the assumptions of the present analysis the gradients in the x-direction would never be constant. The present analysis, however, neglects axial conduction and this would be expected to eventually damp out the oscillations so that the conditions postulated by Mr. Lu could be reached. Thus the thermal entrance length for the oscillating case is not a thermal entrance in the usual sense. Usually it is dependent only on the development of the temperature distribution over the channel cross section which depends on the transverse heat flow. In the oscillating case, however, the entrance length would depend on axial conduction to eliminate the cyclical temperature variations in the axial direction which propagate down from the channel entrance. For axial conduction to be effective it would seem necessary for the channel to be long enough to contain several oscillation cycles. In most cases this would result in very long channels as can be shown, for example, by considering Fig. 9 (*b*). Here even two cycles take an X of about 9. This is equivalent to $x/2a = 9\,\mathrm{Re}(.7)/8$ or approximately, $x = {}^3/_4\,(2a)\mathrm{Re}$. Hence if Re were 1000 it would take on the order of 750 spacing widths in length for only two cycles in the x-direction. For axial conduction to remove the oscillations might require even a longer length. Hence the practical utility of fully developed solutions for oscillatory flow would be questionable in these instances.

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