Non-Equilibrium Thermo-Field Dynamics for a Fourth-Order Hamiltonian

Mizuhiko Saeki\textsuperscript{1,2,\textasteriskcentered)}

\textsuperscript{1}Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan
\textsuperscript{2}Shimoasoushin-machi 859-40, Takaoka 939-1271, Japan\textsuperscript{\textasteriskcentered)}

(Received December 10, 2009; Revised May 7, 2010)

The non-equilibrium thermo-field dynamics proposed by Arimitsu and Umezawa are generalized to the case of a fourth-order unperturbed Hamiltonian which includes not only a second-order (quadratic) part but also a fourth-order part. Fujita’s analysis for effects of the initial particle correlation of a quantum gas is proved generally in terms of TFD. The forms of the quasi-particle operators for a semi-free boson field are derived. It is shown that the energies and life-times of the quasi-particles depend on the adiabatic boson-reservoir interaction which leads to the fourth-order part of the unperturbed Hamiltonian. The form of the two-point Green’s function for the semi-free boson field is evaluated. A form of the admittance for a boson system interacting with its heat reservoir, which includes effects of the initial correlation and memory, is derived using the TCLE method formulated in terms of the generalized non-equilibrium thermo-field dynamics. A calculation method of the higher-order parts of the admittance in powers of the boson-boson interaction is given. Furthermore, a calculation method of the perturbation expansions of the two-point Green’s function for the boson system is given.

Subject Index: 130

\S1. Introduction

The equilibrium thermo-field dynamics\textsuperscript{1)\textendash 7)} formulated by extending quantum field theory to the case of finite temperature, have many properties of quantum field theory, e.g., operator formalism, time-ordered formulation of Green functions, the Feynman diagram method in real time, etc., and is useful to treat many-body systems, because the statistical average is given by the expectation value in the thermal vacuum state. Arimitsu and Umezawa formulated the non-equilibrium thermo-field dynamics by combining the concepts of the coarse-graining with the thermal state and by introducing the thermal-Liouville space, for a quantum system interacting with its heat reservoir.\textsuperscript{8\textendash 10) Although Arimitsu and Umezawa formulated the non-equilibrium thermo-field dynamics in the van Hove limit approximation\textsuperscript{11)} or in the narrowing limit approximation\textsuperscript{12)} for the system-reservoir interaction,\textsuperscript{8\textendash 10)} when the TCLE method formulated in terms of the non-equilibrium thermo-field dynamics\textsuperscript{13\textendash 15)} is used to calculate the admittance for the quantum system, the effects of the deviation from the van Hove limit\textsuperscript{11)} or the narrowing limit\textsuperscript{12)} can be included in the results. Because, these effects are the effects of the initial correlation and memory for the quantum system and heat reservoir, and are represented by the interference

\textsuperscript{*} E-mail: mmasaeki@angel.ocn.ne.jp

\textsuperscript{**} Present address.
terms or the interference thermal states in the TCLE method.\(^{13}\)–\(^{17}\) In the TCLE method, the admittance of the system is directly calculated from time-convolutionless (TCL) equations with external driving terms,\(^{13}\)–\(^{20}\) and can be calculated by inserting the corresponding interference terms into the results in the van Hove limit or in the narrowing limit.\(^{13}\)–\(^{17}\) The effects of the initial correlation and memory are also called the ‘interference effects’, increase the peak heights of the line shapes in the resonance region and decrease the power spectrum in the low-frequency region, and those effects produce effects that cannot be ignored in general.\(^{16}\), \(^{17}\), \(^{21}\)–\(^{24}\)

In the formulation of the non-equilibrium thermo-field dynamics, Arimitsu and Umezawa treated the case of a bilinear unperturbed Hamiltonian for the system and reservoir,\(^{8}\)–\(^{10}\) which includes only the second-order (quadratic) of boson or fermion operator and its tilde conjugate. In the case that the interaction of a boson system with its heat reservoir includes not only a non-adiabatic part but also an adiabatic part,\(^{25}\) the unperturbed Hamiltonian includes not only a second-order part but also a fourth-order part in the lowest Born approximation for the interaction between the boson system and heat reservoir. In such a case, it is necessary to generalize the non-equilibrium thermo-field dynamics proposed by Arimitsu and Umezawa\(^{8}\)–\(^{10}\) to the case of a fourth-order unperturbed Hamiltonian which includes not only a second-order part but also a fourth-order part.

In the present paper, we consider a boson system interacting with its heat reservoir, where the boson-reservoir interaction includes not only a non-adiabatic part but also an adiabatic part, and generalize the non-equilibrium thermo-field dynamics proposed by Arimitsu and Umezawa\(^{8}\)–\(^{10}\) to the case of a fourth-order unperturbed Hamiltonian which includes not only a second-order part but also a fourth-order part. We derive a form of the admittance for the boson system using the TCLE method formulated in terms of the generalized non-equilibrium thermo-field dynamics, where the admittance includes effects of the initial correlation and memory for the boson system and heat reservoir. We also give a calculation method of the higher-order parts of the admittance in powers of the boson-boson interaction. Moreover, we give a calculation method of the perturbation expansions of the two-point Green’s function for the boson system.

In §2, we generalize the non-equilibrium thermo-field dynamics of Arimitsu and Umezawa to the case of a fourth-order unperturbed Hamiltonian for a boson system interacting with its heat reservoir. In §3, we derive the form of the two-point Green’s function for the semi-free boson field. In §4, we derive a form of the admittance for the boson system using the TCLE method. In §5, we give a calculation method of the perturbation expansions of the two-point Green’s function for the boson system. In §6, we give a short summary and some concluding remarks.

§2. A generalized non-equilibrium thermo-field dynamics

We work in the Liouville space or thermal-Liouville space\(^{8}\)–\(^{10}\), \(^{26}\), \(^{27}\) of a quantum system and its quantum heat reservoir. In the Liouville space and thermal-Liouville space, the tilde conjugate \(\tilde{A}\) of an operator \(A\) is defined to satisfy the
following rules,\(^8\)\(^{--}\)\(^10\), \(^28\)

\[
(AB)^\dagger = A^\dagger B^\dagger, \quad (c_1A + c_2B)^\dagger = c_1^*A^\dagger + c_2^*B^\dagger, \quad \hat{A}^\dagger = (A^\dagger)^-, \quad (\hat{A})^- = A, \quad (2.1)
\]

for arbitrary operators \(A\) and \(B\) and for arbitrary complex \(c\)-numbers \(c_1\) and \(c_2\). We also define the renormalized hat-operator \(\hat{A}\) for arbitrary operator \(A\) as\(^{15}\)

\[
\hat{A} \equiv \frac{(A - A^\dagger)}{\hbar}, \quad \hat{A}^\dagger = -\hat{A}. \quad (2.2)
\]

We represent the bra-vector and ket-vector by symbols \(|\rangle\) and \(|\rangle\), respectively, in the Liouville space and thermal-Liouville space, and represent the unit bra-vector and unit ket-vector by \(|1\rangle\) and \(|1\rangle\), respectively, in the Liouville space of the system and reservoir. Then, for arbitrary operators \(A\) and \(B\), we have\(^8\)\(^{--}\)\(^10\), \(^26\), \(^27\)

\[
A|1\rangle = |A\rangle, \quad \langle 1|A = \langle A|, \quad \langle 1|AB|1\rangle = \langle A|B \rangle = \langle AB|1\rangle = \langle 1|AB \rangle = \text{Tr}AB, \quad (2.3)
\]

with \(\text{Tr} = \text{tr}_S \text{tr}_R\), where the notations \(\text{tr}\) and \(\text{tr}_R\) denote the trace operations in the spaces of the system and reservoir, respectively. The unit bra-vector and unit ket-vector in the Liouville space of the system are represented by \(|1_S\rangle\) and \(|1_S\rangle\), respectively, and the unit bra-vector and unit ket-vector in the Liouville space of the reservoir are represented by \(|1_R\rangle\) and \(|1_R\rangle\), respectively. For the renormalized hat-operator \(\hat{A}\) defined by (2.2), we provide the basic requirements (axioms)

\[
\langle 1|A = \langle 1|\hat{A}^\dagger, \quad \langle 1|\hat{A} = 0, \quad (2.4)
\]

as in Ref. 10).

We consider a quantum system interacting with its quantum heat reservoir in an external static field. We take the Hamiltonian \(\mathcal{H}\) of the quantum system and heat reservoir as

\[
\mathcal{H} = \mathcal{H}_S + \mathcal{H}_R + \mathcal{H}_{SR} = \mathcal{H}_0 + \mathcal{H}_{SR}, \quad (\mathcal{H}_0 = \mathcal{H}_S + \mathcal{H}_R) \quad (2.5)
\]

where \(\mathcal{H}_S\) is the Hamiltonian of the quantum system in the external static field, \(\mathcal{H}_R\) is the Hamiltonian of the heat reservoir in the external static field, and \(\mathcal{H}_{SR}\) is the interaction Hamiltonian between the system and reservoir. For the renormalized hat-Hamiltonians defined by (2.2) and (2.5), we provide the basic requirements (axioms)

\[
\mathcal{H}_S |\rho_{TE}\rangle = 0, \quad \mathcal{H}_S |\rho_S\rangle = 0, \quad \mathcal{H}_R |\rho_R\rangle = 0, \quad (2.6)
\]

as in Ref. 10), where \(\rho_{TE}\), \(\rho_S\) and \(\rho_R\) are the normalized, time-independent density operators for the system and reservoir in the external static field, given by

\[
\rho_{TE} = \exp(-\beta \mathcal{H})/\langle 1|\exp(-\beta \mathcal{H})\rangle = \exp(-\beta \mathcal{H})/\text{Tr}\exp(-\beta \mathcal{H}), \quad (2.7)
\]

\[
\rho_S = \exp(-\beta \mathcal{H}_S)/\langle 1_S|\exp(-\beta \mathcal{H}_S)\rangle = \exp(-\beta \mathcal{H}_S)/\text{tr}_S\exp(-\beta \mathcal{H}_S), \quad (2.8)
\]

\[
\rho_R = \exp(-\beta \mathcal{H}_R)/\langle 1_R|\exp(-\beta \mathcal{H}_R)\rangle = \exp(-\beta \mathcal{H}_R)/\text{tr}_R\exp(-\beta \mathcal{H}_R), \quad (2.9)
\]

which are the thermal equilibrium density operators at temperature \(T = (k_B\beta)^{-1}\). In the thermal-Liouville space of the quantum system and heat reservoir, the
time-evolution of the thermal state $|\rho_T(t)\rangle [=\rho_T(t)|1\rangle$ is given by the Schrödinger equation\(^8\)\(^{-10}\)
\[\frac{d}{dt}|\rho_T(t)\rangle = -i\hat{H}|\rho_T(t)\rangle, \tag{2.10}\]
where $\rho_T(t)$ is the density operator of the total system. As an initial condition, the system and reservoir are assumed to be in the thermal state $|\rho_T(0)\rangle$ at the initial time $t=0$. In order to eliminate the irrelevant part associated with the heat reservoir, we introduce the time-independent projection operators $P$ and $Q$ as\(^9\)
\[P = |\rho_R\rangle\langle I_R| = \rho_R |I_R\rangle\langle I_R|, \qquad Q = 1 - P. \tag{2.11}\]
Proceeding in the same way as in Ref. 29) by using the above projection operators, we can obtain the time-convolutionless (TCL) equation of motion for $|\rho_T(t)\rangle$:
\[\frac{d}{dt}P|\rho_T(t)\rangle = -iP\hat{H}P|\rho_T(t)\rangle - i\hat{P}\hat{H}\{\theta(t) - 1\}P|\rho_T(t)\rangle - iP\hat{H}\theta(t)\exp(-iQ\hat{H}Q t)Q|\rho_T(0)\rangle, \tag{2.12}\]
where $\theta(t)$ is defined as
\[\theta(t) = \left\{1 + i\int_0^t d\tau\exp(-iQ\hat{H}Q \tau)\hat{Q}\hat{P}\exp(i\hat{H}\tau)\right\}^{-1}. \tag{2.13}\]
By virtue of the projection operators (2.11), Eq. (2.12) is reduced to the TCL equation of motion for the reduced thermal state $|\rho(t)\rangle [=\langle I_R|\rho_T(t)\rangle]$:
\[\frac{d}{dt}|\rho(t)\rangle = -i\hat{H}_S|\rho(t)\rangle + C(t)|\rho(t)\rangle + |I(t)\rangle, \tag{2.14}\]
where the collision operator $C(t)$ and the thermal state $|I(t)\rangle$ are given by
\[C(t) = -i\langle I_R|\hat{H}_{SR}\{\theta(t) - 1\}|\rho_R\rangle, \tag{2.15}\]
\[|I(t)\rangle = -i\langle I_R|\hat{H}_{SR}\theta(t)\exp(-iQ\hat{H}Q t)Q|\rho_T(0)\rangle. \tag{2.16}\]
Here, we have renormalized the Hamiltonian of the system and the system-reservoir interaction as
\[\hat{H}_S \rightarrow \hat{H}_S + \langle I_R|\hat{H}_{SR}|\rho_R\rangle \Rightarrow \hat{H}_S, \quad \hat{H}_{SR} \rightarrow \hat{H}_{SR} - \langle I_R|\hat{H}_{SR}|\rho_R\rangle \Rightarrow \hat{H}_{SR}, \tag{2.17}\]
\[\hat{H}_S \rightarrow \hat{H}_S + \langle I_R|\hat{H}_{SR}|\rho_R\rangle \Rightarrow \hat{H}_S, \quad \hat{H}_{SR} \rightarrow \hat{H}_{SR} - \langle I_R|\hat{H}_{SR}|\rho_R\rangle \Rightarrow \hat{H}_{SR}. \tag{2.18}\]
Hereafter, we use $\hat{H}_S$, $\hat{H}_{SR}$, $\hat{H}_S$ and $\hat{H}_{SR}$ for the Hamiltonians renormalized according to (2.17) and (2.18). Then, we have
\[\langle I_R|\hat{H}_{SR}|\rho_R\rangle = 0, \quad \langle I_R|\hat{H}_{SR}|\rho_R\rangle = 0. \tag{2.19}\]
The thermal state $|I(t)\rangle$ represents the effects of the initial correlation of the system and reservoir. Because, if the initial state $|\rho_T(0)\rangle$ is given by the decoupled one $|\rho(0)\rangle|\rho_R\rangle$ with $|\rho(0)\rangle$ given by $|\rho(0)\rangle = \langle I_R|\rho_T(0)\rangle$, i.e., $|\rho_T(0)\rangle = |\rho(0)\rangle|\rho_R\rangle$, then $|I(t)\rangle$ vanishes, since $Q|\rho(0)\rangle|\rho_R\rangle = 0$. Hereafter, we assume that the system and reservoir are in the thermal equilibrium state at the initial time $t=0$, i.e., $|\rho_T(0)\rangle = |\rho_{TE}\rangle$. 
We now consider the case that the quantum system is interacting so weakly with its heat reservoir that we can use the lowest Born approximation, and expand Eq. (2.14) up to the second order in powers of the system-reservoir interaction. Then, it reduces to

\[
(d/dt) |\rho(t)\rangle = - i \hat{H}_S |\rho(t)\rangle + C^{(2)}(t) |\rho(t)\rangle + |I^{(2)}(t)\rangle,
\]

where \(C^{(2)}(t)\) and \(|I^{(2)}(t)\rangle\) are given by

\[
C^{(2)}(t) = - \int^t_0 d\tau \langle 1_R | \hat{H}_{SR} \exp(- i \hat{H}_0 \tau) \hat{H}_{SR} \exp(i \hat{H}_0 \tau) |\rho_R\rangle,
\]

\[
|I^{(2)}(t)\rangle = i \langle 1_R | \hat{H}_{SR} \exp(- i \hat{H}_0 t) \int^\beta_0 d\beta' \rho_S \rho_R \exp(\beta' h \hat{H}_0) |\rho_{SR}\rangle = i \langle 1_R | \hat{H}_{SR} \exp(- i \hat{H}_0 t) \rho_S \rho_R (h \hat{H}_0)^{-1} \{ \exp(\beta h \hat{H}_0) - 1 \} |\rho_{SR}\rangle = i \langle 1_R | \hat{H}_{SR} \exp(- i \hat{H}_0 t) \rho_S \rho_R |1\rangle, \]

\[
= - \lim_{\xi \to +0} \int^\infty_t d\tau \langle 1_R | \hat{H}_{SR} \exp(- i \hat{H}_0(t - \tau)) \hat{S}_{SR} \rho_S \rho_R |1\rangle e^{-\xi \tau},
\]

\[
= - \int^\infty_t d\tau \langle 1_R | \hat{H}_{SR} \exp(- i \hat{H}_0(t - \tau)) \hat{S}_{SR} \rho_S \rho_R |1\rangle.
\]

When the system-reservoir interaction \(\hat{H}_{SR}\) can be expanded in a complete set of operators \(\{u_k \dagger v_k\}\) as

\[
\hat{H}_{SR} = \hbar \sum_k g_k u_k v_k, \quad \hat{S}_{SR} = \sum_k g_k (u_k v_k - \vec{v}_k \dagger \vec{u}_k),
\]

the collision operator \(C^{(2)}(t)\) and the thermal state \(|I^{(2)}(t)\rangle\) can be expressed as

\[
C^{(2)}(t) = \sum_{k,l} g_k g_l \int^t_0 d\tau \{ R_{kl}(\tau) v_l(-\tau) - R_{lk}(-\tau) \vec{v}_l(-\tau) \},
\]

\[
|I^{(2)}(t)\rangle = - \sum_{k,l} g_k g_l \int^\infty_t d\tau \{ R_{kl}(\tau) v_l(-\tau) - R_{lk}(-\tau) \vec{v}_l(-\tau) \} \rho_S |1_S\rangle,
\]

with the Heisenberg operators \(v_k(t)\) and \(\vec{v}_k^\dagger(t)\) of the system, which are defined by

\[
v_k(t) = \exp(i \hat{H}_S t) v_k \exp(-i \hat{H}_S t), \quad \vec{v}_k^\dagger(t) = \exp(i \hat{H}_S t) \vec{v}_k^\dagger \exp(-i \hat{H}_S t),
\]

where \(u_k\) and \(v_k\) operate in the spaces of the reservoir and system, respectively, and are assumed to be commutable with each other. Here, \(\hbar g_k\) are the coupling constants between system and reservoir, and \(R_{kl}(t)\) are the reservoir correlation functions given by

\[
R_{kl}(t) = \langle 1_R | u_k(t) u_l |\rho_R\rangle; \quad u_k(t) = \exp(i \hat{H}_R t/\hbar) u_k \exp(-i \hat{H}_R t/\hbar).
\]

In the derivations of (2.25) and (2.26), we have used the axioms (2.4) and (2.6). If the relaxation time \(t\) of the system is much greater than the correlation time \(\tau_c\) of the...
heat reservoir, i.e., $t \gg \tau_c$, the expression (2.26) shows that the thermal state $|I(2)(t)\rangle$ given by (2.22) or (2.23), which represents the effects of the initial correlation of the system and reservoir, becomes small negligibly. In Ref. 30), Fujita discussed the effects of initial particle correlations for a quantum gas, and showed that those effects are negligibly small for a time much greater than the average collision time, which corresponds to the reservoir correlation time $\tau_c$. We have thus proved Fujita’s analysis for a quantum gas$^{30}$ generally in terms of TFD.

In the case that the heat reservoir is damped very rapidly ($\tau_c \to 0$), Eq. (2.20) becomes

$$ \frac{d}{dt} |\rho(t)\rangle = -i \hat{H}_S |\rho(t)\rangle + C^{(2)} |\rho(t)\rangle, \quad (\tau_c \to 0) $$  \hspace{1cm} (2.29)

with

$$ C^{(2)} = C^{(2)}(\infty) = -\int_0^\infty d\tau \langle 1_R| \hat{H}_{SR} \exp(-i \hat{H}_0 \tau) \hat{H}_{SR} \exp(i \hat{H}_0 \tau) | \rho_R \rangle. $$  \hspace{1cm} (2.30)

Equation (2.29) is that obtained in the van Hove limit$^{11}$ or in the narrowing limit$^{12}$ in which the heat reservoir is damped very rapidly, and can be formally solved as

$$ |\rho(t)\rangle = \exp\{-i \hat{H}_S t + C^{(2)} t\} |\rho(0)\rangle, \quad (\tau_c \to 0) $$  \hspace{1cm} (2.31)

For complicated systems or for many-body systems, dividing the Hamiltonian $\mathcal{H}_S$ of the system into the unperturbed part $\mathcal{H}_{S0}$ and the perturbed part $\mathcal{H}_{S1}$, as

$$ \mathcal{H}_S = \mathcal{H}_{S0} + \mathcal{H}_{S1}, \quad \hat{\mathcal{H}}_S \equiv (\mathcal{H}_S - \hat{\mathcal{H}}_{S0}^\dagger)/\hbar = \hat{\mathcal{H}}_{S0} + \hat{\mathcal{H}}_{S1}, $$  \hspace{1cm} (2.32)

we have

$$ \exp\{-i \hat{\mathcal{H}}_S t + C^{(2)} t\} = U_0(t) \exp\{-i \int_0^t d\tau \hat{\mathcal{H}}_{S1}(\tau)\}, $$  \hspace{1cm} (2.33)

where we have defined

$$ U_0(t) = \exp\{-i (\hat{\mathcal{H}}_{S0} - C^{(2)} t)\} = \exp\{-i (\hat{\mathcal{H}}_{S0} + i C^{(2)} t)\}, $$  \hspace{1cm} (2.34)

$$ \hat{\mathcal{H}}_{S1}(t) = U_0^{-1}(t) \hat{\mathcal{H}}_{S1} U_0(t). $$  \hspace{1cm} (2.35)

Then, the reduced thermal state $|\rho(t)\rangle \equiv |\langle 1_R|\rho_T(t)\rangle\rangle$ can be rewritten as

$$ |\rho(t)\rangle = U_0(t) \exp\{-i \int_0^t d\tau \hat{\mathcal{H}}_{S1}(\tau)\} |\rho(0)\rangle, \quad (\tau_c \to 0) $$  \hspace{1cm} (2.36)

and the expectation value of a physical quantity $A$ of the system can be described as

$$ \langle 1_A|\rho_T(t)\rangle = \langle 1_S|A|\rho(t)\rangle = \langle 1_S|AU_0(t) \exp\{-i \int_0^t d\tau \hat{\mathcal{H}}_{S1}(\tau)\} |\rho(0)\rangle, $$  \hspace{1cm} (2.37)

which has a form convenient for the expansion in powers of the interaction $\mathcal{H}_{S1}$.

Let us consider a boson system interacting with a heat reservoir in an external static field. We divide the Hamiltonian $\mathcal{H}_S$ of the boson system into the Hamiltonian...
$\mathcal{H}_0$ of free bosons and the boson-boson interaction $\mathcal{H}_1$, and take the Hamiltonian of the free bosons in the external static field as

$$\mathcal{H}_0 = \sum_k \hbar \epsilon_k a_k^\dagger a_k, \quad (2.38)$$

with the free boson energy $\hbar \epsilon_k$ of wave number $k$, where $a_k$ and $a_k^\dagger$ are the boson operators of wave number $k$. We also take the boson-reservoir interaction as

$$\mathcal{H}_{SR} = \sum_k \hbar g_{1k} (a_k R_k^\dagger + a_k^\dagger R_k) + \sum_k \hbar g_{2k} a_k^\dagger a_k R_k^\dagger R_k, \quad (2.39)$$

where $R_k$ and $R_k^\dagger$ are the reservoir operators of wave number $k$, and $\hbar g_{ik} (i = 1, 2)$ are the coupling constants between the boson system and heat reservoir. In the boson-reservoir interaction (2.39), the first term is non-commutable with the free boson Hamiltonian $\mathcal{H}_0$ and is called the ‘non-adiabatic part’, while the second term is commutable with $\mathcal{H}_0$ and is called the ‘adiabatic part’. We renormalize the free boson Hamiltonian $\mathcal{H}_0$ and the boson-reservoir interaction $\mathcal{H}_{SR}$ according to (2.17) and (2.18) as

$$\mathcal{H}_0 \Rightarrow \mathcal{H}_0 = \hbar \sum_k \epsilon_k a_k^\dagger a_k, \quad \epsilon_k = \epsilon_k + g_{2k} \langle 1_R | R_k^\dagger R_k | \rho_R \rangle, \quad (2.40)$$

$$\mathcal{H}_{SR} \Rightarrow \mathcal{H}_{SR} = \hbar \sum_k \{ g_{1k} (a_k R_k^\dagger + a_k^\dagger R_k) + g_{2k} a_k^\dagger a_k (R_k^\dagger R_k - \langle 1_R | R_k^\dagger R_k | \rho_R \rangle) \}. \quad (2.41)$$

We assume that the thermal equilibrium expectation values of the reservoir operators $R_k$ and $R_k^\dagger$ vanish, i.e., $\langle 1_R | R_k | \rho_R \rangle = \langle 1_R | R_k^\dagger | \rho_R \rangle = 0$, and then we have $\langle 1_R | \mathcal{H}_{SR} | \rho_R \rangle = 0$ and $\langle 1_R | \mathcal{H}_{SR} | \rho_R \rangle = 0$ which are consistent with (2.19). The boson operators and their tilde conjugates satisfy the commutation relations

$$[a_k, a_{k'}^\dagger] = [\tilde{a}_k, \tilde{a}_{k'}^\dagger] = \delta_{kk'}, \quad (2.42)$$

while the other commutators vanish. We provide the basic requirements (axioms) for the boson operators and reservoir operators

$$\langle 1_S | a_k = \langle 1_S | a_k^\dagger, \quad \langle 1_S | a_k^\dagger = \langle 1_S | a_k, \quad (2.43)$$

$$\langle 1_R | R_k = \langle 1_R | R_k^\dagger, \quad \langle 1_R | R_k^\dagger = \langle 1_R | R_k, \quad (2.44)$$

according to (2.4) as in Ref. 10. Substituting (2.41) into (2.30) and using the axioms (2.44), the collision operator $C^{(2)}$ takes the form

$$C^{(2)} = -\sum_k \{ (a_k - a_k^\dagger)(a_k^\dagger \phi_k^{z+}(\epsilon_k) - \tilde{a}_k \phi_k^{z-}(\epsilon_k)^*) + (\tilde{a}_k - a_k^\dagger)(a_k^\dagger \phi_k^{z-}(\epsilon_k)^* - a_k \phi_k^{z+}(\epsilon_k)) + (a_k^\dagger a_k - \tilde{a}_k^\dagger \tilde{a}_k)(a_k^\dagger a_k \phi_k^{zz}(0) - \tilde{a}_k^\dagger \tilde{a}_k \phi_k^{zz}(0)^*) \}, \quad (2.45)$$
In the calculation of the collision operator $C$,

$$C = \int_0^\infty d\tau g_{tk}^2 \langle 1_R | R_k(t) R_k(0) | \rho_R \rangle \exp(-i\epsilon \tau),$$  \hspace{1cm} (2.46)

$$\phi_k^{\pm}(\epsilon) = \int_0^\infty d\tau g_{tk}^2 \langle 1_R | R_k(t) R_k(0) | \rho_R \rangle \exp(i\epsilon \tau),$$ \hspace{1cm} (2.47)

$$\phi_k^{zz}(\epsilon) = \int_0^\infty d\tau g_{tk}^2 \langle 1_R | (R_k(t) R_k(0)) \rangle \exp(i\epsilon \tau),$$ \hspace{1cm} (2.48)

where we have assumed that the reservoir operators for each wave number are mutually independent and that

$$\langle 1_R | R_k(t) R_k(0) | \rho_R \rangle = \langle 1_R | R_k(t) R_k(0) | \rho_R \rangle = 0,$$ \hspace{1cm} (2.49)

$$\langle 1_R | R_k(t) R_k(0) | \rho_R \rangle = \langle 1_R | R_k(t) R_k(0) | \rho_R \rangle = 0.$$ \hspace{1cm} (2.50)

In the calculation of the collision operator $C^{(2)}$, we have ignored the terms that contain the collision-boson interaction $H_{S1}$. The third term in the parentheses $\{}$ of the collision operator $C^{(2)}$ given by (2.45) corresponds to the fourth-order part which comes from the adiabatic part of the boson-reservoir interaction (2.39). The axioms (2.43) lead to

$$\langle 1_S | C^{(2)} = 0, \langle 1_S | U_0(t) = \langle 1_S | U_0^{-1}(t) = \langle 1_S |,$$ \hspace{1cm} (2.51)

for $U_0(t)$ defined by (2.34). As in Refs. 9) and 10), we provide the thermal state conditions at the initial time $t = 0$ for the ket-vector $|\rho(0)\rangle$ of the boson system as

$$a_k |\rho(0)\rangle = h_k(0) a_k^{\dagger} |\rho(0)\rangle, \quad \tilde{a}_k |\rho(0)\rangle = h_k(0) a_k^{\dagger} |\rho(0)\rangle,$$ \hspace{1cm} (2.52)

where the $c$-number function $h_k(0)$ is given by

$$h_k(0) = \frac{n_k(0)}{1 + n_k(0)}, \quad n_k(0) = \frac{1}{1 - h_k(0)},$$ \hspace{1cm} (2.53)

with the equilibrium boson number $n_k(0)$ of wave number $k$, which is defined by

$$n_k(0) = \langle 1_S | a_k^{\dagger} a_k |\rho(0)\rangle = \text{tr} a_k^{\dagger} a_k |\rho(0)\rangle \equiv \langle a_k^{\dagger} a_k |\rho(0)\rangle.$$ \hspace{1cm} (2.54)

Here, we have used the fact that the ket-vector $|\rho(0)\rangle$ is normalized, i.e.,

$$\langle 1_S |\rho(0)\rangle = \text{tr} |\rho(0)\rangle = 1.$$ \hspace{1cm} (2.55)

As in Refs. 9) and 10), we define the Heisenberg operators for the semi-free boson field:

$$a_k(t) = U_0^{-1}(t) a_k U_0(t), \quad a_k^{\dagger}(t) = U_0^{-1}(t) a_k^{\dagger} U_0(t),$$ \hspace{1cm} (2.56)

$$\tilde{a}_k(t) = U_0^{-1}(t) \tilde{a}_k U_0(t), \quad \tilde{a}_k^{\dagger}(t) = U_0^{-1}(t) \tilde{a}_k^{\dagger} U_0(t).$$ \hspace{1cm} (2.57)
which satisfy the canonical commutation relations
\[ [a_k(t), a_{k'}^{\dagger}(t)] = [\tilde{a}_k(t), \tilde{a}_{k'}^{\dagger}(t)] = \delta_{kk'}, \tag{2.58} \]
while the other commutators vanish. Using the axioms (2.43) and (2.51), we have
\[ \langle 1_S | a_k(t) = \langle 1_S | \tilde{a}_k^{\dagger}(t), \quad \langle 1_S | a_k^{\dagger}(t) = \langle 1_S | \tilde{a}_k(t), \tag{2.59} \]
which are the thermal state conditions at time \( t \) for the bra-vector \( \langle 1_S | \) of the boson system.\(^{10}\)
As done in Ref. 10, we extend the thermal state conditions (2.62) at the initial time \( t = 0 \) to the thermal state conditions at time \( t \) as
\[ a_k(t) |\rho(0)\rangle = h_k(t) \tilde{a}_k^{\dagger}(t) |\rho(0)\rangle, \quad \tilde{a}_k(t) |\rho(0)\rangle = h_k(t) a_k^{\dagger}(t) |\rho(0)\rangle, \tag{2.60} \]
where the \( c \)-number function \( h_k(t) \) is given by
\[ h_k(t) = \frac{n_k(t)}{1 + n_k(t)}, \quad n_k(t) = \langle 1_S | a_k^{\dagger}(t) a_k(t) |\rho(0)\rangle. \tag{2.61} \]
We now introduce the annihilation and creation quasi-particle operators, respectively, defined by\(^9,10\)
\[ \alpha_k(t) = Z_k^{1/2}(t)\{a_k(t) - h_k(t) \tilde{a}_k^{\dagger}(t)\}, \quad \tilde{a}_k(t) = Z_k^{1/2}(t)\{\tilde{a}_k(t) - h_k(t) a_k^{\dagger}(t)\}, \tag{2.62} \]
\[ \alpha_k^+(t) = Z_k^{1/2}(t)\{a_k^{\dagger}(t) - \tilde{a}_k(t)\}, \quad \tilde{a}_k^+(t) = Z_k^{1/2}(t)\{\tilde{a}_k^{\dagger}(t) - a_k(t)\}, \tag{2.63} \]
where the normalization factor \( Z_k^{1/2}(t) \) is given by
\[ Z_k(t) = \{1 - h_k(t)\}^{-1} = 1 + n_k(t), \quad h_k(t) = 1 - Z_k^{-1}(t). \tag{2.64} \]
These lead to the canonical commutation relations of the quasi-particle operators:
\[ [\alpha_k(t), \alpha_{k'}^{\dagger}(t)] = [\tilde{a}_k(t), \tilde{a}_{k'}^{\dagger}(t)] = \delta_{kk'}, \tag{2.65} \]
while the other commutators vanish. The thermal state conditions (2.59) and (2.60) give
\[ \langle 1_S | \alpha_k^+(t) = 0, \quad \langle 1_S | \tilde{a}_k^+(t) = 0, \tag{2.66} \]
\[ \alpha_k(t) |\rho(0)\rangle = 0, \quad \tilde{a}_k(t) |\rho(0)\rangle = 0. \tag{2.67} \]
In the meaning of (2.66) and (2.67), \( \langle 1_S | \) and \( |\rho(0)\rangle \) are, respectively, called the thermal vacuum bra-vector and the thermal vacuum ket-vector for the boson system.\(^9,10\)
The inverse transformation of (2.62) and (2.63) gives
\[ a_k(t) = Z_k^{1/2}(t)\{\alpha_k(t) + h_k(t) \tilde{a}_k^{\dagger}(t)\}, \quad \tilde{a}_k(t) = Z_k^{1/2}(t)\{\tilde{a}_k(t) + h_k(t) \alpha_k^{\dagger}(t)\}, \tag{2.68} \]
\[ a_k^{\dagger}(t) = Z_k^{1/2}(t)\{\alpha_k^{\dagger}(t) + \tilde{a}_k(t)\}, \quad \tilde{a}_k^{\dagger}(t) = Z_k^{1/2}(t)\{\tilde{a}_k^{\dagger}(t) + \alpha_k(t)\}. \tag{2.69} \]
We next derive the forms of the quasi-particle operators. For the Heisenberg operators \(a_k(t)\) and \(a_k^\dagger(t)\) defined by (2.56), we have

\[
\frac{d}{dt} a_k(t) = U_0^{-1}(t) \left[ i \hat{\mathcal{H}}_{S0} - C^{(2)}, a_k \right] U_0(t),
\]

\[
= -i \epsilon_k a_k(t) - \phi_k^{+ -} (\epsilon_k) (a_k(t) - \bar{a}_k^\dagger(t)) - \phi_k^{- +} (\epsilon_k) a_k(t) + \phi_k^{+ -} (\epsilon_k)^* \tilde{a}_k^\dagger(t) - a_k(t) \left\{ a_k^\dagger(t) a_k(t) \phi_k^{zz}(0) - \bar{a}_k^\dagger(t) \tilde{a}_k(t) \phi_k^{zz}(0)^* \right\} - \phi_k^{zz}(0) \left\{ a_k^\dagger(t) a_k(t) - \bar{a}_k^\dagger(t) \tilde{a}_k(t) \right\} a_k(t),
\]

(2.70)

\[
\frac{d}{dt} a_k^\dagger(t) = U_0^{-1}(t) \left[ i \hat{\mathcal{H}}_{S0} - C^{(2)}, a_k^\dagger \right] U_0(t),
\]

\[
= i \epsilon_k a_k^\dagger(t) + \phi_k^{+ -} (\epsilon_k) a_k^\dagger(t) - \phi_k^{- +} (\epsilon_k)^* \bar{a}_k(t) + \phi_k^{+ -} (\epsilon_k) (a_k^\dagger(t) - \tilde{a}_k(t)) + a_k^\dagger(t) \left\{ a_k^\dagger(t) a_k(t) \phi_k^{zz}(0) - \bar{a}_k^\dagger(t) \tilde{a}_k(t) \phi_k^{zz}(0)^* \right\} + \phi_k^{zz}(0) \left\{ a_k^\dagger(t) a_k(t) - \bar{a}_k^\dagger(t) \tilde{a}_k(t) \right\} a_k^\dagger(t),
\]

(2.71)

which give the equation of motion for \(n_k(t)\) defined by (2.61):

\[
\frac{d}{dt} n_k(t) = \langle 1s | \left( \frac{d}{dt} a_k^\dagger(t) \right) a_k(t) | \rho(0) \rangle + \langle 1s | a_k^\dagger(t) \left( \frac{d}{dt} a_k(t) \right) | \rho(0) \rangle,
\]

\[
= - \left\{ \phi_k^{+ -} (\epsilon_k) - \phi_k^{- +} (\epsilon_k)^* + \phi_k^{+ -} (\epsilon_k)^* - \phi_k^{- +} (\epsilon_k) \right\} n_k(t) + \phi_k^{+ -} (\epsilon_k) + \phi_k^{+ -} (\epsilon_k)^*,
\]

\[
= -2 \Phi_k(\epsilon_k) n_k(t) + 2 \Phi_k(\epsilon_k) \bar{n}_k(\epsilon_k) = -2 \Phi_k' n_k(t) + 2 \Phi_k' \bar{n}_k,
\]

(2.72)

where we have introduced the quantity \(\Phi_k(\epsilon) = -i \Phi_k^\prime(\epsilon) + i \Phi_k''(\epsilon)\) defined by (9)

\[
\Phi_k(\epsilon) = \phi_k^{+ -} (\epsilon) - \phi_k^{- +} (\epsilon)^* = g_{kk}^2 \int_0^\infty d\tau \langle 1_R | [R_k(\tau), R_k^\dagger(0)] | \rho_R \rangle e^{i \epsilon \tau},
\]

(2.73)

and have put

\[
\Phi_k' = \Phi_k'(\epsilon_k), \quad \Phi_k'' = \Phi_k''(\epsilon_k), \quad \bar{n}_k = \bar{n}_k(\epsilon_k) = \{ \exp(\beta h \epsilon_k) - 1 \}^{-1}.
\]

(2.74)

Then, \(n_k(t)\) defined by (2.61) takes the form

\[
n_k(t) = \{ n_k(0) - \bar{n}_k \} \exp(-2 \Phi_k' t + \bar{n}_k), \quad n_k(0) = \langle 1s | a_k^\dagger a_k | \rho(0) \rangle
\]

(2.75)

which is independent of \(\phi_k^{zz}(0)\) and \(\phi_k^{zz}(0)^*\) and has the same form as in Ref. 10.

Equations (2.70) and (2.71) lead to the equations of motion for the quasi-particle operators:

\[
\frac{d}{dt} Z_k^{1/2}(t) \alpha_k(t) = \frac{d}{dt} Z_k(t) \left\{ a_k(t) + h_k(t) \bar{a}_k^\dagger(t) \right\},
\]

\[
= (d/dt) \left\{ (n_k(t) + 1) a_k(t) - n_k(t) \bar{a}_k^\dagger(t) \right\},
\]

\[
= \{-i (\epsilon_k + \Phi_k'') - \Phi_k'\} \left\{ (n_k(t) + 1) a_k(t) - n_k(t) \bar{a}_k^\dagger(t) \right\} - \{ \phi_k^{+ -} (0) a_k^\dagger(t) a_k(t) - \phi_k^{+ -} (0)^* \bar{a}_k^\dagger(t) \tilde{a}_k(t) \} \left\{ (n_k(t) + 1) a_k(t) - n_k(t) \bar{a}_k^\dagger(t) \right\} - (a_k^\dagger(t) a_k(t) - \bar{a}_k^\dagger(t) \tilde{a}_k(t)) \left\{ (n_k(t) + 1) \phi_k^{+ -} (0) a_k(t) - n_k(t) \phi_k^{+ -} (0)^* \bar{a}_k^\dagger(t) \right\}
\]

(2.76)
as shown in Refs. 14 and 9), we assume that

\[ \{ \phi_k^{\pi\pi}(0) (n_k(t) + 1) a_k(t) - \phi_k^{\pi\pi}(0)^* n_k(t) \tilde{a}_k(t) \}, \quad (2.76) \]

\[ \frac{d}{dt} Z_k^{-1/2}(t) \alpha_k(t) = \frac{d}{dt} \{ a_k^{\dagger}(t) - \tilde{a}_k(t) \}, \]

\[ = \{ i (\epsilon_k + \Phi''_k) + \Phi'_k \} \{ a_k^{\dagger}(t) - \tilde{a}_k(t) \}

+ (a_k^{\dagger}(t) - \tilde{a}_k(t)) \{ \phi_k^{\pi\pi}(0) a_k^{\dagger}(t) a_k(t) - \phi_k^{\pi\pi}(0)^* \tilde{a}_k(t) \}

+ \{ \phi_k^{\pi\pi}(0) a_k^{\dagger}(t) - \phi_k^{\pi\pi}(0)^* \tilde{a}_k(t) \} (a_k^{\dagger}(t) a_k(t) - \tilde{a}_k(t) \tilde{a}_k(t))

+ \phi_k^{\pi\pi}(0) a_k^{\dagger}(t) - \phi_k^{\pi\pi}(0)^* \tilde{a}_k(t), \quad (2.77) \]

which are derived in Appendix A. When \( \phi_k^{\pi\pi}(0) \) is real, i.e., \( \phi_k^{\pi\pi}(0) = \phi_k^{\pi\pi}(0)^* \), Eqs. (2.76) and (2.77) lead to the equations

\[ (d/dt) \langle 1_s | Z_k^{1/2}(t) \alpha_k(t) \rangle = (d/dt) \langle 1_s | \{(n_k(t) + 1) a_k(t) - n_k(t) \tilde{a}_k(t) \}, \]

\[ = \{ -i (\epsilon_k + \Phi''_k) - \Phi'_k - \phi_k^{\pi\pi}(0) \} \{ 1_s | \{(n_k(t) + 1) a_k(t) - n_k(t) \tilde{a}_k(t) \}, \]

\[ = \{ -i (\epsilon_k + \Phi''_k) - \Phi'_k - \phi_k^{\pi\pi}(0) \} \{ 1_s | Z_k^{1/2}(t) \alpha_k(t), \quad (2.78) \]

\[ (d/dt) Z_k^{-1/2}(t) \alpha_k^{\dagger}(t) | \rho(0) \rangle = (d/dt) \{ a_k^{\dagger}(t) - \tilde{a}_k(t) \} | \rho(0) \rangle,

\[ = \{ i (\epsilon_k + \Phi''_k) + \Phi'_k + \phi_k^{\pi\pi}(0) \} \{ a_k^{\dagger}(t) - \tilde{a}_k(t) \} | \rho(0) \rangle,

\[ = \{ i (\epsilon_k + \Phi''_k) + \Phi'_k + \phi_k^{\pi\pi}(0) \} Z_k^{-1/2}(t) \alpha_k^{\dagger}(t) | \rho(0) \rangle, \quad (2.79) \]

where we have used the thermal state conditions (2.59) and (2.60) at time \( t \). Hereafter, we assume that \( \phi_k^{\pi\pi}(0) \) is real, i.e., \( \phi_k^{\pi\pi}(0) = \phi_k^{\pi\pi}(0)^* \). Equations (2.78) and (2.79) can be solved to give the forms of the quasi-particle operators:

\[ \langle 1_s | \alpha_k(t) = Z_k^{1/2}(0) Z_k^{-1/2}(t) \exp \{-i (\epsilon_k + \Phi''_k) t - \Phi'_k t - \Psi_k t \} \langle 1_s | \alpha_k, \quad (2.80) \]

\[ \alpha_k^{\dagger}(t) | \rho(0) \rangle = Z_k^{-1/2}(0) Z_k^{1/2}(t) \exp \{i (\epsilon_k + \Phi''_k) t + \Phi'_k t + \Psi_k t \} \alpha_k^{\dagger} | \rho(0) \rangle, \quad (2.81) \]

with \( \alpha_k = \alpha_k(0) \) and \( \alpha_k^{\dagger} = \alpha_k^{\dagger}(0) \), where we have put

\[ \Psi_k = \phi_k^{\pi\pi}(0) = \int_0^\infty d\tau \sum_{2k} \langle 1_R | (R_k^\dagger(\tau) R_k(\tau) - \langle 1_R | R_k^\dagger R_k | \rho_R \rangle \rangle \times (R_k^\dagger R_k - \langle 1_R | R_k^\dagger R_k | \rho_R \rangle) | \rho_R \rangle. \quad (2.82) \]

From the quasi-particle forms (2.80) and (2.81), we obtain

\[ \langle 1_s | \alpha_k(t) \alpha_k^{\dagger}(s) | \rho(0) \rangle = \frac{Z_k^{1/2}(s)}{Z_k^{1/2}(t)} \exp \{-i (\epsilon_k + \Phi''_k)(t - s) - (\Phi'_k + \Psi_k)(t - s) \}, \quad (2.83) \]

\[ \langle 1_s | \tilde{a}_k(t) \tilde{a}_k^{\dagger}(s) | \rho(0) \rangle = \frac{Z_k^{1/2}(s)}{Z_k^{1/2}(t)} \exp \{i (\epsilon_k + \Phi''_k)(t - s) - (\Phi'_k + \Psi_k)(t - s) \}, \quad (2.84) \]

where we have used the commutation relations (2.65) and the thermal state conditions (2.66) and (2.67). As shown in Refs. 14 and 9), \( \Phi'_k \) is non-negative for non-negative \( \epsilon_k \) in general, and as shown in Appendix B, \( \Psi_k = \phi_k^{\pi\pi}(0) \) is non-negative.
when $\Psi_k$ is real ($\Psi_k = \Psi_k^*$). Considering these, we see from (2.83) and (2.84) that the quasi-particles of wave number $k$ for the semi-free boson field, which are created at time $s$ and are annihilated at time $t$, have the energies $\pm (\epsilon_k + \Phi_k'$) and the life-time $(\Phi'_k + \Psi_k)^{-1}$. As $\epsilon_k$ is given by (2.40) and $\Psi_k$ is given by (2.82), the energies and life-time of the quasi-particles depend on the adiabatic boson-reservoir interaction which leads to the fourth-order part of the unperturbed Hamiltonian.

§3. Two-point Green’s function of the semi-free boson field

The two-point Green’s function of the semi-free field for the boson system is defined using the time-chronological-ordering operator $T$ by

$$G_{\mu k}^{\mu
u}(t, s) = -i \langle 1_s | T\{a^\mu_k(t) \bar{a}^\nu_k(s)\} | \rho(0) \rangle, \quad (\mu, \nu = 1, 2)$$

(3.1)

where $\langle 1_s |$ and $| \rho(0) \rangle$ describe the thermal vacuum states which satisfy the thermal state conditions (2.59) and (2.60), respectively. Here, we have introduced the thermal doublet

$$a^1_k(t) = a_k(t), \quad a^2_k(t) = \bar{a}^\dagger_k(t); \quad \bar{a}^1_k(t) = a^\dagger_k(t), \quad \bar{a}^2_k(t) = -\bar{a}_k(t).$$

(3.2)

We also define the thermal doublet for the quasi-particle operators as

$$\alpha^1_k(t) = \alpha_k(t), \quad \alpha^2_k(t) = \bar{a}^\dagger_k(t); \quad \bar{\alpha}^1_k(t) = a^\dagger_k(t), \quad \bar{\alpha}^2_k(t) = -\bar{\alpha}_k(t),$$

(3.3)

by which the two-point Green’s function of the quasi-particle operators is defined as

$$G_{\mu k}^{\mu
u}(t, s) = -i \langle 1_s | T\{\alpha^\mu_k(t) \bar{\alpha}^\nu_k(s)\} | \rho(0) \rangle,$$

(3.4)

$$= \begin{pmatrix} Z_k^{1/2}(s) Z_k^{-1/2}(t) G_{\mu k}^{\nu\nu}(t - s), & 0 \\ 0, & Z_k^{1/2}(t) Z_k^{-1/2}(s) G_{\nu k}^{\mu\nu}(t - s) \end{pmatrix}^{\mu\nu}$$

(3.5)

with

$$G_{\mu k}^{\nu\nu}(t) = -i \Theta(t) \exp\{-i (\epsilon_k + \Phi_k') t - \Phi_k' t - \Psi_k t\},$$

(3.6)

$$G_{\nu k}^{\mu\nu}(t) = i \Theta(-t) \exp\{-i (\epsilon_k + \Phi_k') t + \Phi_k' t + \Psi_k t\},$$

(3.7)

where $\Theta(t)$ is the step function defined by $\Theta(t) = 1$ for $t > 0$, $\Theta(t) = 0$ for $t < 0$. We can write the relations (2.62), (2.63), (2.68) and (2.69) in the thermal doublet notation as

$$\alpha^\mu_k(t) = \sum_\nu S_k(t)^{\mu\nu} \alpha^\nu_k(t), \quad \bar{\alpha}^\mu_k(t) = \sum_\nu \bar{a}^\nu_k S_k^{-1}(t)^{\nu\mu},$$

(3.8)

$$a^\mu_k(t) = \sum_\nu S_k^{-1}(t)^{\mu\nu} \alpha^\nu_k(t), \quad \bar{a}^\mu_k(t) = \sum_\nu \bar{\alpha}^\nu_k S_k(t)^{\nu\mu},$$

(3.9)

where the matrix $S_k(t)$ is defined by

$$S_k(t) = Z_k^{1/2}(t) \begin{pmatrix} 1, & -h_k(t) \\ -1, & 1 \end{pmatrix}, \quad S_k^{-1}(t) = Z_k^{1/2}(t) \begin{pmatrix} 1, & h_k(t) \\ 1, & 1 \end{pmatrix} = \tau_3 S_k(t) \tau_3,$$

(3.10)
Then, the two-point Green’s function $G_{0k}^{\mu\nu}(t, s)$ of the semi-free field is given by

$$G_{0k}^{\mu\nu}(t, s) = -i \sum_{\lambda} \sum_{\xi} S_{k}^{-1}(t) \mu_{\lambda} \langle 1_{s} | T\{ \alpha_{k}^{\lambda}(t) \bar{\alpha}_{k}^{\xi}(s) \} | \rho(0) \rangle S_{k}(s) \delta_{\lambda},$$

which takes the explicit expression

$$G_{0k}^{\mu\nu}(t, s) = \left\{ \begin{array}{c} \left( 1 + \bar{n}_{k} \right) \mu_{\lambda} G_{0k}^{\rho}(t - s) + \left( - \bar{n}_{k} \right) \mu_{\lambda} G_{0k}^{a}(t - s) \\
\left( 1, -1 \right) \mu_{\lambda} G_{0k}^{r}(t - s) e^{-2 \Phi_{k}^{s} s} - G_{0k}^{r}(t - s) e^{-2 \Phi_{k}^{t} t} \\
\left( 1 + \bar{n}_{k} \right) \mu_{\lambda} G_{0k}^{a}(t - s) - G_{0k}^{a}(t - s) \right\}^{\mu_{\lambda}}$$

where we have used the relation $h_{k}(t) Z_{k}(t) = n_{k}(t)$. By virtue of the relations (2.64) and (2.75), we have

$$G_{0k}^{\mu\nu}(t, s) = \left\{ \begin{array}{c} \left( 1 + \bar{n}_{k} \right) \mu_{\lambda} G_{0k}^{\rho}(t - s) + \left( - \bar{n}_{k} \right) \mu_{\lambda} G_{0k}^{a}(t - s) \\
\left( 1, -1 \right) \mu_{\lambda} G_{0k}^{r}(t - s) e^{-2 \Phi_{k}^{s} s} - G_{0k}^{r}(t - s) e^{-2 \Phi_{k}^{t} t} \\
\left( 1 + \bar{n}_{k} \right) \mu_{\lambda} G_{0k}^{a}(t - s) - G_{0k}^{a}(t - s) \right\}^{\mu_{\lambda}}$$

$$- i \left( n_{k}(t) - \bar{n}_{k} \right) \left\{ \begin{array}{c} \left( 1, 1 \right) \tau_{3} \\
\left( 1, 1 \right) \tau_{3} \right\}^{\mu_{\lambda}}$$

$$\times \left\{ \Theta(t - s) \exp \{- i (\epsilon_{k} + \Phi_{k}^{\mu})(t - s) - \Phi_{k}^{s}(t + s) - \Psi_{k}(t - s) \} \\
+ \Theta(s - t) \exp \{- i (\epsilon_{k} + \Phi_{k}^{\mu})(t - s) - \Phi_{k}^{s}(t + s) - \Psi_{k}(s - t) \} \right\},$$

where we have used (3.6) and (3.7).

§4. Admittance of boson system by the TCLE method

We consider a boson system interacting with a heat reservoir in time-dependent external fields which are composed of a static field and a weak driving field. We take into account the Hamiltonians of the boson system and heat reservoir in the
We consider the case that the external driving field is a classical field described by a c-number function of time $t$ and is a periodic function of the frequency $\omega$, and take the interaction of the boson system with the external driving field as

$$\mathcal{H}_{ed}(t) = -\sum_k \{ a_k F_k^*(t) + a_k^\dagger F_k(t) \} = -\sum_k \{ a_k F_k^*[\omega] e^{i\omega t} + a_k^\dagger F_k[\omega] e^{-i\omega t} \}, \quad (4.1)$$

where $F_k(t)$ is the Fourier transform of wave number $k$ for the external driving field. We assume that the external driving field is turned on adiabatically at the initial time $t = 0$, and that the boson system and heat reservoir in the external static field are in the thermal equilibrium state at the initial time $t = 0$, i.e., $|\rho_T(0)\rangle = |\rho_{TE}\rangle$. Then, using the TCLE method in the lowest Born approximation for the boson-reservoir interaction, the admittance $\chi_{a_k a_k^\dagger}(\omega)$ of the boson system takes the following expressions\textsuperscript{(13)–(15)}

$$\chi_{a_k a_k^\dagger}(\omega) = \langle 1_S | a_k ( i (\hat{H}_S - \omega) - C^{(2)} )^{-1} \{ i \hat{a}_k^\dagger |\rho(0)\rangle + |D^{(2)}_{a_k^\dagger}[\omega]\rangle \},$$

$$= \int_0^\infty dt \langle 1_S | a_k U_0(t) \exp_i \{- i \int_0^t d\tau \hat{H}_{S1}(\tau) \{ i \hat{a}_k^\dagger |\rho(0)\rangle + |D^{(2)}_{a_k^\dagger}[\omega]\rangle \} e^{i\omega t}, \quad (4.2)$$

with $|\rho(0)\rangle$ defined by $|\rho(0)\rangle = \langle 1_R |\rho_{TE}\rangle$. Here, the collision operator $C^{(2)}$ is given by (2.30) and the interference thermal state (interference term) $|D^{(2)}_{a_k^\dagger}[\omega]\rangle$ is given by

$$|D^{(2)}_{a_k^\dagger}[\omega]\rangle = -i \int_0^\infty d\tau \int_0^\tau ds \langle 1_R | \hat{H}_{SR} e^{-i \hat{H}_0 s} \hat{a}_k^\dagger e^{i \hat{H}_0 (s-\tau)} \hat{H}_{SR} |\rho(0)\rangle |\rho_R\rangle e^{i\omega s}$$

$$+ i \int_0^\infty d\tau \int_0^\tau ds \langle 1_R | \hat{H}_{SR} e^{-i \hat{H}_0 \tau} \hat{H}_{SR} e^{i \hat{H}_0 (s-\tau)} \hat{a}_k^\dagger |\rho(0)\rangle |\rho_R\rangle e^{i\omega s}, \quad (4.3)$$

which represents the effects of the initial correlation and memory for the boson system and heat reservoir\textsuperscript{(13)–(15)}. In Appendix C, by using the axioms (2.44) and (2.52), the interference thermal state $|D^{(2)}_{a_k^\dagger}[\omega]\rangle$ is calculated as

$$|D^{(2)}_{a_k^\dagger}[\omega]\rangle = \{ \phi_k^{-+}(\omega) - \phi_k^{-+}(\epsilon_k) - (\phi_k^{-+}(\omega)^* - \phi_k^{-+}(\epsilon_k)^*) \}(a_k^\dagger - \hat{a}_k) |\rho(0)\rangle$$

$$+ \{ a_k (\hat{a}_k e^{zz}(\omega - \epsilon_k) - \hat{a}_k e^{zz}(0)) - \hat{a}_k (\phi_k e^{zz}(\omega + \epsilon_k)^* - \phi_k e^{zz}(0)^*) \} |\rho(0)\rangle \}

/\langle \hbar (\omega - \epsilon_k) \rangle, \quad (4.4)$$

where we have ignored the terms that contain the boson-boson interaction $\mathcal{H}_{S1}$. Substituting the interference thermal state (4.4) into (4.2) and using the axioms (2.51), the admittance $\chi_{a_k a_k^\dagger}(\omega)$ can be rewritten as

$$\chi_{a_k a_k^\dagger}(\omega) = \frac{1}{\hbar} \int_0^\infty dt \langle 1_S | a_k(t) \exp_i \{- i \int_0^t d\tau \hat{H}_{S1}(\tau) \{ a_k^\dagger(0) - \hat{a}_k(0) \} |\rho(0)\rangle e^{i\omega t}$$

$$\times \{ i + X_k(\omega) \}, \quad (4.4)$$
In the derivation of (4.11), we have considered that

\[ \chi(\omega) = \text{with } \hbar \]

where

\[ \Psi_k = \text{real for non-negative } \epsilon_k \text{ in general} \]

and that \( \Psi_k \) is non-negative when \( \Psi_k \) is real (\( \Psi_k = \Psi_k^* \)) as shown in Appendix B. The first-order part \( \chi^{(1)}_{a_k^a k}(\omega) \) is represented as

\[
\begin{align*}
\chi^{(1)}_{a_k a_k^a}(\omega) & = - \frac{i}{h} \int_0^\infty dt \int_0^t dt_1 \frac{Z_{k}^{1/2}(t)}{Z_{k}^{1/2}(0)} \langle 1_s | \alpha_k(t) \tilde{H}_{S1}(t_1) \alpha_k^a | \rho(0) \rangle \{ i + X_k(\omega) \} e^{i \omega t}.
\end{align*}
\]

(4.12)
Solving Eq. (2.78) of motion for \( \langle 1_s | Z_k^{1/2}(t) \alpha_k(t) \rangle \) as
\[
\langle 1_s | Z_k^{1/2}(t) \alpha_k(t) \rangle = \exp\{-i(\epsilon_k + \Phi'_k)(t - t_1) - (\Phi_k + \Psi_k)(t - t_1)\}\langle 1_s | Z_k^{1/2}(t_1) \alpha_k(t_1) \rangle,
\]
and using the axioms (2.4) and (2.51), we obtain
\[
Z_k^{1/2}(t)\langle 1_s | \alpha_k(t) \hat{H}_{S1}(t_1) \rangle = Z_k^{1/2}(t_1)\langle 1_s | [\alpha_k(t_1) , \hat{H}_{S1}(t_1) ] \exp\{-i(\epsilon_k + \Phi'_k)(t - t_1) - (\Phi_k + \Psi_k)(t - t_1)\},
\]
\[
- Z_k(t_1)\langle 1_s | [\alpha_k(t_1) - h_k(t_1) \hat{a}_k^+(t_1) , \hat{H}_{S1}(t_1) ] \times \exp\{-i(\epsilon_k + \Phi'_k)(t - t_1) - (\Phi_k + \Psi_k)(t - t_1)\},
\]
\[
= \frac{1}{2} Z_k(t_1) \sum_q \sum_{k'} V_q \langle 1_s | \{ a_{k'-q}(t_1) a_k^+(t_1) a_{k-q}(t_1) + a_{k'+q}(t_1) a_{k'}(t_1) a_{k+q}(t_1) - h_k(t_1) \{ \hat{a}_{k'-q}(t_1) \hat{a}_k^+(t_1) \hat{a}_{k-q}(t_1) + \hat{a}_{k'+q}(t_1) \hat{a}_{k'}(t_1) \hat{a}_{k+q}(t_1) \} \alpha_k^+(t_1) | \rho(0) \rangle \times \exp\{-i(\epsilon_k + \Phi'_k)(t - t_1) - (\Phi_k + \Psi_k)(t - t_1)\},
\]
which leads to
\[
Z_k^{1/2}(t) Z_k^{-1/2}(\langle 1_s | \alpha_k(t) \hat{H}_{S1}(t_1) \alpha_k^+ | \rho(0) \rangle) = \frac{1}{2} Z_k(t_1) \sum_q \sum_{k'} V_q \langle 1_s | \{ a_{k'-q}(t_1) a_k^+(t_1) a_{k-q}(t_1) + a_{k'+q}(t_1) a_{k'}(t_1) a_{k+q}(t_1) - h_k(t_1) \{ \hat{a}_{k'-q}(t_1) \hat{a}_k^+(t_1) \hat{a}_{k-q}(t_1) + \hat{a}_{k'+q}(t_1) \hat{a}_{k'}(t_1) \hat{a}_{k+q}(t_1) \} \alpha_k^+(t_1) | \rho(0) \rangle \times \exp\{-i(\epsilon_k + \Phi'_k)(t - t_1) - (\Phi_k + \Psi_k)(t - t_1)\},
\]
where we have used (2.81). By using the transformations (2.68) and (2.69), the commutation relations (2.65) and the thermal state conditions (2.66) and (2.67), (4.15) can be calculated as
\[
Z_k^{1/2}(t) Z_k^{-1/2}(\langle 1_s | \alpha_k(t) \hat{H}_{S1}(t_1) \alpha_k^+ | \rho(0) \rangle) = \frac{1}{2} Z_k(t_1) \sum_{k'} (2V_0 + V_{k-k'} + V_{k'-k}) Z_{k'}(t_1) h_{k'}(t_1) \{ 1 - h_k(t_1) \} \times \exp\{-i(\epsilon_k + \Phi'_k)(t - (\Phi_k' + \Psi_k)(t - t_1)\},
\]
which is derived in Appendix D. Substituting (4.16) into (4.12) and using the relations (2.64), we obtain the expression
\[
\chi^{(1)}_{\alpha_k \alpha_k}(t) = \frac{i}{2\hbar} \sum_{k'} \int_0^\infty dt \int_0^t dt_1 n_{k'}(t_1) \exp\{-i(\epsilon_k + \Phi'_k)(t - (\Phi_k' + \Psi_k)(t + i\omega)t\}
\]
\[
\times (2V_0 + V_{k-k'} + V_{k'-k}) \{ i + X_k(\omega) \},
\]
which can be integrated, by using the expression (2.75) of \( n_k(t) \), as
\[
\chi^{(1)}_{\alpha_k \alpha_k}(t) = \frac{i}{2\hbar} \sum_{k'} \{ i(\epsilon_k + \Phi'_k - \omega) + \Phi_k' + \Psi_k \} \{ i(\epsilon_k + \Phi'_k - \omega) + \Phi_k' + \Psi_k + 2\Phi'_k \}
\]
the calculations of the expectation values \( \langle \cdot \rangle \) of the admittance given by (4.5) in powers of the boson-boson interaction \( \mathcal{H}_{S1} \). Since (4.14) can be rewritten, by using the thermal state conditions (2.59) and the relations (2.64), as

\[
\langle 1_S\cdots\vert \rho(0) \rangle \text{ if not connected. Those correspond to the calculations of the connected diagrams in Feynman diagram method. Let us consider the calculations of the expectation values } \langle 1_S\cdots\vert \rho(0) \rangle \text{ in (4.10). By virtue of the transformations (2.62), we have}
\]

\[
[\alpha_k(t), \hat{\mathcal{H}}_{S1}(t)] = \frac{1}{2} Z_{k'}^{1/2}(t) \sum_q \sum_{k'} (V_q + V_{-q}) \left\{ a_{k'}^{\dagger}(t) a_{k'+q}(t) a_{k-q}(t) - h_k(t) \tilde{a}_{k-q}(t) \tilde{a}_{k'+q}(t) \tilde{a}_{k'}(t) \right\},
\]

\[
[\tilde{a}_k(t), \hat{\mathcal{H}}_{S1}(t)] = \frac{1}{2} Z_{k'}^{1/2}(t) \sum_q \sum_{k'} (V_q + V_{-q}) \left\{ h_k(t) a_{k+q}(t) a_{k'}(t) - \tilde{a}_{k+q}(t) \tilde{a}_{k'}(t) \tilde{a}_{k-q}(t) \right\}.
\]

Since (4.14) can be rewritten, by using the thermal state conditions (2.59) and the relations (2.64), as

\[
Z_{k'}^{1/2}(t) \langle 1_S\vert \alpha_k(t) \hat{\mathcal{H}}_{S1}(t_1) \rangle
\]

\[
= \frac{1}{2} Z_k(t_1)(1 - h_k(t_1)) \sum_q \sum_k (V_q + V_{-q}) \langle 1_S\vert a_{k'}^{\dagger}(t_1) a_{k'+q}(t_1) a_{k-q}(t_1) \rangle
\]

\[
\times \exp \left\{ -i (\epsilon_k + \Phi_k')(t - t_1) - (\Phi_k' + \Psi_k)(t - t_1) \right\},
\]

\[
= \frac{1}{2} \sum_q \sum_{k'} (V_q + V_{-q}) \langle 1_S\vert a_{k'}^{\dagger}(t_1) a_{k'+q}(t_1) a_{k-q}(t_1) \rangle
\]

\[
\times \exp \left\{ -i (\epsilon_k + \Phi_k')(t - t_1) - (\Phi_k' + \Psi_k)(t - t_1) \right\},
\]

we can obtain

\[
Z_{k'}^{1/2}(t) \langle 1_S\vert \alpha_k(t) \hat{\mathcal{H}}_{S1}(t_1) \hat{\mathcal{H}}_{S1}(t_2) \rangle
\]

\[
= \frac{1}{2} \sum_q \sum_{k'} (V_q + V_{-q}) Z_{k'}^{1/2}(t_1) Z_{k'+q}(t_1) Z_{k-q}(t_1) \langle 1_S\vert a_{k'}^{\dagger}(t_1) a_{k'+q}(t_1) a_{k-q}(t_1) \rangle
\]

\[
\times \exp \left\{ -i (\epsilon_k + \Phi_k')(t - t_1) - (\Phi_k' + \Psi_k)(t - t_1) \right\},
\]

\[
= \frac{1}{2} \sum_q \sum_{k'} (V_q + V_{-q}) Z_{k'}^{1/2}(t_1) Z_{k'+q}(t_1) Z_{k-q}(t_1) \times \langle 1_S\vert \tilde{a}_{k'}(t_1) a_{k'+q}(t_1) a_{k-q}(t_1) + h_k(t_1)(\delta_q, 0 + \delta_{q,k-k'}) \rangle
\]
\[ \times \hat{H}_{S1}(t_2) \exp \{ -i (\epsilon_k + \Phi_k')(t - t_1) - (\Phi'_k + \Psi_k)(t - t_1) \}, \quad (4.23) \]

\[ = \frac{1}{2} \sum_q \sum_{k'} (V_q + V_{-q}) Z_{k'}^{1/2}(t_2) Z_{k'+q}^{1/2}(t_2) Z_{k-q}^{1/2}(t_2) \langle 1_s | \tilde{\alpha}_{k'}(t_2) \alpha_{k'+q}(t_2) \alpha_{k-q}(t_2) \times \hat{H}_{S1}(t_2) \exp \{ i (\epsilon_{k'} + \Phi_{k'}') - \epsilon_{k'+q} - \Phi_{k'+q}' - \epsilon_{k-q} - \Phi_{k-q}' (t_1 - t_2) \]

\[ - (\Phi'_{k'} + \Phi'_{k'+q} + \Phi'_{k-q} + \Psi_{k'} + \Psi_{k'+q} + \Psi_{k-q})(t_1 - t_2) \]

\[ - i (\epsilon_k + \Phi_k')(t_1 - t_2) - (\Phi'_k + \Psi_k)(t - t_1) \} \]

\[ + \frac{1}{2} Z_{k'}^{1/2}(t_2) \sum_{k'} (2V_0 + V_{-k'} + V_{k'} - V_{k'-1}) n_{k'}(t_1) \langle 1_s | \alpha_{k}(t_2) \hat{H}_{S1}(t_2) \]

\[ \times \exp \{ -i (\epsilon_k + \Phi_k')(t - t_2) - (\Phi'_k + \Psi_k)(t - t_2) \}, \quad (4.24) \]

where we have used the relation (4.13) and its tilde conjugate. Solving the equation (2.79) of motion for \( Z_k^{-1/2}(t) \alpha_k^\dagger(t) |\rho(0)\rangle \) as

\[ Z_k^{-1/2}(t) \alpha_k^\dagger(t) |\rho(0)\rangle = \exp \{ i (\epsilon_k + \Phi_k')(t - t_1) + (\Phi'_k + \Psi_k)(t - t_1) \}

\[ \times Z_k^{-1/2}(t_1) \alpha_k^\dagger(t_1) |\rho(0)\rangle, \quad (4.25) \]

and using the axioms (2.4) and (2.51), we obtain

\[ Z_k^{1/2}(t) Z_k^{-1/2}(t_1) \langle 1_s | \alpha_k(t) \hat{H}_{S1}(t_1) \hat{H}_{S1}(t_2) \alpha_k^\dagger |\rho(0)\rangle \]

\[ = Z_k^{1/2}(t) Z_k^{-1/2}(t_2) \langle 1_s | \alpha_k(t) \hat{H}_{S1}(t_1) \hat{H}_{S1}(t_2) \alpha_k^\dagger(t_2) |\rho(0)\rangle \]

\[ \times \exp \{ -i (\epsilon_k + \Phi_k')(t - t_2) - (\Phi'_k + \Psi_k)(t_2) \}, \quad (4.26) \]

\[ = \frac{1}{2} Z_k^{-1/2}(t_2) \sum_q \sum_{k'} (V_q + V_{-q}) Z_{k'}^{1/2}(t_2) Z_{k'+q}^{1/2}(t_2) Z_{k-q}^{1/2}(t_2) \]

\[ \times \langle 1_s | [\tilde{\alpha}_{k'}(t_2) \alpha_{k'+q}(t_2) \alpha_{k-q}(t_2), \hat{H}_{S1}(t_2) \alpha_k^\dagger(t_2) ] |\rho(0)\rangle \]

\[ \times \exp \{ i (\epsilon_{k'} + \Phi_{k'}') - \epsilon_{k'+q} - \Phi_{k'+q}' - \epsilon_{k-q} - \Phi_{k-q}' (t_1 - t_2) \]

\[ - (\Phi'_{k'} + \Phi'_{k'+q} + \Phi'_{k-q} + \Psi_{k'} + \Psi_{k'+q} + \Psi_{k-q})(t_1 - t_2) \]

\[ - i (\epsilon_k + \Phi_k')(t_1 - t_2) - (\Phi'_k + \Psi_k)(t - t_1) \} \]

\[ + \frac{1}{2} \sum_{k'} (2V_0 + V_{-k'} + V_{k'} - V_{k'-1}) n_{k'}(t_1) \langle 1_s | [\alpha_k(t_2), \hat{H}_{S1}(t_2) ] \alpha_k^\dagger(t_2) |\rho(0)\rangle \]

\[ \times \exp \{ -i (\epsilon_k + \Phi_k')(t - t_2) - (\Phi'_k + \Psi_k)(t - t_2) \}. \quad (4.27) \]

The expectation values \( \langle 1_s | \cdots |\rho(0)\rangle \) in (4.27) can be calculated by using the relations (4.19) and (4.20), the transformations (2.68) and (2.69), the commutation relations (2.65) and the thermal state conditions (2.66) and (2.67). Those results are integrated according to (4.10) to give the second-order part \( \chi_{g_k a_k^\dagger}^{(2)}(\omega) \) of the admittance \( \chi_{g_k a_k^\dagger}(\omega) \). The higher-order parts can be calculated in the same way. The expectation values \( \langle 1_s | \cdots |\rho(0)\rangle \) in (4.10) and the \( n \)-th-order parts \( \chi_{g_k a_k^\dagger}^{(n)}(\omega) \) of the admittance can be calculated in this way.
In this section, we consider the perturbation expansions of the two-point Green’s function for the boson system. We define the two-point Green’s function for the boson system, using the time-chronological-ordering operator $T$, by

$$G_k^{\mu \nu}(t, s) = -i \langle 1_S | T \{ a_{\text{Hk}}^{\mu}(t) \bar{a}_{\text{Hk}}^{\nu}(s) \} | \rho(0) \rangle, \quad (\mu, \nu = 1, 2) \quad (5.1)$$

where $a_{\text{Hk}}^{\mu}(t)$ and $\bar{a}_{\text{Hk}}^{\nu}(s)$ ($\mu, \nu = 1, 2$) are the Heisenberg operators defined by

$$a_{\text{Hk}}^{1}(t) = a_{\text{Hk}}(t) = U^{-1}(t) a_k U(t), \quad a_{\text{Hk}}^{2}(t) = \bar{a}_{\text{Hk}}^{\dagger}(t) = U^{-1}(t) \bar{a}_{k} U(t), \quad (5.2)$$

$$\bar{a}_{\text{Hk}}^{1}(t) = \bar{a}_{\text{Hk}}^{\dagger}(t) = U^{-1}(t) a_k U(t), \quad a_{\text{Hk}}^{2}(t) = -\bar{a}_{\text{Hk}}(t) = -U^{-1}(t) \bar{a}_k U(t). \quad (5.3)$$

Here, $U(t)$ is the time evolution operator defined by

$$U(t) = \exp \{-i \hat{H}_{t} + C^{(2)} t \} = U_0(t) \exp \left( -i \int_0^t d\tau \hat{H}_{S1} (\tau) \right), \quad (5.4)$$

where $U_0(t)$ and $\hat{H}_{S1}(t)$ are defined by (2.34) and (2.35), respectively. The axioms (2.4) and (2.51) lead to

$$\langle 1_S | U(t) = \langle 1_S | U_{-1}(t) = \langle 1_S \rangle. \quad (5.5)$$

The Heisenberg operators $a_{\text{Hk}}^{\mu}(t)$ and $\bar{a}_{\text{Hk}}^{\nu}(s)$ can be rewritten using the Heisenberg operators $a_k^{\mu}(t)$ and $\bar{a}_k^{\nu}(s)$ defined by (2.56) and (2.57) for the semi-free field, as

$$a_{\text{Hk}}^{\mu}(t) = \exp \left( i \int_0^t d\tau \hat{H}_{S1} (\tau) \right) a_k^{\mu}(t) \exp \left( -i \int_0^t d\tau' \hat{H}_{S1} (\tau') \right), \quad (5.6)$$

$$\bar{a}_{\text{Hk}}^{\nu}(s) = \exp \left( i \int_0^s d\tau \hat{H}_{S1} (\tau) \right) \bar{a}_k^{\nu}(s) \exp \left( -i \int_0^s d\tau' \hat{H}_{S1} (\tau') \right), \quad (5.7)$$

with $\mu, \nu = 1, 2$. Then, the two-point Green’s function defined by (5.1) can be expressed using the Heisenberg operators $a_k^{\mu}(t)$ and $\bar{a}_k^{\nu}(s)$ for the semi-free field, as

$$G_k^{\mu \nu}(t, s) = -i \Theta(t - s) \langle 1_S | a_{\text{Hk}}^{\mu}(t) \bar{a}_{\text{Hk}}^{\nu}(s) | \rho(0) \rangle - i \Theta(s - t) \langle 1_S | \bar{a}_{\text{Hk}}^{\nu}(s) a_{\text{Hk}}^{\mu}(t) | \rho(0) \rangle,$$

$$= -i \Theta(t - s) \langle 1_S | a_k^{\mu}(t) \exp \left( -i \int_0^t d\tau \hat{H}_{S1} (\tau) \right) \bar{a}_k^{\nu}(s) \exp \left( -i \int_0^s d\tau' \hat{H}_{S1} (\tau') \right) | \rho(0) \rangle,$$

$$- i \Theta(s - t) \langle 1_S | \bar{a}_k^{\nu}(s) \exp \left( -i \int_0^s d\tau \hat{H}_{S1} (\tau) \right) a_k^{\mu}(t) \exp \left( -i \int_0^t d\tau' \hat{H}_{S1} (\tau') \right) | \rho(0) \rangle, \quad (5.8)$$

with the step function $\Theta(t)$, where we have used the axioms (2.4) and (2.51). In the calculations of the expectation values $\langle 1_S | \cdots | \rho(0) \rangle$ in (5.8), the operators $a_k^{\mu}(t)$ and $\bar{a}_k^{\nu}(s)$ are connected with all the renormalized hat-Hamiltonians $\hat{H}_{S1}(\tau)$ included in $\langle 1_S | \cdots | \rho(0) \rangle$, because $\langle 1_S \hat{H}_{S1}(\tau) = 0$ according to the axioms (2.4) and (2.51) if not connected. Those correspond to the calculations of the connected diagrams in Feynman diagram method.
We consider the calculations of the Green’s function $G^{11}_{k}(t, t + 0)$ as an example. We take account of the Hamiltonian (4.7) of the boson-boson interaction. The Green’s function $G^{11}_{k}(t, t + 0)$ takes the forms

$$G^{11}_{k}(t, t + 0) = -i \langle 1_{S} | T \{ a^{\dagger}_{hk}(t) a_{hk}(t + 0) \} | \rho(0) \rangle = -i \langle 1_{S} | a^{\dagger}_{hk}(t) a_{hk}(t) | \rho(0) \rangle,$$

$$= -i \langle 1_{S} | a^{\dagger}_{k}(t) a_{k}(t) \exp \left(-i \int_{0}^{t} d \tau \hat{H}_{S1}(\tau) \right) | \rho(0) \rangle, \quad (5.9)$$

$$= -i \cdot n_{k}(t) + G^{11(1)}_{k}(t, t + 0) + G^{11(2)}_{k}(t, t + 0) + \cdots, \quad (5.10)$$

with $n_{k}(t) = \langle 1_{S} | a^{\dagger}_{k}(t) a_{k}(t) | \rho(0) \rangle$ defined by (2.61), where the $n$th-order parts $G^{11(n)}_{k}(t, t + 0) (n = 1, 2, \cdots)$ are given by

$$G^{11(n)}_{k}(t, t + 0) = -i \sum_{n=1}^{\infty} (-i)^{n} \int_{0}^{t} dt_{1} \int_{0}^{t_{1}} dt_{2} \cdots \int_{0}^{t_{n-1}} dt_{n} \langle 1_{S} | a^{\dagger}_{k}(t) a_{k}(t) \hat{H}_{S1}(t_{1}) \hat{H}_{S1}(t_{2}) \cdots \hat{H}_{S1}(t_{n}) | \rho(0) \rangle. \quad (5.11)$$

Using the transformations (2.68) and (2.69), the commutation relations (2.65) and the thermal state conditions (2.66), we can obtain

$$\langle 1_{S} | a^{\dagger}_{k}(t) a_{k}(t) \hat{H}_{S1}(t_{1}) \hat{H}_{S1}(t_{2}) \cdots \hat{H}_{S1}(t_{n}) | \rho(0) \rangle$$

$$= Z_{k}(t) \langle 1_{S} | \{ \tilde{\alpha}_{k}(t) a_{k}(t) + h_{k}(t) \} \hat{H}_{S1}(t_{1}) \hat{H}_{S1}(t_{2}) \cdots \hat{H}_{S1}(t_{n}) | \rho(0) \rangle,$$

$$= Z_{k}(t_{1}) \langle 1_{S} | \{ \tilde{\alpha}_{k}(t_{1}) a_{k}(t_{1}) , \hat{H}_{S1}(t_{1}) \} \hat{H}_{S1}(t_{2}) \cdots \hat{H}_{S1}(t_{n}) | \rho(0) \rangle \times \exp \{- 2 (\Phi'_{k} + \Psi_{k})(t - t_{1}) \}, \quad (5.12)$$

where the axioms (2.4), (2.51) and the relations (4.13) and its tilde conjugate have been used. By using the relations (4.19) and (4.20), the transformations (2.68) and (2.69), the commutation relations (2.65) and the thermal state conditions (2.66) and (2.67), we can rewrite $\langle 1_{S} | \{ \tilde{\alpha}_{k}(t_{1}) a_{k}(t_{1}) , \hat{H}_{S1}(t_{1}) \}$ in (5.12) in terms of the quasi-particle operators at the time $t_{1}$. The results can be transcribed in terms of the quasi-particle operators at the time $t_{2}$ by virtue of the relations (4.13) and its tilde conjugate, and we can calculate the commutator between those results and $\hat{H}_{S1}(t_{2})$. The expectation values $\langle 1_{S} | \cdots | \rho(0) \rangle$ in (5.12) can be calculated in this way, and the results are integrated according to (5.11) to give the $n$th-order parts $G^{11(n)}_{k}(t, t + 0)$. We can thus calculate the Green’s function $G^{11}_{k}(t, t + 0)$.

First, let us calculate the first-order part $G^{11(1)}_{k}(t, t + 0)$ of the Green’s function $G^{11}_{k}(t, t + 0)$. By virtue of the relation (4.19) and the thermal state conditions (2.60) and (2.67), we obtain

$$\langle 1_{S} | a^{\dagger}_{k}(t) a_{k}(t) \hat{H}_{S1}(t_{1}) | \rho(0) \rangle$$

$$= Z_{k}(t_{1}) \langle 1_{S} | \{ \tilde{\alpha}_{k}(t_{1}) a_{k}(t_{1}) , \hat{H}_{S1}(t_{1}) \} | \rho(0) \rangle \exp \{- 2 (\Phi'_{k} + \Psi_{k})(t - t_{1}) \}, \quad (5.13)$$

$$= \frac{1}{2} \sum_{q} \sum_{k'} Z^{3/2}_{k}(t_{1}) (V_{q} + V_{-q}) \langle 1_{S} | \tilde{\alpha}_{k}(t_{1}) \{ a^{\dagger}_{k}(t_{1}) a_{k'+q}(t_{1}) a_{k-q}(t_{1}) \}$$
\[-h_k(t_1)\tilde{a}_k^{\dagger}(t_1)\tilde{a}_k(t_1)|\rho(0)\rangle \exp\{-2(\Phi'_k + \Psi_k)(t-t_1)\},
\]
\[= \frac{1}{2}\sum_q\sum_{k'} Z_k^{3/2}(t_1)(V_q + V_{-q}) \left\{ 1 - h_k(t_1) h_{k'}(t_1) h_{k'-q}^{-1}(t_1) h_{k-q}^{-1}(t_1) \right\}
\times \langle 1_s| \tilde{a}_k(t_1) a_{k'}^{\dagger}(t_1) a_{k'-q}(t_1) a_{k-q}(t_1)|\rho(0)\rangle \exp\{-2(\Phi'_k + \Psi_k)(t-t_1)\}, \quad (5.14)
\]
which can be calculated, by using the transformations (2.68) and (2.69), the commutation relations (2.65) and the thermal state conditions (2.66) and (2.67), as

\[\langle 1_s| a_k^{\dagger}(t) a_k(t) \hat{H}_{s1}(t_1)|\rho(0)\rangle \]
\[= \frac{1}{2}\sum_q\sum_{k'} Z_k^{3/2}(t_1)(V_q + V_{-q}) \left\{ 1 - h_k(t_1) h_{k'}(t_1) h_{k'-q}^{-1}(t_1) h_{k-q}^{-1}(t_1) \right\}
\times \langle 1_s| \tilde{a}_k(t_1) \tilde{a}_{k'}(t_1) \tilde{a}_{k'-q}(t_1) \tilde{a}_{k-q}(t_1)|\rho(0)\rangle,
\quad (5.15)
\]
\[= \frac{1}{2}\sum_q\sum_{k'} Z_k^{3/2}(t_1)(V_q + V_{-q}) \left\{ 1 - h_k(t_1) h_{k'}(t_1) h_{k'-q}^{-1}(t_1) h_{k-q}^{-1}(t_1) \right\}
\times \langle 1_s| (\delta_{k',k+q} \delta_{k,-q} + \delta_{k',k'+q} \delta_{k,q})|\rho(0)\rangle,
\quad (5.16)
\]
\[= 0. \quad (5.17)
\]
Thus for the first-order part $G_k^{1(1)}(t,t + 0)$, we have

\[G_k^{1(1)}(t,t + 0) = 0. \quad (5.18)
\]

Next, let us calculate the second-order part $G_k^{1(2)}(t,t + 0)$ of the Green's function $G_k^{11}(t,t + 0)$. By virtue of the relations (4.19) and (4.20) and the thermal state conditions (2.59), we obtain

\[Z_k(t_1)\langle 1_s| \tilde{a}_k(t_1) a_k(t_1) \hat{H}_{s1}(t_1) = Z_k(t_1)\langle 1_s| [\tilde{a}_k(t_1) a_k(t_1), \hat{H}_{s1}(t_1)],
\]
\[= Z_k(t_1)\langle 1_s| [\tilde{a}_k(t_1), \hat{H}_{s1}(t_1)] a_k(t_1) + \tilde{a}_k(t_1) [a_k(t_1), \hat{H}_{s1}(t_1)],
\]
\[= \frac{1}{2}\sum_q\sum_{k'} (V_q + V_{-q}) Z_k^{3/2}(t_1)
\times \{ \langle 1_s| h_k(t_1) a_{k-q}^{\dagger}(t_1) a_{k'-q}^{\dagger}(t_1) a_{k'}(t_1) a_k(t_1) \rangle 
- \langle 1_s| h_k(t_1) a_{k'-q}^{\dagger}(t_1) a_{k-q}^{\dagger}(t_1) a_{k}(t_1) a_{k'}(t_1) \rangle \}
\times \{ \langle 1_s| h_k(t_1) a_{k-q}^{\dagger}(t_1) a_{k'-q}^{\dagger}(t_1) a_{k'}(t_1) a_k(t_1) \rangle 
- \langle 1_s| h_k(t_1) a_{k'-q}^{\dagger}(t_1) a_{k-q}^{\dagger}(t_1) a_{k}(t_1) a_{k'}(t_1) \rangle \},
\quad (5.19)
\]
\[= \frac{1}{2}\sum_q\sum_{k'} (V_q + V_{-q}) Z_k^{3/2}(t_1)
\times \{ (h_k(t_1) - 1) \langle 1_s| a_{k-q}^{\dagger}(t_1) a_{k'-q}^{\dagger}(t_1) a_{k'}(t_1) a_k(t_1) \rangle 
+ (1 - h_k(t_1)) \langle 1_s| a_{k'}^{\dagger}(t_1) a_{k'+q}(t_1) a_{k-q}(t_1) \tilde{a}_k(t_1) \rangle 
\times \{ (h_k(t_1) - 1) \langle 1_s| a_{k-q}^{\dagger}(t_1) a_{k'-q}^{\dagger}(t_1) a_{k'}(t_1) a_k(t_1) \rangle 
+ (1 - h_k(t_1)) \langle 1_s| a_{k'}^{\dagger}(t_1) a_{k'+q}(t_1) a_{k-q}(t_1) \tilde{a}_k(t_1) \rangle \}
\]
\[ + \langle 1_s | \{ \tilde{\alpha}_k(t_1), a_{k'}^{\dagger}(t_1) a_{k'-q}(t_1) a_{k-q}(t_1) - h_k(t_1) \tilde{a}_{k-q}^{\dagger}(t_1) \tilde{a}_{k'-q}^{\dagger}(t_1) \tilde{a}_{k'}(t_1) \} \rangle, \]

which can be calculated, by using the transformations (2.68) and (2.69), the commutation relations (2.65) and the thermal state conditions (2.66), as

\[
Z_k(t_1) \langle 1_s | \tilde{\alpha}_k(t_1) \alpha_k(t_1) \tilde{\mathcal{H}}_{S_1}(t_1) \]

\[
= \frac{1}{2} \sum_q \sum_{k'} (V_q + V_{-q}) Z_k^{3/2}(t_1) Z_{k-q}^{1/2}(t_1) Z_{k'-q}^{1/2}(t_1) Z_{k'q}^{1/2}(t_1) \]

\[
\times \{ (h_k(t_1) - 1) \langle 1_s | \{ \tilde{\alpha}_k(t_1) \alpha_0(t_1) \alpha_k(t_1) + \alpha_0(t_1) \tilde{\alpha}_k(t_1) h_k(t_1) (\delta_{q,0} + \delta_{q,k-k'}) \}

+ (1 - h_k(t_1)) \langle 1_s | \{ \alpha_k(t_1) \alpha_0^{\dagger}(t_1) \tilde{\alpha}_k(t_1) \tilde{\alpha}_k(t_1) + \alpha_k(t_1) \tilde{\alpha}_k(t_1) h_k(t_1) (\delta_{q,0} + \delta_{q,k-k'}) \}

+ \tilde{h}_k(t_1) \langle 1_s | \{ \alpha_k(t_1) \alpha_0^{\dagger}(t_1) + h_k(t_1) \tilde{\alpha}_k(t_1) \tilde{\alpha}_k(t_1) (\delta_{q,0} + \delta_{q,k-k'}) \}

- \tilde{h}_k(t_1) \langle 1_s | \{ \alpha_k^{\dagger}(t_1) \alpha_0(t_1) + h_k(t_1) \tilde{\alpha}_k^{\dagger}(t_1) \tilde{\alpha}_k^{\dagger}(t_1) (\delta_{q,0} + \delta_{q,k-k'}) \}

- \tilde{h}_k(t_1) \langle 1_s | \{ \alpha_k(t_1) \alpha_0^{\dagger}(t_1) + h_k(t_1) \tilde{\alpha}_k(t_1) \tilde{\alpha}_k(t_1) (\delta_{q,0} + \delta_{q,k-k'}) \}

\]

\[
= \frac{1}{2} \sum_q \sum_{k'} (V_q + V_{-q}) Z_k^{1/2}(t_1) Z_{k-q}^{1/2}(t_1) Z_{k'-q}^{1/2}(t_1) Z_{k'q}^{1/2}(t_1) \]

\[
\times \{ \langle 1_s | \{ \alpha_k(t_1) \alpha_0^{\dagger}(t_1) \tilde{\alpha}_0(t_1) \tilde{\alpha}_k(t_1) + \alpha_k(t_1) \tilde{\alpha}_0^{\dagger}(t_1) \tilde{\alpha}_0(t_1) (\delta_{q,0} + \delta_{q,k-k'}) \}

\times \exp \{ -i (\epsilon_{k-q} + \tilde{\Phi}_{k-q}')(t_1 - t_2) - (\tilde{\Phi}_{k-q}' + \Phi_{k-q}')(t_1 - t_2) \}

\times \exp \{ -i (\epsilon_{k'-q} + \tilde{\Phi}_{k'-q}')(t_1 - t_2) - (\tilde{\Phi}_{k'-q}' + \Phi_{k'-q}')(t_1 - t_2) \}

\times \exp \{ i (\epsilon_{k'} + \tilde{\Phi}_{k'}')(t_1 - t_2) - (\tilde{\Phi}_{k'} + \Phi_{k}')(t_1 - t_2) \}

\times \exp \{ i (\epsilon_{k} + \tilde{\Phi}_{k}')(t_1 - t_2) - (\tilde{\Phi}_{k} + \Phi_{k}')(t_1 - t_2) \}

\times \{ \langle 1_s | \alpha_k(t_1) \alpha_0^{\dagger}(t_1) \tilde{\alpha}_0(t_1) \tilde{\alpha}_k(t_1) + \alpha_k(t_1) \tilde{\alpha}_0^{\dagger}(t_1) \tilde{\alpha}_0(t_1) (\delta_{q,0} + \delta_{q,k-k'}) \}

\times \{ h_k(t_1) a_{k-q}^{\dagger}(t_2) a_{k-q}^{\dagger}(t_2) a_{k-q}(t_2) - \tilde{a}_{k-q}^{\dagger}(t_2) \tilde{a}_{k-q}^{\dagger}(t_2) \} \langle 1_s | \alpha_k(t_1) \alpha_0^{\dagger}(t_1) \tilde{\alpha}_0(t_1) \tilde{\alpha}_k(t_1) + \alpha_k(t_1) \tilde{\alpha}_0^{\dagger}(t_1) \tilde{\alpha}_0(t_1) (\delta_{q,0} + \delta_{q,k-k'}) \}

\]

where the relations (2.64), (4.13) and its tilde conjugate have been used. Using the relations (4.19) and (4.20) and the thermal state conditions (2.60) and (2.67), we can calculate as

\[
\langle 1_s | \{ \alpha_{k-q}(t_2) \alpha_{k'-q}(t_2) \tilde{\alpha}_{k}(t_2) \tilde{\alpha}_{k}(t_2) \} \hat{\mathcal{H}}_{S_1}(t_2) | \rho(0) \rangle

= \langle 1_s | \{ \alpha_{k-q}(t_2) \alpha_{k'-q}(t_2) \tilde{\alpha}_{k}(t_2) \tilde{\alpha}_{k}(t_2) , \tilde{\mathcal{H}}_{S_1}(t_2) \} | \rho(0) \rangle, \]

\[
= \langle 1_s | \{ \alpha_{k-q}(t_2) \alpha_{k'-q}(t_2) \tilde{\alpha}_{k}(t_2) + \tilde{\mathcal{H}}_{S_1}(t_2) \} | \rho(0) \rangle, \]

\[
= \frac{1}{2} Z_k^{1/2}(t_2) \sum_{q'} \sum_{k''} (V_{q'} + V_{-q'}) \langle 1_s | \{ \alpha_{k-q}(t_2) \alpha_{k'-q}(t_2) \tilde{\alpha}_{k}(t_2) \} \langle 1_s | \{ \alpha_{k-q}(t_2) \alpha_{k'-q}(t_2) \tilde{\alpha}_{k}(t_2) \} | \rho(0) \rangle, \]

\[
\times \{ h_k(t_2) a_{k-q}^{\dagger}(t_2) a_{k'-q}^{\dagger}(t_2) a_{k-q}(t_2) - \tilde{a}_{k-q}^{\dagger}(t_2) \tilde{a}_{k-q}^{\dagger}(t_2) \} \langle 1_s | \{ \alpha_{k-q}(t_2) \alpha_{k'-q}(t_2) \tilde{\alpha}_{k}(t_2) \} \langle 1_s | \{ \alpha_{k-q}(t_2) \alpha_{k'-q}(t_2) \tilde{\alpha}_{k}(t_2) \} | \rho(0) \rangle, \]

\]
\[
\frac{1}{2} Z_k^{1/2}(t_2) \sum_{q'} Z_k^{1/2}(t_2) \sum_{k''} \left( V_{q'} + V_{-q'} \right) h_k(t_2) \left\{ 1 - \frac{h_{k-q'}(t_2) h_{k'+q'}(t_2)}{h_k(t_2) h_{k''}(t_2)} \right\} \\
\times \langle 1_S | \alpha_{k-q}(t_2) a^{\dagger}_{k-q}(t_2) a_{k'q}(t_2) a^{\dagger}_{k'q}(t_2) a_{k''q}(t_2) | \rho(0) \rangle,
\]
\[
= \frac{1}{2} Z_k^{1/2}(t_2) \sum_{q'} \sum_{k''} \left( V_{q'} + V_{-q'} \right) h_k(t_2) h_{k''}(t_2) \left\{ 1 - \frac{h_{k-q'}(t_2) h_{k'+q'}(t_2)}{h_k(t_2) h_{k''}(t_2)} \right\} \\
\times \langle 1_S | \alpha_{k-q}(t_2) a^{\dagger}_{k+q}(t_2) \alpha_{k}(t_2) a^{\dagger}_{k'+q}(t_2) \alpha_{k''}(t_2) | \rho(0) \rangle \times Z_k^{1/2}(t_2) Z_k^{1/2}(t_2) Z_k^{1/2}(t_2),
\]
\[
= (1/2) \left( V_q + V_{-q} + V_{k-k'-q} + V_{k'+q-k} \right) \left( h_k(t_2) h_{k'}(t_2) - h_{k-q}(t_2) h_{k'+q}(t_2) \right) \\
\times Z_k^{1/2}(t_2) Z_k^{1/2}(t_2) Z_k^{1/2}(t_2) Z_k^{1/2}(t_2),
\]
and similarly we can calculate as
\[
\langle 1_S | \tilde{\alpha}_{k-q}(t_2) \tilde{\alpha}_{k'+q}(t_2) \alpha_{k}(t_2) \alpha_{k}(t_2) \tilde{\mathcal{H}}_{S_1}(t_2) | \rho(0) \rangle \\
= (1/2) \left( V_q + V_{-q} + V_{k-k'-q} + V_{k'+q-k} \right) \left( h_{k-q}(t_2) h_{k'+q}(t_2) - h_k(t_2) h_{k'}(t_2) \right) \\
\times Z_k^{1/2}(t_2) Z_k^{1/2}(t_2) Z_k^{1/2}(t_2) Z_k^{1/2}(t_2).
\]
From (5.11), (5.12), (5.23), (5.28) and (5.29), the second-order part of the Green’s function \(G_k^{11}(t, t+0)\) takes the following form,
\[
G_k^{11(2)}(t, t+0) = i \int_0^t dt_1 \int_0^{t_1} dt_2 \langle 1_S | a^{\dagger}_{k'}(t_1) a_k(t_2) \tilde{\mathcal{H}}_{S_1}(t_1) \tilde{\mathcal{H}}_{S_1}(t_2) | \rho(0) \rangle,
\]
\[
= \frac{i}{4} \int_0^t dt_1 \int_0^{t_1} dt_2 \sum_q \sum_{k'} \left( V_q + V_{-q} \right) \left( V_q + V_{-q} + V_{k-k'-q} + V_{k'+q-k} \right) \\
\times \{ n_k(t_2) n_{k'}(t_2) n_{k-q}(t_2) + n_k(t_2) n_{k'}(t_2) n_{k'+q}(t_2) + n_k(t_2) n_{k'}(t_2) \\
- n_k(t_2) n_{k-q}(t_2) n_{k'+q}(t_2) - n_{k-q}(t_2) n_{k'}(t_2) n_{k'+q}(t_2) - n_{k-q}(t_2) n_{k'+q}(t_2) \} \\
\times \exp\{-i (\epsilon_{k-q} + \Phi'_{k-q})(t_1 - t_2) - (\Phi'_{k-q} + \Psi_{k-q})(t_1 - t_2) \} \\
\times \exp\{-i (\epsilon_{k'+q} + \Phi'_{k'+q})(t_1 - t_2) - (\Phi'_{k'+q} + \Psi_{k'+q})(t_1 - t_2) \} \\
\times \exp\{i (\epsilon_{k'} + \Phi'_{k'})(t_1 - t_2) - (\Phi'_{k'} + \Psi_{k'})(t_1 - t_2) \} \\
\times \exp\{i (\epsilon_k + \Phi'_{k})(t_1 - t_2) - (\Phi'_{k} + \Psi_{k})(t_1 - t_2) - 2 (\Phi'_{k} + \Psi_{k})(t_1 - t_2) \} \\
- (\text{complex conjugate}),
\]
where by virtue of the relations (2.64), the following relation has been used.
\[
Z_{k-q}(t_2) Z_{k'+q}(t_2) Z_{k}(t_2) Z_{k}(t_2) \{ h_{k}(t_2) h_{k'}(t_2) - h_{k-q}(t_2) h_{k'+q}(t_2) \} \\
= n_k(t_2) n_{k'}(t_2) (n_{k-q}(t_2) + 1) (n_{k'+q}(t_2) + 1) \\
- (n_k(t_2) + 1) (n_{k'}(t_2) + 1) n_{k-q}(t_2) n_{k'+q}(t_2).
\]
The above result can be integrated by using the form (2.75) of \(n_k(t)\). The forms of the higher-order parts can be obtained in the same way.
§6. Summary and concluding remarks

We have considered a boson system interacting with its heat reservoir, where the boson-reservoir interaction includes not only a non-adiabatic part but also an adiabatic part, and have generalized the non-equilibrium thermo-field dynamics proposed by Arimitsu and Umezawa \(^8\)–\(^10\) to the case of a fourth-order unperturbed Hamiltonian which includes not only a second-order (quadratic) part but also a fourth-order part. We have derived the forms of the quasi-particle operators for a semi-free boson field, and have shown that the energies and life-times of the quasi-particles depend on the adiabatic boson-reservoir interaction which leads to the fourth-order part of the unperturbed Hamiltonian. We have also evaluated the form of the two-point Green’s function for the semi-free boson field. We have besides derived a form of the admittance for the boson system using the TCLE method formulated in terms of the generalized non-equilibrium thermo-field dynamics, where the admittance includes effects of the initial correlation and memory for the boson system and heat reservoir, and have given a calculation method of the higher-order parts of the admittance in powers of the boson-boson interaction. Moreover, we have given a calculation method of the perturbation expansions of the two-point Green’s function for the boson system.

When the boson-reservoir interaction is given by (2.39), which includes not only the non-adiabatic part but also the adiabatic part, the collision operator \(C^{(2)}\) takes the expression (2.45), which includes not only the second-order part but also the fourth-order part. Therefore, the unperturbed Hamiltonian \(\hat{H}_{0} + iC^{(2)}\) in the non-equilibrium thermo-field dynamics takes a non-bilinear form which includes not only the second-order part but also the fourth-order part, where the fourth-order part comes from the adiabatic part of the boson-reservoir interaction (2.39). When \(\Psi_{k} \equiv \phi_{k}^{zz}(0)\) given by (2.82) is real, the quasi-particle operators for the semi-free boson field take the forms (2.80) and (2.81), where \(\Phi'_{k}\) is non-negative for non-negative \(\epsilon_{k}\) and \(\Psi_{k}\) is non-negative in general. The quasi-particles of wave number \(k\) for the semi-free boson field have the energies \(\pm (\epsilon_{k} + \Phi''_{k})\) and the life-time \((\Phi'_{k} + \Psi_{k})^{-1}\). Then, the two-point Green’s function for the semi-free boson field is given by (3.14).

The admittance for the boson system takes the form (4.5), which has been obtained using the TCLE method formulated in terms of the generalized non-equilibrium thermo-field dynamics, and includes effects of the initial correlation and memory for the boson system and heat reservoir. The zeroth-order part \(\chi^{(0)}_{a_{k}a_{k}^\dagger}(\omega)\) of the admittance (4.5) in powers of the boson-boson interaction is given by (4.11), and the first-order part \(\chi^{(1)}_{a_{k}a_{k}^\dagger}(\omega)\) is given by (4.18). We have shown a calculation method of the second-order part of the admittance in powers of the boson-boson interaction. We have calculated the second-order part of the admittance in our another paper.\(^31\) The higher-order parts can be calculated in the same way.

We have given calculation methods of the perturbation expansions of the two-point Green’s function and the admittance for the boson system. The calculations correspond to those of the connected diagrams in Feynman diagram method, but the calculation methods are not diagrammatic. It is necessary to improve the calculation
methods of the Green’s functions and of the admittance diagrammatically. This is a subject of future study.

Recently, we have studied the linear response of a ferromagnetic spin system interacting with a phonon reservoir in the spin-wave approximation using the TCLE method. In the forthcoming paper, we will discuss the line shapes of the transverse magnetic susceptibility for a ferromagnetic spin system in the resonance region, by assuming the spin-phonon interaction similar to (2.39).

Acknowledgements

The author is very grateful to Professor S. Miyashita (The University of Tokyo) and to the Miyashita research group for stimulating and valuable discussions.

Appendix A

Derivation of Eqs. (2.76) and (2.77)

Equations (2.76) and (2.77) can be derived using Eqs. (2.70), (2.71) and (2.75) as follows,

\[
(d/dt) Z_k^{1/2}(t) \alpha_k(t) = (d/dt) Z_k(t) \{ a_k(t) - h_k(t) \tilde{a}_k^\dagger(t) \},
\]

\[
= (d/dt) \{(n_k(t) + 1) a_k(t) - n_k(t) \tilde{a}_k^\dagger(t) \},
\]

\[
= -2 \Phi'_k (n_k(t) - \bar{n}_k) \{ a_k(t) - \tilde{a}_k^\dagger(t) \} + (n_k(t) + 1) \{- i \epsilon_k a_k(t) - \phi_k^{+\dagger}(\epsilon_k) a_k(t) + \phi_k^{-\dagger}(\epsilon_k) a_k(t) \}
\]

\[
- a_k(t) \{ \phi_k^{zz}(0) a_k^\dagger(t) a_k(t) + \tilde{\phi}_k^{zz}(0) a_k^\dagger(t) \tilde{a}_k(t) \}
\]

\[
- \phi_k^{zz}(0) \{ a_k^\dagger(t) a_k(t) - \tilde{a}_k^\dagger(t) \tilde{a}_k(t) \} \}
\]

\[
= -i \epsilon_k \{(n_k(t) + 1) a_k(t) - n_k(t) \tilde{a}_k^\dagger(t) \} - \phi_k^{-\dagger}(\epsilon_k) a_k(t)
\]

\[
-2 \Phi'_k (n_k(t) - \bar{n}_k) \{ a_k(t) - \tilde{a}_k^\dagger(t) \} + \phi_k^{+\dagger}(\epsilon_k) \tilde{a}_k(t)
\]

\[
- \{(n_k(t) + 1) \phi_k^{+\dagger}(\epsilon_k) - n_k(t) \phi_k^{-\dagger}(\epsilon_k) \} \{ a_k(t) - \tilde{a}_k^\dagger(t) \}
\]

\[
+ (n_k(t) + 1) \{- a_k(t) \{ \phi_k^{zz}(0) a_k^\dagger(t) a_k(t) \}
\]

\[
- \phi_k^{zz}(0) \{ a_k^\dagger(t) a_k(t) - \tilde{a}_k^\dagger(t) \tilde{a}_k(t) \} \}
\]

\[
= -i (\epsilon_k + \Phi''_k - \Phi'_k) (n_k(t) + 1) a_k(t) - n_k(t) \tilde{a}_k^\dagger(t) \}
\]

\[
- \{(n_k(t) + 1) a_k(t) - n_k(t) \tilde{a}_k^\dagger(t) \} \{ \phi_k^{zz}(0) a_k^\dagger(t) a_k(t) - \phi_k^{-\dagger}(\epsilon_k) a_k(t) \}
\]

\[
- \phi_k^{zz}(0) \{ a_k^\dagger(t) a_k(t) - \tilde{a}_k^\dagger(t) \tilde{a}_k(t) \} \]

\[
= \{(n_k(t) + 1) a_k(t) - n_k(t) \tilde{a}_k^\dagger(t) \} \{ \phi_k^{zz}(0) a_k^\dagger(t) a_k(t) - \phi_k^{-\dagger}(\epsilon_k) a_k(t) \}
\]

\[
- \phi_k^{zz}(0) \{ a_k^\dagger(t) a_k(t) - \tilde{a}_k^\dagger(t) \tilde{a}_k(t) \} \]

\[
= \{(n_k(t) + 1) a_k(t) - n_k(t) \tilde{a}_k^\dagger(t) \} \{ \phi_k^{zz}(0) a_k^\dagger(t) a_k(t) - \phi_k^{-\dagger}(\epsilon_k) a_k(t) \}
\]

\[
- \phi_k^{zz}(0) \{ a_k^\dagger(t) a_k(t) - \tilde{a}_k^\dagger(t) \tilde{a}_k(t) \} \]
\begin{align}
&- (a_k^{\dagger}(t) a_k(t) - \tilde{a}_k^{\dagger}(t) \tilde{a}_k(t)) \{(n_k(t) + 1) \phi_k^{zz}(0) a_k(t) - n_k(t) \phi_k^{zz}(0)^* \tilde{a}_k^{\dagger}(t)\}, \\
&= \{ -i (\epsilon_k + \Phi_k^\nu) - \Phi_k' \} \{(n_k(t) + 1) a_k(t) - n_k(t) \tilde{a}_k^{\dagger}(t)\}
&\quad - \{ \phi_k^{zz}(0) a_k^{\dagger}(t) a_k(t) - \phi_k^{zz}(0)^* \tilde{a}_k^{\dagger}(t) \tilde{a}_k(t)\} \{(n_k(t) + 1) a_k(t) - n_k(t) \tilde{a}_k^{\dagger}(t)\}
&\quad - (a_k^{\dagger}(t) a_k(t) - \tilde{a}_k^{\dagger}(t) \tilde{a}_k(t)) \{(n_k(t) + 1) \phi_k^{zz}(0) a_k(t) - n_k(t) \phi_k^{zz}(0)^* \tilde{a}_k^{\dagger}(t)\}
&\quad - \{ \phi_k^{zz}(0) (n_k(t) + 1) a_k(t) - \phi_k^{zz}(0)^* n_k(t) \tilde{a}_k^{\dagger}(t)\},
\end{align}

(A-3)

\[
(d/dt) Z_k^{-1/2}(t) a_k^{\dagger}(t) = (d/dt) \{ a_k^{\dagger}(t) - \tilde{a}_k(t)\},
\]

\[
= i \epsilon_k a_k^{\dagger}(t) + \phi_k^{+}(\epsilon_k) a_k^{\dagger}(t) - \phi_k^{-}(\epsilon_k) \tilde{a}_k(t) + \phi_k^{+}(\epsilon_k) (a_k^{\dagger}(t) - \tilde{a}_k(t))
\]

\[
+ \phi_k^{zz}(0) (a_k^{\dagger}(t) a_k(t) - a_k^{\dagger}(t) \tilde{a}_k(t)) a_k^{\dagger}(t)
\]

\[
- i \epsilon_k \tilde{a}_k(t) + \phi_k^{+}(\epsilon_k) (\tilde{a}_k(t) - a_k^{\dagger}(t)) + \phi_k^{-}(\epsilon_k) \tilde{a}_k(t) - \phi_k^{+}(\epsilon_k) a_k^{\dagger}(t)
\]

\[
+ \tilde{a}_k(t) \{ a_k^{\dagger}(t) a_k(t) \phi_k^{zz}(0)^* - a_k^{\dagger}(t) a_k(t) \phi_k^{zz}(0)\}
\]

\[
+ \phi_k^{zz}(0)^* (a_k^{\dagger}(t) \tilde{a}_k(t) - a_k^{\dagger}(t) a_k(t) \tilde{a}_k(t)),
\]

(A-5)

\[
= i \epsilon_k \{ a_k^{\dagger}(t) - \tilde{a}_k(t)\} + (\phi_k^{+}(\epsilon_k) - \phi_k^{-}(\epsilon_k)^*) \{ a_k^{\dagger}(t) - \tilde{a}_k(t)\}
\]

\[
+ (a_k^{\dagger}(t) - \tilde{a}_k(t)) \{ \phi_k^{zz}(0) a_k^{\dagger}(t) a_k(t) - \phi_k^{zz}(0)^* \tilde{a}_k^{\dagger}(t) \tilde{a}_k(t)\}
\]

\[
+ (a_k^{\dagger}(t) a_k(t) - \tilde{a}_k^{\dagger}(t) \tilde{a}_k(t)) \{ \phi_k^{zz}(0) a_k^{\dagger}(t) - \phi_k^{zz}(0)^* \tilde{a}_k(t)\},
\]

(A-6)

\[
= \{ i (\epsilon_k + \Phi_k^\nu) + \Phi_k' \} \{ a_k^{\dagger}(t) - \tilde{a}_k(t)\}
\]

\[
+ (a_k^{\dagger}(t) - \tilde{a}_k(t)) \{ \phi_k^{zz}(0) a_k^{\dagger}(t) a_k(t) - \phi_k^{zz}(0)^* \tilde{a}_k^{\dagger}(t) \tilde{a}_k(t)\}
\]

\[
+ \{ \phi_k^{zz}(0) a_k^{\dagger}(t) - \phi_k^{zz}(0)^* \tilde{a}_k(t)\} \{ a_k^{\dagger}(t) a_k(t) - \tilde{a}_k^{\dagger}(t) \tilde{a}_k(t)\}
\]

\[
+ \phi_k^{zz}(0)^* a_k^{\dagger}(t) - \phi_k^{zz}(0)^* \tilde{a}_k(t),
\]

(A-7)

---

**Appendix B**

---

**Proof of Non-Negative of \(\Psi_k\) (\(= \phi_k^{zz}(0)\))**

When \(\Psi_k \equiv \phi_k^{zz}(0)\) is real (\(\Psi_k = \Psi_k^*\)), by putting \(\Delta R_k^0 = g_{2k} (R_k^I R_k - \langle 1_R | R_k^I R_k | \rho_R \rangle)\), which is Hermitian, \(\Psi_k\) can be calculated as follows,

\[
\Psi_k = \int_0^\infty d\tau g_{2k}^2 \langle 1_R | R_k^I(\tau) R_k(\tau) - \langle 1_R | R_k^I R_k | \rho_R \rangle \rangle \langle R_k^I R_k - \langle 1_R | R_k^I R_k | \rho_R \rangle \rangle | \rho_R \rangle,
\]

\[
= \int_0^\infty d\tau \langle 1_R | \Delta R_k^0(\tau) \Delta R_k^0 | \rho_R \rangle = \int_0^\infty d\tau \langle 1_R | \Delta R_k^0 \Delta R_k^0(\tau) | \rho_R \rangle,
\]

(B-1)

\[
= \frac{1}{2} \int_0^\infty d\tau \sum_{m,n} \{ \langle m | \Delta R_k^0 | n \rangle \langle n | \Delta R_k^0 | m \rangle \langle \rho_R \rangle_m \exp{i (E_m - E_n) \tau / h} \}
\]

\[
+ \langle m | \Delta R_k^0 | n \rangle \langle n | \Delta R_k^0 | m \rangle \langle \rho_R \rangle_m \exp{i (E_n - E_m) \tau / h} \},
\]

(B-2)
\[ = \frac{1}{2} \int_{-\infty}^{\infty} d\tau \sum_{m,n} |\langle m|\Delta R_k^0|n\rangle|^2 (\rho_{R})_m \exp\{i (E_m - E_n) \tau / \hbar\}, \]  
\[ = \pi \hbar \sum_{m,n} |\langle m|\Delta R_k^0|n\rangle|^2 (\rho_{R})_m \delta(E_m - E_n), \]

which is non-negative, where \( \delta(x) \) is the delta-function. Here, \( (\rho_{R})_m \) are the diagonal elements of the density operator \( \rho_{R} \) given by (2.9) for the reservoir, and are given by

\[ (\rho_{R})_m = \exp(-\beta E_m) / \sum_n \exp(-\beta E_n). \]

### Appendix C

**Calculation of the Interference Thermal State** \( |D_{a_k}^{(2)}[\omega]| \)

By substituting (2.41) into (4.3) and by using the axioms (2.44), the interference thermal state \( |D_{a_k}^{(2)}[\omega]| \) can be calculated as follows,

\[ |D_{a_k}^{(2)}[\omega]| = \frac{i}{\hbar} \int_{0}^{\infty} d\tau \int_{0}^{\tau} d\tau' g_{1k}^2 \left( \langle 1_R | R_k(\tau_1) R_k^\dagger(0) | \rho_R \rangle - \langle 1_R | R_k^\dagger(0) R_k(\tau) | \rho_R \rangle \right) e^{i \omega s} \]
\[ \times (a_k^\dagger - \bar{a}_k) |\rho(0)\rangle \exp\{i \epsilon_k(\tau - s)\} \]
\[ - \frac{i}{\hbar} \int_{0}^{\infty} d\tau \int_{0}^{\tau} d\tau' g_{2k}^2 \left( a_k a_k^\dagger - \bar{a}_k \bar{a}_k^\dagger \right) \exp(-i \epsilon_k s) e^{i \omega s} \]
\[ \times \left\{ \left[ a_k^\dagger, a_k^\dagger a_k \right] \langle 1_R | (R_k^\dagger(\tau_1)R_k(\tau) - \langle 1_R | R_k^\dagger R_k | \rho_R \rangle) (R_k^\dagger R_k - \langle 1_R | R_k^\dagger R_k | \rho_R \rangle) | \rho_R \rangle \right\} \]
\[ = \frac{1}{\hbar} \int_{0}^{\infty} d\tau \ g_{1k}^2 \left( \langle 1_R | R_k(\tau_1) R_k^\dagger(0) | \rho_R \rangle - \langle 1_R | R_k^\dagger(0) R_k(\tau) | \rho_R \rangle \right) \]
\[ \times (a_k^\dagger - \bar{a}_k) |\rho(0)\rangle \exp\{i \omega \tau\} - \exp(i \epsilon_k \tau)\} / (\omega - \epsilon_k) \]
\[ + \frac{1}{\hbar} \int_{0}^{\infty} d\tau \ g_{2k}^2 \left( a_k a_k - \bar{a}_k \bar{a}_k^\dagger \right) \exp\{i (\omega - \epsilon_k) \tau\} - 1\} / (\omega - \epsilon_k) \]
\[ \times \left\{ \left( a_k^\dagger \langle 1_R | (R_k^\dagger(\tau_1)R_k(\tau) - \langle 1_R | R_k^\dagger R_k | \rho_R \rangle) (R_k^\dagger R_k - \langle 1_R | R_k^\dagger R_k | \rho_R \rangle) | \rho_R \rangle \right\} |\rho(0)\rangle, \]

where we have ignored the terms that contain the boson-boson interaction \( \mathcal{H}_{S1} \).

Using (2.46) - (2.48), the interference term \( |D_{a_k}^{(2)}[\omega]| \) can be written as

\[ |D_{a_k}^{(2)}[\omega]| \]
\[ = \{ \phi_k^+(\omega) - \phi_k^{++}(\epsilon_k) - (\phi_k^{++}(\epsilon_k) | a_k^\dagger - \bar{a}_k) |\rho(0)\rangle / (\hbar (\omega - \epsilon_k)) \]
\[ + (a_k a_k - \bar{a}_k a_k^\dagger) \{ a_k^\dagger (\phi_k^{zz}(\omega - \epsilon_k) - \phi_k^{zz}(0)) \} \]
− \tilde{a}_k (\phi_k^{zz}(\omega + \epsilon_k)^* − \phi_k^{zz}(0)^*) \rangle \rho(0) / (\hbar (\omega − \epsilon_k), \tag{C.3}
= \{ \phi_k^+(\omega) − \phi_k^+(\epsilon_k) − (\phi_k^+(\omega)^* − \phi_k^+(\epsilon_k)^*) \} \langle a_k^\dagger − \tilde{a}_k \rangle \rho(0) / (\hbar (\omega − \epsilon_k))
+ \{ a_k^\dagger (\phi_k^{zz}(\omega − \epsilon_k) − \phi_k^{zz}(0)) − \tilde{a}_k (\phi_k^{zz}(\omega + \epsilon_k)^* − \phi_k^{zz}(0)^*) \} \langle \rho(0) \rangle / (\hbar (\omega − \epsilon_k)), \tag{C.4}

where we have used the thermal state conditions (2.52), which lead to the relation

\begin{equation}
\langle a_k^\dagger a_k \rangle = \tilde{a}_k^\dagger \tilde{a}_k \langle \rho(0) \rangle. \tag{C.5}
\end{equation}

### Appendix D

**Derivation of (4.16)**

By using the transformations (2.68) and (2.69), the commutation relations (2.65) and the thermal state conditions (2.66) and (2.67), we can calculate the expectation values \( \langle 1_s | \cdots | \rho(0) \rangle \) in (4.15) as follows,

\begin{equation}
\langle 1_s | a_{k'}^{\dagger q} (t_1) a_{k'} (t_1) a_{k-q} (t_1) \alpha_k^\dagger (t_1) \langle \rho(0) \rangle
= Z_k^{1/2} (t_1) Z_k^{1/2} (t_1) Z_k^{1/2} (t_1) \langle 1_s | \tilde{a}_{k'-q} (t_1) \{ \alpha_{k'} (t_1) + h_{k'} (t_1) \tilde{a}_{k'}^\dagger (t_1) \}
\times \{ \alpha_{k-q} (t_1) + h_{k-q} (t_1) \tilde{a}_{k-q}^\dagger (t_1) \} \alpha_k^\dagger (t_1) \langle \rho(0) \rangle, \tag{D.1}
\end{equation}

\begin{equation}
\langle 1_s | a_{k-q}^{\dagger} (t_1) \tilde{a}_{k+q}^\dagger (t_1) \tilde{a}_{k'} (t_1) \alpha_k^\dagger (t_1) \langle \rho(0) \rangle
= Z_k^{1/2} (t_1) Z_k^{1/2} (t_1) Z_k^{1/2} (t_1) \langle 1_s | \alpha_{k-q} (t_1) \{ \tilde{a}_{k+q}^\dagger (t_1) + \alpha_{k+q} (t_1) \}
\times \{ \alpha_{k'} (t_1) + h_{k'} (t_1) \alpha_{k'}^\dagger (t_1) \} \alpha_k^\dagger (t_1) \langle \rho(0) \rangle, \tag{D.2}
\end{equation}

Substituting the above results into (4.15), we have

\begin{equation}
Z_k^{1/2} (t_1) Z_k^{1/2} (t_1) \langle 1_s | \alpha_k (t) \tilde{H}_{\alpha 1} (t_1) \alpha_k^\dagger \langle \rho(0) \rangle
= \frac{1}{2} Z_k (t_1) \sum_{k'} \{ 2 V_0 Z_{k'} (t_1) h_{k'} (t_1) + (V_{k-k'} + V_{k'-k}) Z_{k'} (t_1) h_{k'} (t_1)
− 2 V_0 Z_{k'} (t_1) h_{k} (t_1) h_{k'} (t_1) − (V_{k-k'} + V_{k'-k}) Z_{k'} (t_1) h_{k} (t_1) h_{k'} (t_1) \}
\times \exp \{ − i (\epsilon_k + \Phi''_k) t − (\Phi'_k + \Psi_k) t \} \tag{D.3}
\end{equation}
which is equal to (4.16).

References

11) L. van Hove, Physica 23 (1957), 441.
26) U. Fano, Rev. Mod. Phys. 29 (1957), 74.