Asymptotic edge-and-vertex diffraction theory

Andrzej Hanyga
Institute for Solid Earth Physics, University of Bergen, Allégaten 41, N5007 Bergen, Norway

Accepted 1995 May 9. Received 1995 May 9; in original form 1994 July 21

SUMMARY
Uniformly asymptotic formulae for edge-and-vertex diffraction in the time-domain, involving elementary functions of time, traveltimes and GTD amplitudes, are derived. Explicit expressions for diffraction at a pyramid and a triangle are constructed. They can be applied to the numerical calculation of a field reflected and diffracted at 3-D objects with sharp edges and to reflection from triangulated surfaces. The computational cost is very low.

Key words: diffraction, edge waves, seismic modelling.

1 INTRODUCTION
Diffracted waves account for a substantial part of the visible energy in a seismic wavefield reflected from a complex subsurface. They allow correct identification of faults and discontinuities in layering. On the other hand, for reasons related to computer capacity, 3-D seismic modelling still relies on asymptotic ray theory (ART). It is therefore important to be able to include diffraction in ray-theoretical computational schemes.

A general approach to the asymptotic solution of diffraction problems is laid out in Hanyga (1993, 1994, 1995). It is based on a coupling of point-to-point ray tracing with uniformly valid asymptotic formulae in the time or frequency domain. Ray tracing must include diffracted rays, accounting for evanescent fields in deep shadow (and complex rays in caustic shadow; Hanyga & Seredyńska 1991; Hanyga & Helle 1990, 1994). This approach leads to computationally effective algorithms which combine powerful ray-tracing algorithms (Hanyga 1988, 1991) with simple analytic formulae for the evaluation of the traces. In Hanyga (1995), it is shown that this approach is very general, and that in all the cases of practical interest—the exception of umbilic caustics—the time-domain asymptotic formulae involve elementary functions and elliptic integrals.

This paper is devoted to the diffraction effects associated with shadows caused by sharp objects (edge-and-vertex diffraction). The method was, however, originally applied to caustics. Frequency-domain asymptotic expressions for caustic diffraction were introduced in Ludwig (1966). Time-domain asymptotic expressions for cuspoid caustics were introduced in Burridge (1962) and Stickler, Ahluwalia & Ting (1981), and in a more complete form in Hanyga & Seredyńska (1991). The numerical test described in Hanyga & Helle (1990, 1994) provides a good example of their unexpected accuracy.

The first step towards the development of ray theory for edge diffraction was made in Keller (1958, 1966). More detailed formulae were obtained by Lewis & Boersma (1969), and a generalization to higher-order edges and vertices was presented in Kaminetzky & Keller (1972). The theory developed by Keller is called the geometrical theory of diffraction (GTD). It is then assumed that the formulae for diffraction coefficients remain valid in more general boundary value problems with inhomogeneities and curved interfaces and edges (Keller 1958; James 1976; Achenbach, Gautesen & McMaken 1982). This is a version of the 'locality principle' for edge and vertex diffraction.

Edge and vertex diffraction coefficients depend on the angle between the launching direction of the diffracted ray and the shadow boundary. At the shadow boundary, the diffracted rays coincide with reflected rays and the diffraction coefficients assume infinite values. It is shown in Hanyga (1989a, 1994) that the coalescence of two different kinds of rays and the blow-up of diffraction coefficients at a point in the wavefield are closely interrelated. Furthermore, it is possible to determine the singularity of diffraction coefficients at the shadow boundary without even calculating them for any specific problem.

In order to overcome this difficulty, Keller and his co-workers applied boundary-layer methods to edge diffraction (Buchal & Keller 1960; Bakker 1990; Klem-Musatov & Aizenberg 1984). A different approach was developed in Orlov (1974), and independently in Hanyga (1989a), in the frequency
domain. It links uniform asymptotics to ray tracing, which is of great importance for numerical applications. It also encompassed more complex singularities involving reflected-wave caustics and diffracted-wave caustics tangent to the shadow boundary. The derivation of the basic formulae is different in Orlov (1974) from that in Hanyga (1989a)—in the latter reference, the topological aspects of the singularity were shown to be independent of the propagation mechanism, which affects only the kinematic and dynamic parameters of the ray field. More importantly, Hanyga (1989a; and also Hanyga 1993, 1994) contains uniformly asymptotic time-domain expressions, which turn out to be more appropriate for numerical applications since they are often expressible in terms of elementary functions and elliptic integrals. [A systematic analysis of this aspect of asymptotic diffraction theory is given in Hanyga (1995).] On the other hand, frequency-domain expressions involve transcendental functions defined in terms of oscillatory integrals. Unless the function depends on a single variable—which is the case for simple caustics and shadow boundaries only—it has to be calculated by numerical integration over a complex contour (Hanyga 1989a; Connor & Curtis 1982). Such an integration typically has to be repeated 1024 or 2048 times in order to evaluate a seismic trace by fast Fourier transform (FFT). As a result, frequency-domain expressions are inappropriate for the numerical computation of diffracted signals. An extensive review of the Soviet work on frequency-domain expressions for caustics and simple shadow boundaries can be found in Kravtsov & Orlov (1993).

In this paper we present uniformly asymptotic time-domain expressions for edge diffraction (this is merely the Fourier transform of the familiar frequency-domain solution involving the Fresnel function) and a brand new theory of edge- and vertex diffraction which is applicable to objects having multiple edges joined at the vertices. It will be seen that the time-domain expressions in both cases involve inverse trigonometric functions, and hence that they are extremely convenient in numerical applications involving ray tracing.

2 BASIC CONCEPTS

We consider a wavefield reflected and diffracted at a multifaceted interface. In the high-frequency asymptotic approximation, such a wavefield can be considered as a superposition of the contributions of the individual faces. The contribution of a single face $\mathcal{S}$ can be expressed in terms of an oscillatory integral

$$U_{\mathcal{S}}(x, \omega) = f(\omega) \int_{\mathcal{S}} u(y, x)e^{i(\omega \phi - \omega \gamma)}dS(y), \quad (1)$$

where $y = (y_1, y_2)$ are the parameters of the (non-specular) 'reflection point' on $\mathcal{S}$, $\mathcal{S}$ denotes their range, the phase function $\phi(x, y)$ is the sum of the traveltimes along two Fermat rays, from the source to $y$ and from $y$ to $x$, and $dS(y)$ is a measure on $\mathcal{S}$. The factor $f(\omega)$ has been included for the sake of convenience. It will be specified later in such a way that the reflected signal is a delta spike.

An expression of the form of eq. (1) can be derived by the Kirchhoff-Helmholtz approximation, but its detailed form and origin are irrelevant for the following derivations. Expression (1) does not exhibit any singularities but its numerical implementation would require tracing a large number of rays and applying a quadrature that is appropriate for oscillatory integrals. It also involves an error, since the amplitude function $a(y, x)$ involves a pseudo-reflection coefficient applied to non-specular 'reflection' points. In the next sections we shall simplify eq. (1) in such a way that the new expression is still uniformly asymptotic but involves only specular reflections, edge diffractions and vertex diffractions.

A rigorous and general argument in Hanyga (1995) leads to the conclusion that the reflected-diffracted field can always be represented by an oscillatory integral (eq. 1) over a domain $\mathcal{D}$ with a boundary, and this fact is sufficient for the derivations that follow.

We conclude this section with a review of the above relationship between eq. (1) and the geometrical theory of diffraction (GTD). GTD, originally introduced as an ansatz by Kellner (1966), can be derived from eq. (1) by applying the stationary phase or saddle point method. The contribution of the interior stationary point is identical to the contribution of the specularly reflected ray (see Hanyga & Seredyńska 1991). For $x$ in the shadow, the specular reflection point lies outside the face $\mathcal{S}$ on the surface which is the analytic continuation of $\mathcal{S}$. In the same way, boundary stationary points of the second kind correspond to edge-diffracted rays (Hanyga 1989a, 1993) and the asymptotic contributions of the corners of $\mathcal{S}$ correspond to vertex-diffracted rays (Hanyga 1994).

As the point $x$ is allowed to vary, some of these points coalesce and their asymptotic contributions tend to infinity, which limits the range of validity of both the stationary phase method and the GTD. More specifically, two or more interior stationary points coalesce when $x$ lies on a reflected-wave caustic, and two more boundary stationary points coalesce if $x$ tends to a diffracted-wave caustic. The amplitude singularity appears in the factor $[\det \tilde{\phi}/\partial y^2]^{-1/2}$ or, equivalently, in the ray spreading factor $J^{-1/2}$ of the ray amplitude, since $J \propto [\det \tilde{\phi}/\partial y^2]$. In the caustic shadow, the ray field has a complex-analytic continuation (Hanyga & Seredyńska 1991; Hanyga 1989b) and complex rays correspond to saddles of $\phi$ in the complex $y$-space. At a simple shadow boundary, a single non-degenerate interior stationary point coalesces with a single non-degenerate boundary stationary point. The corresponding reflected and diffracted rays also coalesce into a single ray lying on the shadow boundary. In this case, the amplitude of the diffracted ray displays a singularity $(n \cdot \phi/\partial y)^{-1}$, where $n$ denotes a normal to the corresponding part of the boundary of $\mathcal{S}$. This singularity appears in the edge diffraction coefficient (Hanyga 1989a, 1993).

A vertex in the face $\mathcal{S}$ corresponds to a corner of the integration domain $\mathcal{D}$ of eq. (1). The two edges of $\mathcal{S}$ through the vertex generate two shadow boundaries, $\mathcal{S}_1$ and $\mathcal{S}_2$. At the intersection of these two shadow boundaries, the interior stationary point coalesces with both boundary stationary points and, consequently, also with the vertex. At $\mathcal{S}_1 \cap \mathcal{S}_2$, the vertex diffraction coefficient also has a singularity.

In the following sections we shall simplify eq. (1) in such a way that non-specular reflections are eliminated but no singularities are introduced. The resulting expressions will depend exclusively on the kinematic and dynamic parameters of the reflected and diffracted rays evaluated at the given point.

3 SIMPLE SHADOW BOUNDARY

In this section we consider the Fresnel diffraction associated with a single edge. The time-domain expression derived below
is a correct asymptotic approximation of the field at a single shadow boundary. The corresponding frequency-domain expression is well known, and involves a Fresnel function (see Achenbach, Gautesen & McMaken 1982; Klem-Musatov & Aizenberg 1984).

Both expressions will now be derived by a non-linear mapping of a neighborhood of the diffraction point B on the boundary $\partial \mathcal{D}$ of $\mathcal{D}$ onto a subset of the half-plane $(u, v)$, $u \geq 0$. The mapping induces the transformation of the phase function $\psi(y_1, y_2, x)$ to the polynomial $\psi = u^2 + v^2 + q_1(x)u + q_2(x)$, $\varepsilon_1, \varepsilon_2 = \pm 1$ (Hanyga 1989a, 1993). The contribution $U_B$ of a neighborhood of B to the integral (1) can be expressed in terms of the new integration variables

$$U_B = f(\omega) \int_0^\infty \int_0^\infty b(u, v, q) e^{i \omega \psi(u, v, q)} du dv, \quad (2)$$

where $b = |\partial \psi(y_1, y_2, x)/\partial (u, v)|$. The stationary phase method can be applied to the integral over $v$, yielding

$$U_B = f(\omega) \left\{ \frac{\pi}{|\omega|} \int_0^\infty b(u, 0, x) e^{i \omega \psi(u, 0, x)} du + O[\omega^{-1}] \right\}. \quad (3)$$

Noting that $\partial \psi/\partial u = 2u + q_2$, we can expand the slowly varying factor $b(u, v, x)$ in eq. (3) as follows:

$$b(u, 0, x) = b_0(x) + b_1(x, u) \partial \psi(u, 0, x)/\partial u. \quad (4)$$

$b_0(x)$ denotes the value of $b(u, 0, x)$ at the stationary point of $\psi(u, 0, x)$, and $b_1(x, u)$ is some smooth function (Hanyga 1993). For $\varepsilon = 1$ and $\omega > 0$, we have

$$U_B = f(\omega) \left\{ \frac{\pi}{|\omega|} \int_0^\infty b_0(x) e^{i \omega \psi(u, 0, x)} du + O[\omega^{-1}] \right\} e^{i \omega \psi_1(x)}$$

$$= f(\omega) \left\{ \frac{\pi}{|\omega|} \int_0^\infty b_0(x) \text{Fr}(\sqrt{\omega q_1}/2) + O[\omega^{-1/2}] \right\} e^{i \omega \psi_1(x)}, \quad (5)$$

where

$$\text{Fr}(\xi) = e^{-i \xi^2} \int_{\xi}^\infty e^{iz^2} dz$$

is a Fresnel function (Achenbach, Gautesen & McMaken 1982) with the asymptotic behaviour

$$\text{Fr}(z) \sim i/2z + O[z^{-3}] \quad \text{for} \quad z \to \infty$$

and the symmetry $\text{Fr}(-z) = \sqrt{\pi} e^{-i z^2} e^{i \pi/4} - \text{Fr}(z)$. The result for $\varepsilon = -1$ is analogous with the $\text{Fr}(\sqrt{\omega q_1}/2)$ replaced by $\text{Fr}(-\sqrt{\omega q_1}/2)^*$, where the asterisk denotes the complex conjugate.

The shadow corresponds to $\epsilon_1 > 0$ (in this case there is no stationary point in $(u \geq 0)$). The reflected traveltime $T_R$ equals the value of $\psi(u, 0, q)$ at the stationary point. The diffracted traveltime $T_D$ corresponds to the endpoint contribution: $T_D = \psi(0, 0, q) = q_2$. The excess traveltime is given by the formula

$$T_B - T_R = q_2^2/4.$$  

We now choose the frequency filter $f(\omega)$ in eq. (5) in such a way that the reflected signal in the time-domain expression becomes $b_0(x) \delta(t - T_R(x))$:

$$f(\omega) = \exp\left[ -i \pi (\varepsilon + 1) \text{sgn} \omega/4 \right] \frac{|\omega|}{2\pi^2}. \quad (6)$$

Applying the Fourier transformation to eq. (5) with respect to the frequency–time pair, a remarkably simple time-domain expression is obtained:

$$\hat{U}_E(x, t) = b_0(x) \frac{\partial \hat{G}(x, t)}{\partial t}, \quad (7)$$

where

$$\hat{G} = \begin{cases} 0 & \text{if } t = 0 \text{ and } \varepsilon(t - T_R) < 0 \\ 1 & \text{if } t = 1 \text{ and } \varepsilon(t - T_R) < 0 \\ \frac{1}{2} \frac{i}{\pi} \sin^{-1} \left[ \sqrt{(T_B - T_R)/(t - T_R)} \right] & \text{if } \varepsilon(t - T_R) > 0, \end{cases}$$

and

$$t = \varepsilon \sin \frac{q_1}{2} = \begin{cases} -1 & \text{in the light,} \\ 1 & \text{in the shadow} \end{cases} \quad (9)$$

(see Appendix A for the derivation). The function $G(\cdot, x)$ is shown in Figs 1 and 2 for $\varepsilon = 1$. The function $G$ for $\varepsilon = -1$ is obtained by inversion with respect to the vertical and horizontal axes. The signal $\hat{U}_B$ for $\varepsilon = -1$ is a time-inverted version of $\hat{U}_E$ for $\varepsilon = 1$.

For a general reflected signal $s(t - T_R)$, the wavefield near the shadow boundary is given by the expression $s(t) \ast \hat{G}(x, t)/\partial t$. The coefficient $b_0$ should match the ART asymptotics, whence

$$b_0 = A/J^{1/2} = A_{inc} R/J^{1/2},$$

where $J$ denotes the ray spreading of the reflected signal, $A$ is its reduced amplitude, $R$ is the reflection coefficient and $A_{inc}$ denotes the incident amplitude (Hanyga, Lenartowicz & Pajchel 1984; Hanyga 1988).

If the point $x$ lies in the shadow then the traveltime $T_R$ and the amplitude $b_0$ should be calculated by tracing a virtual ray. The virtual ray is defined in the same way as the reflected rays, except for the fact that the reflection point lies on a non-physical surface obtained by extrapolation of the reflector. The extrapolation of the reflector is defined in terms of its analytic continuation. The same considerations apply, *mutatis mutandis*, to transmitted waves.

### 4 TIME-DOMAIN KERNEL FOR EDGE-AND-VERTEX DIFFRACTION

We now consider the case of a reflecting surface with edges and vertices. A vertex is the intersection point of several edges. Ray field singularities are expected at those points $x$ for which the reflection points, edge diffraction points and vertex diffraction points coalesce. Since two different vertices cannot
Figure 1. Diffraction at an edge in the illuminated area. The reflected signal is a step.

Figure 2. Diffraction at an edge in the shadow. The reflected signal is a step.

coalesce, it is possible to reduce the problem to a single vertex $V$ and a cluster of stationary points that coalesce with it for some values of $x$. In the simplest case, the cluster consists of a single interior stationary point $R$ and two boundary stationary points $B_1, B_2$ on the two edges $E_1, E_2$, joined at the vertex $V$. Three different kinds of shadows correspond to $R \notin \mathcal{S}, B_1 \notin E_1$ and $B_2 \notin E_2$. Two shadow boundaries $\mathcal{S}_1, \mathcal{S}_2$ are defined by the condition $R \in E_1, R \in E_2$.

Singularities occur if one of the following conditions is satisfied:

- $R = B_1 \quad (x \in \mathcal{S}_1)$,
- $R = B_2 \quad (x \in \mathcal{S}_2)$,
- $R = B_1=B_2=V \quad (x \in \mathcal{S}_1 \cap \mathcal{S}_2)$.

We now introduce a coordinate transformation

© 1995 RAS, GJI 123, 277–290
(y_1, y_2). The corner of $S$ corresponding to V is mapped to (0, 0) and a neighbourhood of it in $S$ is mapped to $u \geq 0$, $v \geq 0$. We identify $E_1$ with $u = 0$ and $E_2$ with $v = 0$. The simplest polynomial phase function for which the situations listed above can occur is

$$\psi(u, v, q) = \epsilon_1 u^2 + e_2 v^2 + \mu uv + q_1 u + q_2 v + q_3,$$

where $\epsilon_1, \epsilon_2 \in \{1, -1\}$. For the case under consideration, the theory expounded by Siersma (1981) and Bruce (1981) guarantees the existence of a non-linear mapping $W : (u, v) \rightarrow (u', v')$, which maps the quarter-plane to itself and transforms the phase $\phi$ to the polynomial (11), where $q = [q_1, q_2, q_3]$ is a vector function of $x$, $\mu$ is a constant and $\mu \neq \pm 2$. Substituting the transformation in eq. (1) and expanding the slowly varying factors in powers of $u$, we have

$$b(u, v, x) = a(y_1, y_2, x) \hat{\delta}(y_1, y_2) \hat{\delta}(u, v),$$

where $b_0(x)$ denotes the value of $b$ at R. Applying integration by parts to the last two terms, we obtain the frequency-domain expression

$$U_{\omega} = f_2(\omega)b_0(x) \int_0^\infty \int_0^\infty e^{i\omega y_1(u, v, q)} du dv + O[\omega^{-1/2}].$$

The $O[\omega^{-1/2}]$ terms involve simple integrals over the edges $u = 0$ or $v = 0$.

In order to calculate the time-domain counterpart of eq. (13), we choose the frequency filter $f_2(\omega)$ in such a way that the reflected signal is $b_0(\delta(t - T_2))$:

$$f_2(\omega) = |\omega| \sqrt{\Delta} \exp\left[-i\mu(x_1 + i_2) \text{sgn} \omega / 4\right] (2\pi)^2$$

where $\Delta = 4\epsilon_1 \epsilon_2 - \mu^2$ and

$$\begin{align*}
\tau_1 &= \tau_2 = 1 & \text{if } \Delta > 0, \epsilon_1 = \epsilon_2 = 1, \\
\tau_1 &= \tau_2 = -1 & \text{if } \Delta > 0, \epsilon_1 = \epsilon_2 = -1, \\
\tau_1 &= -\tau_2 & \text{if } \Delta < 0,
\end{align*}$$

and apply the Fourier transformation. For $\Delta > 0$ and $\epsilon_1 = \epsilon_2 = 1$, we obtain the following expression:

$$\hat{U}_{\omega}(t, x) = \frac{\partial G_2}{\partial t},$$

where

$$G_2(t, x) = \sqrt{\Delta} \frac{b_0(x)}{2\pi} \int_0^\infty \int_0^\infty \delta[\psi(u, v, q) - c] du dv.$$  (16)

For the second and third lines of eq. (14), we obtain the time-domain responses $-\partial G_2/\partial t$ and $-\hat{\partial} G_2/\partial t$, respectively, where $\hat{\partial}$ denotes the Hilbert transformation.

The response with a given reflected wavelet $s(t)$ is obtained by convolution of kernel (16) with $s(t)$. Since $-\hat{\partial} G_2/\partial t * H_0 = \partial G_2/\partial t * s$, the case of $\Delta < 0$ involves a reflected signal that was subject to a caustic phase shift prior to edge-and-vertex diffraction. The remaining case $\tau_1 = \tau_2 = -1$ corresponds to a double caustic phase shift.

Expression (16) can be rewritten in terms of elementary functions. Using the formula

$$\int_0^\infty \delta(x^2 - t) dx = \frac{1}{2} (H(t - w^2) t_+^{1/2} + H(t + w^2 - t) t_+^{1/2})$$

where $H(t)$ denotes the unit step function

$$t_+ = \begin{cases} 0 & \text{for } t \leq 0, \\ t & \text{for } t > 0,
\end{cases}$$

the integral in eq. (16) can be expressed in the form

$$I = \int_0^\infty \int_{\tau_1/2}^\infty dx \delta(x^2 - 2b_2/4 + \omega),$$.  (17)

where $b = \mu + q_1, c = \epsilon_2 v^2 + q_2 v + q_3 - t$ and $t = \epsilon_1$, which leads to

$$I = \int_0^\infty \int_{\tau_1/2}^\infty dx \delta(x^2 - 2b_2/4 + \omega)$$

for $\mu \geq 0$, and

$$I = \int_0^\infty \int_{\tau_1/2}^\infty dx \delta(x^2 - 2b_2/4 + \omega)$$

for $\mu < 0$, where $M = \max \{0, -\tau_1/\mu\}$. Applying the formula

$$\int dz/\sqrt{p(z)} = -\tan^{-1}[(u/2 - 2)/\sqrt{p(u)}],$$

where $p(z) = z^2 + bx + c$, the integrals in eq. (19) can be expressed in terms of an arctangent. The explicit expressions depend on the position of the space–time point $(x, t)$ with respect to the wavefronts and shadow boundaries, and their presentation would require tedious case splitting. A FORTRAN code for the computation of the function $G_2$ is available from the author.

The reflected and edge-diffracted wavefronts are defined by vanishing of the discriminants of the polynomials $b^2 - 4c$ and $c$, as well as by the pairs of equations $b = c = 0$ and $b^2 - 4c = y = 0$. In the first case, the integral $I$ jumps from zero to a finite value—on account of the singularity of the integrand—and the arrival is a delta-spiked reflected signal. In the remaining cases, it begins to grow gradually or to decrease at an initially infinite rate.

In Figs 3–6 the diffracted signal from two edges and a vertex is shown. It corresponds to a reflected signal $H(t - T_k)$ and is given by $G_2(t, x)$ (eq. 16). The traveltimes are marked. The signal arriving at a point on the intersection of the two shadow boundaries is just $0.25H(t - T_k)$.

### 5 COMPUTATION OF THE PARAMETERS $q$ AND $b_0$ FROM THE KINEMATIC PARAMETERS OF THE RAY FIELD

Eq. (16) contains three variable parameters $q = [q_1, q_2, q_3]$ and a constant $\mu$. These parameters are linked to the four traveltimes $T_0, T_1, T_2, T_3$ by the fact that $T_0$ is the critical value of $\psi(u, v, q)$, $T_1$ is the critical value of $\psi(u, 0, q)$, $T_2$ is the critical value of $\psi(0, v, q)$, and

$$T_V = \psi(0, 0, q).$$

By straightforward algebra, one derives the following relations:

$$q_1 = T_V,$$  (22)

$$q_2 T_1/4 = T_V - T_2,$$  (23)

$$q_2 T_3/4 = T_V - T_1.$$  (24)
Figure 3. Example of a signal diffracted at a vertex with two edges in the area illuminated by the reflected and both edge-diffracted signals. The reflected signal is a step function. Travel times are marked: $T_R = -6.33\,s$, $T_R = 7.75\,s$, $T_R = 3.75\,s$, and $T_V = 10.0\,s$.

![Signal Diffraction Diagram](image)

Figure 4. Example of a signal diffracted at a vertex with two edges, at a shadow boundary. Travel times are marked: $T_R = 3.75\,s$, $T_R = 8.44\,s$, and $T_V = 10.0\,s$.

![Shadow Boundary Diagram](image)

Eqs (22)-(24) allow determination of the parameters $q$ if the travel times and $\mu$ are known. The signs of $e_1q_1$ and $e_2q_2$ are positive in the shadow of the edge-diffracted arrivals at $T_R$ and $T_R$, and negative otherwise. The last statement follows from the fact that the boundary stationary points $u = 0, v = -q_2/2e_2$ and $v = 0, u = -q_1/2e_1$ belong to the integration domain if and only if their coordinates are non-negative.

Alternatively, eqs (22), (25) and (26) can be used to determine the parameters $q$. In this case, the signs of $2q_1 - \mu e_2q_2$ and $2q_2 - \mu e_1q_1$ can be determined from the inequality $u_R, v_R \geq 0$ being valid for the region illuminated by the reflected rays, where $u_R$ and $v_R$ are given by eq. (B2).

The reflected and edge-diffracted travel times and amplitudes...
in the shadows are defined by analytic continuation of the travel times and amplitudes of the visible signals. The corresponding non-physical rays are obtained by analytic continuation of the reflected and edge-diffracted rays. More specifically, they are constructed by reflection and diffraction at the analytic continuation of the reflector and its edges. Such non-physical rays do not contribute GTD signals to the composite signal $U_E$ or $U_D$ by virtue of the asymptotic properties of these kernels.

Comparing the asymptotic contribution of the interior stationary point in eq. (13) with the reflected arrival in ART, we find that $b_0(x)$ is the amplitude of the reflected ray. If the point $x$ lies in the shadow, then the amplitude is calculated for a ray reflected from the analytic continuation of the interface.
The impedance contrast—or the media on two sides of the extrapolated interface—is calculated by analytic continuation too. Following the method in Appendix A of Hanyga & Seredyńska (1991), it can be proved that expression (13) satisfies a partial differential equation in an asymptotic sense, provided that:

1. eqs (22)–(26) are satisfied and the traveltime functions $T_R(x), T_B(x)$ and $T_T(x)$ satisfy appropriate eikonal equations;
2. $b_0(x)$ satisfies the transport equation.

The boundary conditions are satisfied by linking $b_0(x)$ to the reflection coefficient of the reflected ray passing through $x$, which leads to eq. (10).

6 GEOMETRIC INTERPRETATION OF THE CONSTANT PARAMETER $\mu$

The constant parameter $\mu$ is related to the angle of the face $S$ at the vertex. Indeed, consider a wavefield from a point source $x_0 = (x_0, y_0, z_0)$ incident on a sector $u = y - ax \geq 0, v = y \geq 0$ of the $z = 0$ plane. We denote by $\phi = (x_1, y_1, 0)$ the reflection point of a virtual (non-Newtonian) ray and by $x_c = (x_c, y_c, z_c)$ the point at which the field has to be evaluated. Assuming for simplicity that the incident and reflected fields propagate in a homogeneous medium with the propagation speed $c = 1$, we have

$$\phi = |x_1 - x_0| + |x_2 - x_1|.$$  (27)

We consider the case of $|x_1, y_1| \ll \min(|x_0|, |x_2|)$, so that

$$\phi \approx |x_0| + |x_2| - (x_0|x_0| + x_2|x_2|)y_1 - (y_0|x_0| + y_2|x_2|)y_1
+ A|x_1|^2/2 + By_1^2/2 - Cx_1y_1 + \ldots
= q_3 + q_1u + q_2v + \frac{1}{2}(A|x|^2)u^2
+ \left(\frac{1}{2}B + \frac{1}{A}A^2/C - \frac{1}{A^2} \right)v^2 - (A|x|^2 - C/2)uv,$$

where

$$A = (1 - \frac{1}{2}y_0^2/|x_0|^2)|x_0| + (1 - \frac{1}{2}y_2^2/|x_2|^2)|x_2|,$$

$$B = (1 - \frac{y_0^2}{|x_0|^2})|x_0| + (1 - \frac{y_2^2}{|x_2|^2})|x_2|,$$

$$C = \frac{1}{2}(x_0|y_0|/|x_0|^3 + x_2|y_2|/|x_2|^3),$$

$$q_3 = T_V = |x_0| + |x_2|,$$

and the specific form of the coefficients $q_1, q_2$ is not relevant in what follows. Since $u, v \geq 0$, the value of $\mu$ is readily obtained by comparing the coefficients of the quadratic terms. For $|x_0|, |y_0|, |x_1|, |y_2| \leq H < \min(|z_0|, |z_2|)$ a relatively simple formula is obtained:

$$\mu = -2/\sqrt{1 + x^2 + O[H^2(z^2 + z_1^2)]}.$$  (28)

The corrections involving $x_0, x_2$ in eq. (28) appear because our choice of transformations $(x_1, y_1) \rightarrow (u, v)$ is limited to linear ones. It is always possible to find a non-linear coordinate transformation $(x_1, y_1) \rightarrow (u, v)$ in such a way that $\mu$ is independent of $x_2, x_0$, for example in terms of a Taylor series [see Gilmore (1981) in a somewhat different context]. Such a choice is implicit in the derivation of eq. (16). For $x \rightarrow 0$, the parameter $\mu \rightarrow -2$. Setting $v = -y_1$, rather than $v = y_1$, we would get $\mu \rightarrow 2$. From eq. (28) it is seen that $\mu = -2\cos \theta$, where $\theta$ denotes the angle at the vector.

A more direct approach to an interpretation of $\mu$ is provided by the equations determining the shadow boundaries:

$$2\varepsilon_3q_1 - 2\mu q_3 = 0,$$

$$2\varepsilon_1q_2 - 2\mu q_1 = 0.$$

They can be readily derived from eqs (B2). It is readily seen that the shadow boundaries are parallel if $\mu = \pm 2$. In view of the assumed geometry, these values must be excluded as a degenerate case.

It is worth noting that the value of $\mu$ can be calculated from the values of all the four travel times, with some ambiguity. Indeed, by straightforward algebra, the following equation is obtained:

$$\zeta \mu^2 - 4\mu \sqrt{|\zeta|} + 4\varepsilon_1\varepsilon_2(\xi + \eta - \zeta) = 0,$$  (29)

where $\zeta = T_V - T_R$, $\xi = T_V - T_B$, $\eta = T_V - T_B$, and $t = \text{sgn}(q_1, q_2)$. On account of the identities $\text{sgn} \xi = \text{sgn} \eta$, $\text{sgn} \eta = \text{sgn} \xi$, the discriminant of eq. (29), $D = \varepsilon_1\varepsilon_2(C^2 - (\xi + \eta + \xi \zeta)$, has exactly two roots, $\zeta = \zeta_1, \zeta_2 = \zeta$ and $\eta$. Consequently, for $q_1, q_2 = 1$, eq. (29) has solutions provided either $T_R < T_B$, or $T_B < T_R$, or $T_R > T_B$, and $T_B$, as expected. For $q_1, q_2 = -1$, the solutions exist provided $T_R$ lies between $T_B$: and $T_B$.

For the cases of $\xi = -2$ (a cuspidal sector or its complement) and $\mu = 2$ (a higher-order sector), polynomials $v$ of codimension $\geq 4$ (with at least four parameters $q_i$) have to be considered. They can be found in the list of Siersma (1981). This means that the computation of the diffraction pattern in this case requires more precise information about the sector. The corresponding three-domain kernels can be expressed in terms of elliptic integrals.

It is not known whether the case of $\Lambda < 0$ corresponds to a physical diffraction pattern.

7 FRACTIONAL-ORDER CORRECTIONS TO DIFRACTED SIGNALS

So far, we have neglected any higher-order corrections $O(\omega^{-n})$, $n > 0$ (or $t^{-n} \ast 1(t)$, $\lambda > 0$ in the time domain). Fractional-order corrections ($0 < n < 1$, $0 < \lambda < 1$) play a fundamentally different role from the $O(\omega^{-n})$ corrections. In edge- and vertex diffraction, the fractional-order terms arise from polynomial expansions of the slowly varying factor $h$ and from the boundary contributions resulting from integration by parts of the $O(\omega^{-n})$ term. In the cases considered in this paper, only the latter produce fractional-order corrections.

The $O(\omega^{-n})$ corrections, $n = 1, 2, \ldots$ are determined by higher-order transport equations. On the other hand, fractional-order terms provide corrections to the amplitudes of the diffracted signals in the dominant term (Hanyga 1989a, 1994a) (obviously, there are analogous fractional-order corrections to the $O(\omega^{-n})$ terms). Without fractional-order terms, the amplitudes of the diffracted signals are linked to the amplitude of the reflected signal, while physical considerations lead to quite different diffracted amplitudes which are proportional to diffraction coefficients (Keller 1958; James 1976; Achenbach, Gautesen & McMaken 1982). The total number of the amplitudes to be matched is equal to the number of elementary signals. The latter is always equal to the number $N$ of non-linear parameters $q_i$ ($i = 1, \ldots, N$). It can be shown (Hanyga 1989a, 1994) that the number of linear parameters $b_0, b_1, \ldots$
is equal to $N$ if and only if all the fractional-order contributions are simultaneously taken into account. Unfortunately, the coefficients of the fractional-order terms depend on the amplitudes of diffracted signals, which in turn are proportional to the diffraction coefficients. Diffraction coefficients can, however, be calculated by explicit formulae only in certain particular cases (cracks, rigid diffracting objects, impedance boundary conditions, perfectly conducting inclusions in an EM field, etc.). In particular, they cannot be calculated for transparent diffracting objects, which are typical in seismic applications. For these reasons, we shall limit our discussion of fractional-order corrections to the simplest case: the simple boundary (see Hanyga 1989a, 1993, 1994 for more general discussions).

The $O(t_0^{-1/2})$ term in eq. (5) is equal to $-b_1(0,x)/|v| \text{sgn} \omega$. Substituting this expression in (5) and applying the Fourier transformation, the following time-domain correction is obtained:

$$ \frac{ab_1(0,x)}{\pi} \frac{i}{(v \omega)^{1/2}}. $$

(30)

The last result follows easily from the following formula, which we shall use repeatedly:

$$ \int_{-\infty}^{\infty} e^{-i\omega t} e^{i x \text{sgn} \omega/4} |\omega|^{-1/2} d\omega = 2\sqrt{\pi} t^{1/2} $$

(31)

(Gel'fand & Shilov 1964). It should be noted that eq. (30) has the same form as the singularity of $\hat{U}_{\nu}(t, x)$ for $t \sim t_0$.

We now express $b_1(0,x)$ in terms of the reflected and diffracted signal amplitudes, by asymptotic evaluation of the contributions of the interior stationary point and the boundary stationary point. Comparing the asymptotic contribution of the interior stationary point with GTD, we get eq. (10). The contribution of the boundary stationary point can be calculated as the endpoint contribution in eq. (5) (see Fedoryuk 1993):

$$ I_B = -f(\omega) \int_{-\infty}^{\infty} e^{i\omega t} e^{i x \text{sgn} \omega/4} \frac{1}{i\omega \psi_1(0,0, q)} e^{i\omega T_B} $$

$$ \frac{e^{i(t_0 - \text{sgn} \omega)q}}{2\pi t_0^{1/2}} \left[ b_1(0,x) + \frac{b_0(x)}{q_1} \right] e^{i\omega T_B}, $$

(32)

with $|q_1| = 2\sqrt{T_B - T_b}$. Incidentally, eq. (32) also illustrates the relation between coalescence and amplitude blow-up.
Eq. (32) can be compared with the standard GTD expression for an edge-diffracted signal (Keller 1958; Lewis & Boersma 1969; James 1976; Achenbach, Gautesen & McMaken 1982):

\[
A_{\text{inc}}^{(b)} \left[ \frac{J_B^{(0)}}{J_B} \right]^{1/2} e^{i \omega_B 0/4} \frac{c^{1/2} D}{|\omega|^{1/2}} e^{i \omega_B r},
\]  

(33)

where \( J_B \) denotes the ray spreading of the diffracted rays, \( J_B^{(0)} = \lim_{r \to 0} J_B/r, r \) is the distance from the edge along the ray, \( c \) denotes the propagation speed of the diffracted waves, \( D \) is the diffraction coefficient, and \( A_{\text{inc}}^{(b)} \) denotes the amplitude of the incident ray at the diffraction point. The factor \( c^{1/2} \) is traditionally included for dimensional reasons. The singularity of the diffraction coefficient is explicitly represented by the \( 1/q_1 \) term in eq. (32). Consequently,

\[
b_1(0, x) = \frac{A_{\text{inc}}^{(b)}}{q_1} q_1 c^{1/2} D (J_B^{(0)}/J_B)^{1/2} - b_0\frac{q_1}{q_1},
\]

(34)

where \( b_0 \) is given by eq. (10). It is shown in Hanyga (1989a, 1994) that \( q_1 D \) is finite at the shadow boundary. It follows from the derivation of eqs (12), (32) and (34) that the numerator of eq. (34) is \( O[q_1] \) (Hanyga 1994).

Figure 9. A seismogram involving reflected and diffracted waves for the receiver line shown in Fig. 11.
In Fig. 7 a reflected-diffracted pulse with a correction term is shown. Without the correction term, the reflected-diffracted signal asymptotically tends to one-half of its height after the arrival of the reflected step pulse, as seen in Fig. 1.

It is worth noting that Trorey's numerical experiments (Trorey 1970) indicate that the approximation obtained by discarding fractional-order terms can in some cases be quite satisfactory. The same approximation is implicit in Klem-Musatov & Aizenberg (1984).

In the case of edge-and-vertex diffraction, both the edge-diffracted and vertex-diffracted signals should be corrected. The correction for the vertex-diffracted signal has the form \( H(t - T_v) \ast s(t) \) and its amplitude involves the vertex diffraction coefficient.

8 REFLECTION–DIFFRACTION AT A PYRAMID

We now consider a pyramid having \( n \) facets \( S_i \) \( (i = 1, \ldots, n; \) Fig. 8). The total wavefield, reflected–diffracted by the pyramid in the area of the vertex, is assumed to be the superposition of the wavefields that are reflected–diffracted by the \( n \) plane sectors \( S_i \). We assume that the phase function \( \psi \) is continuous on the surface of the pyramid. Choosing \( u_i \) and \( u_i+1 \) to be the coordinates of the quarter-plane associated with \( S_i \) \((i = 1, \ldots, n, \) modulo \( n \)), we then have

\[
\psi(0) = \psi|_{S_i} = u_i^2 + \mu_i u_i u_{i+1} + u_i^3 + q_i u_i + q_{i+1} u_{i+1} + q_0,
\]

for \( i = 1, \ldots, n \) modulo \( n \), where \( \mu_i \) denotes the value of the...
constant $\mu$ appropriate for $S_i$. We have identified the coordinates $v$ and $u$ pertaining to the common edge of two adjacent facets $S_i$ and $S_i + 1$, $u_{i+1} = u_i$, as well as the corresponding parameters $q_i = q_i^{i+1} = q_2^1$.

The parameters $q_i$, $(i = 1, \ldots, n)$ can be determined from the traveltimes of the reflected and diffracted waves using Eqs (22)- (26). Given $\mu_i$, $(i = 1, \ldots, n)$, it is enough to know either the $n$ reflected traveltimes $T_{R_i}$ (possibly virtual) and the vertex-diffracted traveltime $T_V = q_0$, or the $n$ edge-diffracted traveltimes and $T_V$. Knowledge of all the traveltimes allows the determination of the constants $\mu_i$, $(i = 1, \ldots, n)$.

The total wavefield reflected-diffracted at the pyramid has the form

$$\hat{U} = \sum_{i=1}^{n} \hat{U}_{S_i}(t, T_{R_i}, T_{B_{i+1}}, T_{R_{i+1}}),$$

where $\hat{U}_{S_i}$ are defined by Eq. (15) with $b_0$ replaced by an appropriate reflected-wave amplitude $b_i$, $(i = 1, \ldots, n)$.

### 9 DIFFRACTION AT A TRIANGLE

A triangle $V_iV_jV_k$ can be viewed as the intersection of three sectors with the vertices $V_i$, $V_j$ and $V_k$. A uniformly asymptotic expression for the field reflected or transmitted at the triangle can be constructed by superposing the contributions of the three sectors and subtracting those arrivals that are either represented twice or generated at a point of a sector outside the triangle. While subtracting the edge contributions, the possible coalescence with the reflected ray and the resulting singularity of the diffraction amplitude must be taken into account. For the sector associated with $V_i$, we take the explicit form of Eq. (15), with $e_1 = e_2 = 1$, $\mu_i = -2 \cos \theta_i$ at $V_i$, where $\theta_i$ denotes the angle at $V_i$, $q$ is given by Eqs (22)- (26), $T_V = T_{V_i}$, and $T_{R_i}$ and $T_{B_i}$ are replaced by appropriate edge-diffracted traveltimes. The joint contribution of the reflected and edge-diffracted waves is given by Eqs (7) and (8), where an appropriate edge-diffracted traveltimes is substituted for $T_{B_i}$.

A simple count of the reflected and diffracted signals in the light and shadows leads to the following formula for the asymptotic reflected-diffracted wavefield associated with the triangle:

$$\hat{U}_T(t, T_{R_i}, T_{B_{i+1}}, T_{R_{i+1}}, \mu_1, \mu_2, \mu_3)$$

$$= \hat{s}(t) \left[ -\sum_{i=1}^{3} \hat{U}_{S_i} - \sum_{i=1}^{3} \hat{U}_{E_i}(t, T_{R_i}, T_{B_i}) + b_0(x)H(t - T_{B_i}) \right],$$

where $\hat{U}_{S_i}$ is the contribution of the sector at $V_i$ and $\hat{U}_{E_i}$ is the joint contribution of the reflected and edge-diffracted signals at $B_i$ as given by Eq. (7). In Appendix B, it is shown that the double diffraction contributions in the first term of the right-hand side of Eq. (35) cancel with the edge diffractions in the second term.

It is easy to correct the diffracted amplitudes in Eq. (35) following the explanations given in Section 7, provided the diffraction coefficients are known. For a curvilinear triangle, Eq. (35) applies provided the sides are sufficiently small in comparison with the radii of curvature of the triangle and of its edges.

### 10 CONCLUDING REMARKS

Using expressions (8) and (16), a uniformly valid asymptotic approximation of a transient wavefield can be computed at shadow boundaries and their intersections. The expressions can be evaluated numerically at a point $x$ if the reflected and diffracted traveltimes and the amplitude of the reflected arrival are available. The expressions are particularly simple in the time domain.

Using Eq. (35), the reflection at a piecewise plane triangulated surface can be calculated. The sign factor $i$ in Eqs (B1) and (B3) ensures that the contribution from the common edge of two adjacent triangles cancels out if the triangles are co-planar. Eq. (35) has an important application. Instead of computing a wavefield reflected from a curved surface obtained by interpolating discrete data, one can compute the wavefield reflected-diffracted from a piecewise planar triangulated surface. In this case, edge-and-vertex diffractions correctly simulate curvature effects (Aizenberg & Klem-Musatov 1984).

Eq. (35) also allows the computation of the reflected-diffracted field from a corner of a model block (see Figs 9–11), or, more generally, from a pyramid.

### ACKNOWLEDGMENTS

The author is greatly indebted to an anonymous reviewer who devoted a lot of time to discovering the ambiguities in the previous draft and cross-checked some derivations. Dr Jan Pajchel attracted my attention to the triangles and triangulated surfaces in seismic modelling.

### REFERENCES


© 1995 RAS, GJI 123, 277–290


APPENDIX A: DERIVATION OF EQS (7) AND (8)

The frequency-dependent factor of the integral (3) can be written in the form

$$-i\omega(2\pi)^{-3/2}\sqrt{2/|\omega|}e^{i\epsilon x \text{sgn} \omega}.$$  \hspace{1cm} (A1)

Using eq. (4), the leading term of the Fourier transform of the integral in eq. (3) is equal to

$$2\pi \delta b_0(x) \int_0^{\infty} \delta(au^2 + q_1 u + q_2 - t) du,$$

while the Fourier transform of eq. (A1) is

$$\frac{e^{-\frac{1}{2\pi} \int_0^{|\omega|} |t|^{-1/2} dt}}{t}$$

(eq. 31). The leading term of the Fourier transform of eq. (5) is the convolution of these two time-domain functions divided by $2\pi$:

$$\hat{U} = \hat{b}_0 \frac{e^{-\frac{1}{2\pi} \int_0^{|\omega|} |t|^{-1/2} dt}}{t}$$

$$= \frac{e^{-\frac{1}{2\pi} \int_0^{|\omega|} |t|^{-1/2} dt}}{t} \int_0^{|\omega|} e^{i\omega t} dt$$

$$= \frac{e^{-\frac{1}{2\pi} \int_0^{|\omega|} |t|^{-1/2} dt}}{t} \sqrt{\frac{i}{\pi}},$$\hspace{1cm} (A2)

for $t = T_R > 0$, with $|U| < 0$ in the opposite case. The lower limit of the integration $\zeta = \text{sgn}(\omega) \sqrt{(T_R - T_0)/(t - T_R)}$, and the prime indicates that the integration is restricted to positive values of the expression under the square root sign. The last expression yields eqs (7) and (8).

APPENDIX B: CHECKING EQ. (35)

The contribution of an edge diffraction to the sector response (15) is equal to the Fourier-transformed contribution of a boundary stationary point of second type (Fedoryuk 1993) to eq. (13). The contribution of $u = -\epsilon_1 q_1/2, v = 0$ to eq. (13) is given by the formula

$$b_0(x)\epsilon_1 \sqrt{|\Delta|/(4\pi)} \left[ \frac{\mu q_1 \epsilon_1}{2} - q_2 \right]^{-1} e^{i\epsilon_1 q_1 x \text{sgn} \omega} \left| \epsilon_1 \right|^{-1/2} e^{i\epsilon_1 T_0}.$$  \hspace{1cm} (B1)

Its Fourier transform can be calculated by applying eq. (31):

$$b_0(x) \sqrt{|\Delta|} \frac{q_1 \epsilon_1}{2\pi} \left[ \frac{\mu q_1 \epsilon_1}{2} - q_2 \right]^{-1} \left[ \epsilon_1 (t - T_{0b}) \right]^{-1/2} \left[ \epsilon_1 (t - T_{0b}) \right]^{-1/2},$$ \hspace{1cm} (B1)

where $t = -1$ if the reflection point lies on the 'physical side'

of the edge $E_2$ and 1 otherwise. The relation between $t$ and the
light/shadow alternative follows from the formula for the value
$(u_R, v_R)$ of $(u, v)$ at the interior stationary point:

$$
\begin{align*}
    u_R &= \left(\mu q_2 - 2 \varepsilon_2 q_1\right)/\Delta, \\
    v_R &= \left(\mu q_1 - 2 \varepsilon_1 q_2\right)/\Delta.
\end{align*} \tag{B2}
$$

The contribution of an edge diffraction to expression (8)
can be calculated more directly by applying the expansions

$$
\sqrt{(T_B - T_K)/(t - T_K)} \sim 1 - \frac{1}{2}(t - T_K)/(T_B - T_K)
$$

and

$$
\cos^{-1}(1 - y) \sim \sqrt{2y}, \quad y > 0 \quad \text{for} \quad \varepsilon(t - T_B) > 0, \quad t \sim T_B,
$$

which yields

$$
-\pi \varepsilon^{-1} \sqrt{\varepsilon(t - T_B)/(T_B - T_K)}
$$

for $G$, and

$$
\hat{U}_E \approx -i(2\pi)^{-1}b_0(x)|T_B - T_K|^{-1/2}[\varepsilon(t - T_B)]^{-1/2}, \tag{B3}
$$

with $t = -1$ in the light and $t = 1$ in the shadow. For $\Delta > 0$
and $\varepsilon_1 = \varepsilon = 1$, expression (B1) coincides with (B3), as expected.

In order to cancel spurious edge-diffracted contributions in
eq. (35), we set $\hat{U}_{RR}(t, T_K, T_K)$ equal to expression (7), provided
$\Delta > 0$ and $\varepsilon = \varepsilon_1$. 

© 1995 RAS, GJI 123, 277–290