Analytical solutions for 3-D flexural deformation of semi-infinite elastic plates

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SUMMARY

The theory of elastic-plate flexure has played a prominent role in isostatic studies and investigations of lithospheric deformation caused by vertical tectonism. Simple analytical, axisymmetrical solutions exist when the boundaries of the plate are far removed from the area where the load is applied. When deformation is driven by end loads (i.e. bending moments and shear forces) the problem is usually simplified to yield a 2-D solution that approximates a cross-section of the 3-D solution. We present analytical Green's functions for the point-force response of semi-infinite elastic plates resting on an inviscid substratum and being acted upon by constant in-plane forces and arbitrary end loads. This solution may be used to study deformation and stresses caused by loads close to a free geological boundary (e.g. a trench or weak fracture zone). We also present Green's functions for the case when two semi-infinite plates of different rigidities are mechanically coupled and subject to a point load. This solution may be useful when studying deformation near mechanically strong fracture zones or at ocean-continent transitions. The analytical solutions provide simple alternatives to (and calibration of) complex numerical finite-element or finite-difference solutions, and furthermore give additional physical insight into the geological process to be examined. The analytical methods are used to demonstrate the different stress patterns that may arise near weak and strong geological boundaries. We also illustrate how the observed scatter in 2-D determinations of elastic thickness at subduction zones may partly have its origin in 3-D effects.

Key words: flexure of the lithosphere, Green's functions.

1 INTRODUCTION

The nature of lithospheric deformation caused by vertical tectonics is largely controlled by the rheology of the Earth's outer layers. The combination of a predominantly brittle/elastic crust and a viscous mantle implies that the Earth's response to an applied load will vary both spatially and temporally. In some studies, such as those of postglacial rebound, the objective is to infer a viscoelastic rheology that is consistent with the observed time-dependent deformation (e.g. Lambeck 1990; Peltier 1991). However, for many geological processes enough time has elapsed since the load initially was applied that all temporal variations have ceased and no record of the time history of deformation is available. In such situations where isostatic equilibrium has been reached an appropriate rheological model is that of an elastic plate overlying an inviscid fluid (e.g. Watts, Bodine & Steckler 1980). Although the elastic theory ignores any plastic yielding that will take place in the lithosphere, it is still a good approximation in most cases where inelastic yielding is not the dominant mode of deformation.

Some of the earliest attempts to use elastic-plate flexure to study the nature of isostasy were completed half a century ago. Both Vening-Meinesz (1941) and Gunn (1943) employed elastic-plate theory to explain the regional variations in the Earth's gravity field surrounding large loads, since conventional Airy- or Pratt-type isostasy proved inadequate. Although the existence of a strong lithosphere and a weak asthenosphere had been known for some time (Barrell 1914), it was not until the plate tectonic revolution that Earth scientists acquired a simple framework within which the large-scale features of oceans and continents could be explained. Following the general acceptance of plate tectonics, the elastic-plate theory was applied swiftly to a variety of geological problems in both oceanic and continental settings. Most work has been directed towards understanding the rheology of the oceanic lithosphere, due to its relatively simple thermal structure (Parsons & Sclater 1977). Studies have examined the deformation beneath seamounts and oceanic islands (e.g. Walcott 1970; Watts & Cochran 1974; Watts 1978; McNutt & Menard 1978; Lambeck & Nakiboglu 1980; ten Brink & Watts 1985; Calmant 1987;
Wessel 1993), the bending of oceanic plates at subduction zones (e.g. Hanks 1971; Watts & Talwani 1974; Caldwell et al. 1976; Bodine & Watts 1979; McAdoo & Martin 1984; McNutt 1991; Levitt & Sandwell 1995), the evolution of oceanic fracture zones (e.g. Sandwell & Schubert 1982; Sandwell 1984; Parmentier & Haxby 1986; Wessel & Haxby 1990; Christeson & McNutt 1992), and intraplate deformation attributed to regional compressive stresses (e.g. McAdoo & Sandwell 1985; Karner & Wessel 1990; Karner, Driscoll & Wessel 1993). Furthermore, elastic-plate theory has been used to model continental deformation (e.g. Karner & Watts 1983; Diament & Kogan 1988; Burov & Diament 1992; Kruse & Royden 1994; Burov & Diament 1995) and the subsidence history of sedimentary basins (e.g. Walcott 1972; Cochran 1973; Beaumont 1979; Steckler & Watts 1981; Karner & Steckler 1982; Karner & Watts 1982; Driscoll & Karner 1994). These investigations have proved extremely useful and have placed strong constraints on the thermomechanical properties and evolution of both oceanic and continental lithosphere.

The majority of 3-D elastic-plate studies of deformation in the interior of the plates have assumed that the loads are located far enough from the plate boundaries that simple, axisymmetrical solutions to the plate-flexure equation may be used. When conditions at the plate edge drive the deformation (e.g. at oceanic subduction zones) the modelling has typically been reduced to a 2-D calculation. In situations where these conditions are not met, a numerical evaluation is usually employed. Approximate numerical calculations of plate flexure with complicated boundary conditions are difficult to set up, slow to converge, and most importantly deprive the scientist of the physical intuition that accompanies analytical results. In this paper, we derive analytical solutions to the flexure of thin elastic plate overlying an inviscid substratum and being subjected to a normal load \( p(x, y) \) and in-plane forces \( N_x \), \( N_y \), and \( N_{xy} \) is given by

\[
DV^2 w(x, y) - N_x \frac{\partial^2 w(x, y)}{\partial x^2} - 2N_{xy} \frac{\partial^2 w(x, y)}{\partial x \partial y} - N_y \frac{\partial^2 w(x, y)}{\partial y^2} + \Delta \rho g w(x, y) = p(x, y),
\]

(1)

where \( \Delta \rho \) is the density contrast between the substratum and the material filling in the depression \( w(x, y) \), \( g \) is normal gravity, and \( D \) is the flexural rigidity defined as

\[
D = \frac{ET_0^2}{12(1 - v^2)},
\]

(2)

Here, \( T_0 \) is the plate thickness (assumed constant), and \( E \) and \( v \) are Young's modulus and Poisson's ratio, respectively (e.g. Timoshenko & Woinowsky-Krieger 1959). Negative values of \( N \) imply in-plane compression; positive values represent tension. For simple (i.e. periodic) boundary conditions, the solution to (1) is readily obtained by taking the 2-D Fourier transform of (1) and algebraically solving for \( \tilde{w}(k_x, k_y) \) in the wavenumber domain:

\[
\tilde{w}(k_x, k_y) = \frac{\tilde{p}(k_x, k_y)}{D(k_x^2 + k_y^2)^2 - k_x^2 N_x - 2k_x k_y N_{xy} - k_y^2 N_y + \Delta \rho g}.
\]

(3)

The deflections are then obtained by evaluating (3) and performing a numerical inverse Fourier transform on the result. In the absence of in-plane forces, the point-load response of the system [i.e. when \( p(x, y) \to \delta(0,0) \)] is axisymmetric and its solution was first determined in 1884 by Heinrich Hertz (1884) to be

\[
w(r) = -\frac{1}{2\pi D r^2} k_0 i \gamma r,
\]

(4)

where \( \gamma = (\Delta \rho g / D)^{1/4} \) is the 3-D flexural wavenumber. Eq. (4) has been used frequently to study the flexural response of loads that are approximately axisymmetrical, such as seamounts (e.g. McNutt & Menard 1978; Lambeck & Nakiboglu 1980; ten Brink 1991; Harris & Chapman 1994; Wessel & Keating 1994).

When the boundaries are not at infinity, the axisymmetry is broken and (4) no longer applies. In the following we will explore solutions to (1) for several boundary conditions that arise in tectonic settings where plate flexure traditionally has been applied.

2.1 Loads close to a free boundary

We seek the response \( w(x, y) \) of a semi-infinite elastic plate (defined for \( x < \ell \)) overlying an inviscid substratum to the vertical load \( p(x, y) \). The boundary-value problem (1) is then subject to free boundary conditions:

\[
M_x(x, y) = -D \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right) = m(y)|_x=\ell,
\]

(5a)

\[
V_x(x, y) = -D \left( \frac{\partial^3 w}{\partial x^3} + (2 - v) \frac{\partial^3 w}{\partial x \partial y^2} \right) - N_x \frac{\partial w}{\partial x} - 2N_{xy} \frac{\partial w}{\partial y} = q(y)|_x=\ell,
\]

(5b)

i.e. both the bending moment \( M_x \) and total vertical force \( V_x \) at the end of the plate must match any applied moments \( m(y) \) and forces \( q(y) \). Here, \( q(y) \) contains the contribution from the twisting moment by virtue of Kirchhoff's supplemented shear force (e.g. Timoshenko & Woinowsky-Krieger 1959). We assume that \( w(x, y) \) is of the form

\[
w(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(x) e^{i\alpha y} dk,
\]

(6)

while we consider \( p(x, y) \) to be composed of a series of \( n \) variable line loads:

\[
p(x, y) = \sum_{j=0}^{n-1} \delta(x - x_j) \phi_j(y),
\]

(7)

\[
= \sum_{j=1}^{n-1} \delta(x - x_j) \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_j(k) e^{i\alpha y} dk.
\]

(7)

For \( n = 0 \) and \( \phi_0(y) = \delta(y - y_0) \) we obtain the point-force response or Green's function, but the choice of (7) will simplify
the evaluation of the complete solution since the spatial convolution over \( y \) is replaced by the multiplication of \( \Phi(k) \) in the wave number domain. With these prescriptions, the complete solution to (1) will be a sum of individual responses to the line loads of the form (7) for all \( x_0 \). Fig. 1 indicates the geometry of the situation.

Without loss of generality, let \( n = x_0 = 0 \) and drop the subscript on \( \phi \). Substituting (6)–(7) in (1) gives (for all \( k \)):

\[
D\Psi^{(m)}_k(x) - (2Dk^2 + N_x)\Psi_k'(x) - 2kN_y\Psi_k(x) + (Dk^4 + N_yk^2 + \Delta \rho g)\Psi_k(x) = \delta(x)\Phi(k), \tag{8}
\]

where the primes denote differentiation with respect to \( x \). Transforming (5a–b) results in the boundary conditions:

\[
M_a(x, k) = -D[\Psi^{(m)}_k(x) - vk^2\Psi_k(x)] = M(k)|_{x=0}, \tag{9a}
\]

\[
V_a(x, k) = -D[\Psi^{(m)}_k(x) - ((2 - \nu)k^2 - N_x/D)\Psi_k(x)] - 2N_yvk\Psi_k(x) = Q(k)|_{x=0}. \tag{9b}
\]

For simple load geometry and end loads, the functions \( \Phi(k), M(k), \) and \( Q(k) \) can be obtained analytically; otherwise they may be represented by a discrete Fourier series. Solving (8) algebraically is cumbersome as long as \( N_{xy} \neq 0 \) and may as well be done numerically. We will therefore limit the remainder of our discussion to cases where the in-plane shearing force can be ignored and thus require \( N_{xy} = 0 \). The solution to (8) will consist of matching solutions at \( x = 0 \), since the delta function implies a discontinuous shear force there. We find (see Appendix for details):

\[
\Psi_k(x) = \begin{cases} 
\exp(a_kx)(A_k \sin b_kx + B_k \cos b_kx), & x \leq 0 \\
\exp(a_kx)(C_k \sin b_kx + D_k \cos b_kx) \\
+ \exp(-a_kx)(E_k \sin b_kx + F_k \cos b_kx), & x \geq 0 
\end{cases} \tag{10}
\]

where

\[
a_k = a_k \sqrt{1 + e_k}, \tag{11a}
\]

\[
b_k = a_k \sqrt{1 - e_k}, \tag{11b}
\]

and

\[
a_k = \left( k^4 + a_k^2 + k^2 N_x \right)^{1/4}, \tag{12a}
\]

\[
a_0 = \left( \frac{\Delta \rho g}{D} \right)^{1/4}, \tag{12b}
\]

\[
a_k = \frac{k^2 + N_x/D}{2\alpha_k^2}. \tag{12c}
\]

Here, \( \alpha_k \) is the 2-D flexural wavenumber; its reciprocal is the flexural parameter \( \lambda_0 \) which represents the intrinsic length-scale of 2-D deformation. The complete solution is arrived at via (6); essentially it becomes a sum over all wavenumbers \( k \). The fundamental mode \( k = 0 \) is the 2-D flexural solution for a beam acted upon by a line load \( \Phi(0) \), the in-plane force \( N_x \), and end loads \( M(0) \) and \( Q(0) \). The solutions for \( k \neq 0 \) are higher-order modes of decreasing flexural wavelengths that superpose to give the complete 3-D solution. The derivation of the six coefficients \( A_k - F_k \) can be found in the Appendix.

As can be seen from (3), there may be values of \( N_x \) and \( N_y \) that will cause the solution of \( W \) to become undefined; these critical values indicate the onset of buckling. Thus, for our solution to be valid, both \( N_x \) and \( N_y \) must be larger (i.e. less compressive) than the critical compressional value, which is equal to \(-4\alpha_0^2\).

As mentioned, when \( \phi(y) = \delta(y) \) the solution to (1) is the Green’s function or the plate’s response to a point load. Fig. 2 shows its form for the case when the point force is applied at a distance \( 2x_0 \) from the edge. In this example, \( N_x = N_y = 0 \), with \( m(y) = m_0 \) and \( q(y) = q_0 \) which therefore only contribute for \( k = 0 \). The Green’s function changes with distance from the edge; Fig. 3 displays a series of solutions as the point load approaches the axisymmetrical case (4) and becomes identical to it as \( \xi \to \infty \).

\[\text{Figure 1. Forces and bending moments acting on a semi-infinite plate.}\]

Variable bending moments \( m(y) \) and shear forces \( q(y) \) act at the plate’s edge \( (x = 0) \). In-plane tension or compression \( N \) may be specified in the principal directions, with \( N_{xy} = 0 \). The normal load is a variable line load. In addition, buoyancy forces will resist the plate’s deflection.

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Figure 2. Flexural deformation of a semi-infinite elastic plate subjected to the combined action of a point load at \((0, 0)\), and constant bending moment \(m_0\) and shear force \(q_0\) at the edge \(x = 2\). Here, \(N_x = N_y = 0\). The coordinates are in units of the flexural parameter \(\lambda_0\).

Fig. 4 shows an example where we have applied a constant bending moment \(m_0\) and shear force \(q_0\) in the region \(|y| < \lambda_0\). Thus \(M(k)\) and \(Q(k)\) in (14) will be proportional to \(\text{sinc}(k\lambda_0)\). For arbitrary bending moments and shear force variations, a Fourier series approximation may be used instead.

2.3 Response of two mechanically coupled plates
In this scenario we no longer have the free edge conditions we enjoyed in the previous sections. The two plate segments are now mechanically coupled, which implies that deflections, slopes, bending moments, and shear forces must be continuous at \(x = \xi\). Fig. 5 outlines the geometry for this situation. Because of the discontinuity in plate rigidity \((D = L\) for \(x < \xi\); \(D = R\) for \(x > \xi\)), the boundary-value problem becomes

\[
L \Psi''_x(x) - (2Lk^2 + N_x)\Psi_x(x)
\]
\[
+ (Lk^4 + N_xk^2 + \Delta pg)\Psi_x(x) = \delta(x)\Phi(k), \quad x \leq \xi
\]
\[
R \Psi''_x(x) - (2Rk^2 + N_x)\Psi_x(x)
\]
\[
+ (Rk^4 + N_xk^2 + \Delta pg)\Psi_x(x) = 0, \quad x \geq \xi
\]

subject to the conditions that (i) the deflections remain finite as \(x \to \pm \infty\), (ii) (A5) is applied at \(x = 0\), and (iii) \(\Psi, \Psi', M_x, V_x\) are all continuous across the seam:

\[
\Psi_x(\xi^-) = \Psi_x(\xi^+),
\]
\[
\Psi'_x(\xi^-) = \Psi'_x(\xi^+),
\]
\[
- L[''\Psi'_x(\xi^-) - vk^2\Psi_x(\xi^-)] = - R[''\Psi'_x(\xi^+) - vk^2\Psi_x(\xi^+)]
\]

From our earlier discussion, it is clear that \(\Psi_x(x)\) must be written

\[
\Psi_x(x) = \begin{cases} \exp(a_kx)(A_k \sin b_kx + B_k \cos b_kx), & x \leq 0 \\ \exp(a_kx)(C_k \sin b_kx + D_k \cos b_kx) + \exp(-a_kx)(E_k \sin b_kx + F_k \cos b_kx), & 0 \leq x \leq \xi \\ \exp[-g_k(x-\xi)] \times [G_k \sin h_k(x-\xi) + H_k \cos h_k(x-\xi)], & x \geq \xi \end{cases}
\]

where \(a_k, b_k, g_k, h_k\) are defined as in (11) for \(D = R\). The solution for the eight coefficients \(A_k, B_k, C_k, D_k, E_k, F_k, G_k, H_k\) is given in the Appendix. An example of this Green’s function is presented in Fig. 6, which shows the response of a 20-km thick plate and a 40-km thick plate joined at \(x = 0\) to a point load on the thin plate at \((-\lambda_0, 0)\). While the solution and its slope are continuous across the joint, higher derivatives are not, as a consequence of \(16c-d\).

3 GEOLOGICAL EXAMPLES

3.1 Seamount loads near a weak fracture zone
As a possible geological application of the free-edge solution discussed in Section 2.1, we will consider a linear hotspot seamount chain approaching an oceanic fracture zone (FZ).
There is evidence that some fracture zones, such as the Molokai and Marquesas FZs, may be considered to be weak and not able to maintain the high stresses otherwise present at FZs (McNutt et al. 1989; Wessel & Haxby 1990; Christieon & McNutt 1992; Jordahl et al. 1995). The FZ may be further weakened by reheating caused by the hotspot. We will investigate the effect the edge condition has on plate stresses as the hotspot builds seamounts closer and closer to the FZ. The load geometry has been reduced to be a chain of conical seamounts of equal size (basal radius = 40 km; height = 5 km) equidistantly spaced every 60 km along a specified hotspot track making a 30° angle with the FZ. The grey circles in Fig. 7(a) indicate the basal extent of the model seamount chain. The most recent seamount is 100 km (~1.6λ₀) from the FZ (lower x-axis); the beginning of the hotspot track is far enough outside the area shown for the chain to be considered semi-infinite. Fig. 7(a) presents a contour map of the deflections which are only slightly affected by the free edge condition. Figs 7(b) and (c) give contour maps of the principal stresses $\sigma_{xx}$ and $\sigma_{yy}$, respectively (the stresses are proportional to derivatives of the deflection, see the Appendix for details). It has been argued that the position and spacing of individual volcanoes in hotspot chains reflect elastic parameters of the plate: the next volcano appears to erupt at a certain distance from the previous volcano where the stresses due to the load have decreased significantly (ten Brink 1991). If this is true, it is possible that the stress patterns illustrated in Fig. 7 will affect the linearity of the hotspot track. Perturbations to this
Figure 4. Flexural deformation of a semi-infinite elastic plate subjected to the combined action bending moment $m(y)$ and shear force $q(y)$ at the edge $x = 0$. These end-loads are only applied for $|y| < \lambda_0$; hence $M(k)$ and $Q(k)$ have analytical expressions proportional to $\text{sinc}(k\lambda_0)$.

Figure 5. Forces acting on two semi-infinite plates of different thicknesses mechanically coupled at $x = \xi$, which implies that the deflection, its slope, bending moments and shear forces must be continuous across the seam. In-plane tension or compression $N$ may be specified in the principal directions, with $N_{xy} = 0$. The normal load is a variable line load at $x = x_0$. In addition, buoyancy forces will resist the plate’s deflection.

Figure 6. Flexural deformation of two semi-infinite elastic plates mechanically coupled at $x = 0$ and acted upon by a point load at $(-1, 0)$. The coordinates are in units of the flexural parameter $\lambda_0$, which change at $x = 0$ since the two plates have different flexural rigidities.

Linearity has been documented at several intersections between hot-spot tracks and oceanic fracture zones (e.g. Epp 1984).

3.2 Seamount loads near a strong fracture zone

Most oceanic fracture zones appear capable of maintaining the stresses that accumulate during their evolution (Sandwell 1984; Wessel & Haxby 1990). It is therefore of interest to contrast the previous example with one in which the FZ is completely locked. From Section 2.3 we know that, unless the jump in rigidity is large, the deflections are likely to depart only moderately from the simple axisymmetrical solution. In this example, we assigned the upper and lower plate segments an elastic thickness of 25 km and 27 km, respectively, representing the typical change across the 10 Myr offset Molokai FZ at its intersection with the Hawaiian Islands. The load was identical to the one used in Fig. 7. The results of this calculation are shown in Fig. 8. We notice that the deflections show slightly less deviation than the corresponding surface in Fig. 7. The most pronounced change, however, is seen in the stress fields.
Figure 7. (a) Contour map of the deflections (in m) caused by a simulated island chain made up of identical conical loads (whose basal extent is indicated in grey) emplaced close to a free-slip fracture zone. The fracture zone is at \( y = 0 \). (b) Contour map of \( \sigma_{xx} \) (in MPa) at the surface of the plate. Negative values indicate compression. Because of the free boundary, these stresses remain quite high in the region close to the edge. (c) Similar plot for \( \sigma_{yy} \), which approaches zero along the edge.

Figure 8. (a) Contour map of the deflections in the case of a mechanically strong fracture zone. (b) Contour map of \( \sigma_{xx} \) (in MPa) at the surface of the plate. Because of the continuous boundary conditions on the bending moment \( M_y \) normal to the FZ, these stresses are much lower in the region close to the edge. (c) Similar plot for \( \sigma_{yy} \), which no longer goes to zero along the edge.

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For comparison, the difference in average stress \((\sigma_{xx} + \sigma_{yy})/2\), a stress invariant, between the two scenarios is shown in Fig. 9. It can be seen that the stress differences become large in the region between the last seamount load and the FZ. Hence, if plate stresses do indeed modulate the hotspot volcanism, one would perhaps expect a weak FZ to be more susceptible to such deviations than a strong FZ. In the case presented here no shear stresses were transmitted across the FZ. However, Christeson & McNutt (1992) estimate that perhaps 25 per cent of the shear stresses may be transmitted across the FZ. Furthermore, the angle between the FZ and hotspot track may also play a significant role. Clearly, more work is needed to document the alleged stress dependence.

3.3 Variable end loads at subduction zones

There have been numerous investigations of flexural deformation at subduction zones since the early studies of Hanks (1971), Watts & Talwani (1974) and Caldwell et al. (1976). These studies employed the 2-D thin-elastic-plate theory to study the bending of oceanic plates under the action of in-plane forces and end-loads. Later work made use of more realistic rheologies that allowed for yielding at the high curvatures often encountered in these tectonic settings (Goetze & Evans 1979; Bodine, Steckler & Watts 1981). More recently, several authors have returned to the purely elastic plate model, with the understanding that one can usually convert the elastic thickness to mechanical thickness by assuming a given rheology (Judge & McNutt 1991; Levitt & Sandwell 1995). Regardless of the approach taken, the apparent conclusion from these studies is that the inherent scatter in the thickness estimates, even after converting to mechanical thickness, obscures the anticipated systematic increase of mechanical thickness with lithospheric age (Levitt & Sandwell 1995). Thus, the promise of flexural modelling at subduction zones to constrain the thermomechanical evolution of the oceanic lithosphere remains unfulfilled. The high degree of scatter is often attributed to uncertainties inherent in estimating the flexural wavelength from bathymetric (or gravimetric) profiles, or perhaps inelastic effects or accumulated thermoelastic stresses (Levitt & Sandwell 1995; McQueen & Lambeck 1989). Although rarely pointed out, one would anticipate that the process of modelling a cross-section of a 3-D surface that varies along strike is likely to introduce uncertainties as well. We will limit our discussion to a simple case that illustrates the effect of along-strike variation in bending at oceanic trenches.

Fig. 10(a) shows the results of a flexural calculation using the theory presented in Section 2.2. Here, we have applied a periodic bending moment and shear force both proportional to \((1 + 0.75 \sin(2\pi y/500))\) to a 20-km thick elastic plate. The along-strike variation in end loads results in a flexural surface that shows isolated outer-rise highs, similar to the pattern seen off Peru (e.g. Judge & McNutt 1991). When we extract profiles perpendicular to the edge and follow the procedures of Levitt & Sandwell (1995) or Judge & McNutt (1991), we find that the 2-D estimates of elastic-plate thickness also vary along strike (Fig. 10b), and in some regions it has been overestimated by as much as 30 per cent. The pattern is similar to the along strike variation in elastic thickness documented for the Peru trench using 2-D methods (Judge & McNutt 1991). It appears that in those locations the larger bending moments to either side of the profile may combine to give a 3-D deflection that, when projected onto the profile, gives the appearance of a longer wavelength of deformation. This effect is likely to be more pronounced in settings where the trench axis is arcuate. We note that the Peru trench is not as straight as the Chile trench further south.

4 CLOSING REMARKS

We have presented analytical solutions to several 3-D flexural problems that involve a geological boundary in close proximity to the applied loads. In addition to the cases presented here, the method may be used to study 3-D deformation of sedimentary basins close to or straddling an ocean–continent boundary with its expected change in flexural rigidity (e.g. Watts 1988). The method can also be generalized to involve several plate segments, all mechanically coupled to the adjoining segments. [A 2-D example of such a model was presented by Sandwell & Schubert (1982) in their study of oceanic fracture zones.] Such a solution may be used to approximate the behaviour of a region with variable rigidity and still provide analytical results. The solutions discussed in this paper are relatively simple to implement compared to complex numerical schemes. Because the convolution over all point loads has been considerably simplified by the notation in (7), the solutions are obtained much faster than in finite-difference schemes. The methods presented in this paper have been implemented using Matlab; these functions are available electronically from the author upon request to wessel@soest.hawaii.edu.

The solutions discussed in this paper inherit all the limitations of the thin-plate approximation. Specifically, this means that stresses directly below point loads will be overestimated since the radial bending moment, being proportional to \(\partial^2 \sigma/\partial r^2\) of (4) evaluated at \(r = 0\), is \(-\infty\). In the event that the exact stresses in the plate directly below concentrated loads are required, one must necessarily employ the exact thick-plate theory. In the case of simple boundary conditions (i.e. 2-D or axisymmetry), analytical Fourier representations exist (e.g.
Figure 10. (a) Flexural deformation (in m) of a 20 km thick elastic plate caused by periodic bending moments and shear forces along the edge. The 3-D deformation includes isolated outer-rise highs often found seawards of subduction zones. (b) Variation in 2-D estimates of elastic thickness from profiles taken perpendicular to the trench in (a). The standard estimation process overestimates the true elastic thickness by up to 20 per cent.

McNutt & Menard 1982; Comer 1983; Zhou 1991). The other limitation in the thin-plate theory lies in its disregard of deformation due to the transverse stresses (except through boundary conditions), which implies that the vertical shear modulus is infinite. This simplification is justified as long as we are not very close to an edge (say, within \( \lambda_0/4 \) of the edge). In such narrow strips along the edge, the predictions of the thin-plate theory will be somewhat inaccurate. Thus, if we continued to place seamounts even closer to the edge in Fig. 7 then the stresses would only qualitatively be correct. However, one must keep in mind that these inaccuracies are usually smaller than the uncertainties associated with our incomplete knowledge of the plate's rheology, temperature structure, and state of stress. The case of two coupled plates is not affected as it is a special case of variable rigidity and does not involve a free end where shear deformation may be significant. An alternative approach would be to utilise modified thin-plate theories such as Reissner's (1945) theory, which may be implemented in a fashion analogous to the present development.
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APPENDIX A: SOLUTION METHOD

It is known that the solution to (8) will be of the form $e^{rx}$; inserting this into the homogeneous part of (8) gives the characteristic equation

$$Dr^4 - (2Dk^2 + N_x)r^2 + (Dk^4 + k^2N_x + \Delta\rho g) = 0$$

(A1)

Its four complex roots may be written

$$r = \pm \sqrt{2a_k \pm \sqrt{\Delta}} = \pm a_k \pm ib_k,$$

where $a_k$ is given in (12) and

$$\tan 2\theta_k = \frac{\sqrt{4a_k^4 - k^2(N_x - N)}D - N_x^2/4D^2}{k^2 + N_x/2D}.$$  

(A3)

Trigonometric identities may be used to determine that

$$a_k = a_k \sqrt{2 \cos \theta_k} = a_k \sqrt{1 + \cos 2\theta_k} = a_k \sqrt{1 + \epsilon_k},$$

$$b_k = a_k \sqrt{2 \sin \theta_k} = a_k \sqrt{1 - \cos 2\theta_k} = a_k \sqrt{1 - \epsilon_k},$$

(A4)

where $a_k$ is given in (12). Because the complex roots appear in conjugate pairs, the general solution $\Psi_k(x) = A' \exp(r_+x) + B' \exp(r_-x) + C' \exp(r_+x) + D' \exp(r_-x)$ can be rewritten as in (10). Assuming that the plate is infinite in the $-x$ direction, the general solution to (8) is given by (10). The boundary conditions at $x = 0$ are

$$\Psi_k(0^+) - \Psi_k(0^-) = 0,$$

$$\Psi'_k(0^+) - \Psi'_k(0^-) = 0,$$

$$-D[\Psi''_k(0^+) \mp vki^2 \Psi'_k(0^+)] + D[\Psi''_k(0^-) \mp vki^2 \Psi'_k(0^-)] = 0,$$

$$-D[\Psi''_k(0^+) - ((2 - \nu)k^2 - N_x/D)\Psi'_k(0^+)] + D[\Psi''_k(0^-) - ((2 - \nu)k^2 - N_x/D)\Psi'_k(0^-)] = 1.$$  

(A5)

The six conditions (9) and (A5) uniquely determine the coefficients. From (A5) we obtain

$$A_k = C_k - E_k,$$

$$B_k = D_k + F_k,$$

$$E_k = -\Phi(k) \frac{8a_k b_k^2 D}{8a_k b_k^2 D},$$

$$F_k = \Phi(k) \frac{8a_k b_k^2 D}{8a_k b_k^2 D}.$$  

(A6)

and we need to look at the boundary conditions at the plate edge (9) to determine the two final coefficients $C_k$ and $D_k$. Using $\omega = \sin(b_k \xi)$, $c = \cos(b_k \xi)$, $\eta_1 = \nu k^2$, and $\eta_2 = (2 - \nu)k^2$, we find that

$$C_1[[2a_k b_k^2 - \eta_1]s + 2a_k b_k c] + D_1[[2a_k b_k^2 - \eta_1]c - 2a_k b_k s] = -\Phi(k) \frac{6a_k b_k^2 D}{8a_k b_k^2 D} + M(k) \frac{\exp(-\alpha_k \xi)}{D},$$

(A7a)

$$C_2[a_k(a_k - 3b_k^2 - \eta_2)s - b_k(b_k - 3a_k^2 + \eta_2)c] + D_2[a_k(a_k - 3b_k^2 - \eta_2)c + b_k(b_k - 3a_k^2 + \eta_2)s] = -\Phi(k) \frac{6a_k b_k^2 D}{8a_k b_k^2 D} + Q(k) \frac{\exp(-\alpha_k \xi)}{D}.$$  

(A7b)

These equations are of the form

$$c_1 C_k + c_2 D_k = d_1,$$

$$c_3 C_k + c_4 D_k = d_2,$$

with solutions

$$C_k = \frac{c_1 d_1 - c_2 d_2}{c_1 c_4 - c_2 c_3},$$

$$D_k = \frac{c_1 d_2 - c_3 d_1}{c_1 c_4 - c_2 c_3}.$$  

(A9)

To solve the problem outlined in Section 2.3, we insert (17) into (16) and evaluate the expressions for $x = \xi$. Because (A6) still applies, we obtain the following algebraic equations:

$$C_1 s + D_1 c - \exp(-\alpha_k \xi) H_k = \exp(-2a_k \xi) (a_k s + b_k c) \Phi(k),$$

$$c_1 c_4 - c_2 c_3$$

(A10a)

$$C_1(a_k s + b_k c) + D_1(a_k c - b_k s) - \exp(-\alpha_k \xi) (b_k G_k - g_k H_k) = -\exp(-2a_k \xi) \frac{\Phi(k)}{4a_k b_k L} L,$$

$$R$$

(A10b)

$$C_1[\{2a_k^2 - \eta_1\} s + 2a_k b_k c] + D_1[\{2a_k^2 - \eta_1\} c - 2a_k b_k s] + R \exp(-\alpha_k \xi)[2a_k b_k G_k - (g_k^2 - h_k^2 - \eta_1) H_k]$$

$$= \exp(-2a_k \xi) \{[2a_k^2 - \eta_1] a_k s - (2a_k + \eta_1) b_k c \} \Phi(k),$$

$$8a_k^2 b_k L$$

(A10c)

$$C_1[\{a_k^2 - 3b_k^2 - \eta_k\} s - (b_k^2 - 3a_k^2 + \eta_k) b_k c] + D_1[\{a_k^2 - 3b_k^2 - \eta_k\} c + (b_k^2 - 3a_k^2 + \eta_k) b_k s] + R L \exp(-\alpha_k \xi)[(h_k^2 - 3g_k^2 + \eta_k) b_k G_k + (g_k^2 - 3h_k^2 - \eta_k) g_k H_k]$$

$$= -\exp(-2a_k \xi) \{[2a_k^2 - \eta_1] a_k s - 2a_k b_k c \} \Phi(k),$$

$$4a_k b_k L.$$  

(A10d)
where \( \eta_L = \eta_2 - \frac{N_2}{L} \) and \( \eta_R = \eta_2 - \frac{N_4}{R} \). It is simpler to solve this 4 by 4 linear system using a matrix solver than to find the coefficients algebraically. However, when the line load is exactly on the seam (i.e. \( \xi = 0 \)) we only have two domains, and

\[
\Psi_k(x) = \begin{cases} 
\exp(a_k x)(A_k \sin b_k x + B_k \cos b_k x), & x \leq 0 \\
\exp(-g_k x)(G_k \sin h_k x + H_k \cos h_k x), & x \geq 0 
\end{cases} \tag{A11}
\]

since \( C_k = D_k = E_k = F_k = 0 \). Applying (16) and introducing \( e_k = g_k / a_k, f_k = h_k / b_k \), and \( r = R/L \) gives

\[
G_k = \frac{b A_k + (a_k + g_k) B_k}{h_k}, \quad H_k = B_k \tag{A12}
\]

and

\[
2a_k b_k (1 + c_k r) A_k + [a_k^2 (1 + 2 c_k r + c_k^2 r^2) - b_k^2 (1 - rf_k^2)] - \eta_2 (1 - r) B_k = 0, \tag{A13a}
\]

\[
[3a_k^2 (1 - re_k^2) - b_k^2 (1 - rf_k^2) - \eta_2 (1 - r)] b_k A_k + (a_k^2 (1 - 2re_k^2 - re_k^2) - b_k^2 [3 + (2e_k - 1)rf_k^2]) - \eta_2 (1 - r) a_k B_k = -\frac{\Phi(k)}{L}. \tag{A13b}
\]

Again, these equations are of the form

\[
c_1 A_k + c_2 B_k = 0, \tag{A14}
\]

\[
c_3 A_k + c_4 B_k = d,
\]

with the solution

\[
A_k = \frac{d_c^2}{c_1 c_4 - c_2 c_3}, \quad B_k = \frac{d c_1}{c_1 c_4 - c_2 c_3}. \tag{A15}
\]

The bending stresses in the plate may be derived from the deflections (e.g. Timoshenko & Woinowsky-Krieger 1959). We may write

\[
\sigma_{xx} = \frac{6M_x}{T_x^2}, \quad \sigma_{yy} = \frac{6M_y}{T_y^2}, \quad \tau_{xy} = \frac{6M_{xy}}{T_x^2}, \tag{A16}
\]

where

\[
M_x(x,y) = -D \left[ \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right],
\]

\[
M_y(x,y) = -D \left[ \frac{\partial^2 w}{\partial y^2} + v \frac{\partial^2 w}{\partial x^2} \right],
\]

\[
M_{xy}(x,y) = D(1 - v) \frac{\partial^2 w}{\partial x \partial y}, \tag{A17}
\]

which in the \( (x, k) \) domain becomes

\[
M_x(x,k) = -D[\Psi_x' - v k^2 \Psi_k'],
\]

\[
M_y(x,k) = -D[\nu \Psi_x' - k^2 \Psi_k'],
\]

\[
M_{xy}(x,k) = ikD(1 - v) \Psi_k'. \tag{A18}
\]

The stress components may therefore be obtained using analytical expressions involving simple derivatives of the solutions such as (10), (13), or (17), summed over all \( k \) as in (6).