Propagation of Ordinary and Extraordinary Modes in Ultra-Relativistic Maxwellian Electron Plasma

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Modes of ultra relativistic electron plasma embedded in a strong magnetic field are investigated for perpendicular propagation. Using Boltzmann-Vlasov equation, a general expression for the conductivity tensor is derived. An ultra-relativistic Maxwellian distribution function is employed to derive different modes for strong magnetic field limit. In particular, the dispersion relations for the ordinary mode and the extra ordinary mode (O-mode and X-mode) are obtained. Graphs of these dispersion relations and the imaginary parts of the frequency are drawn for some specific values of the parameters. It is observed that the damping rate increases gradually, reaches some maximum point and then decreases for larger wavenumbers. Further, increasing the strength of the magnetic field lowers the maximum value of the damping rate.

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§1. Introduction

Highly relativistic plasmas existed in the early universe and are also frequently encountered in active galactic nuclei and in pulsar magnetosphere.1) Pulsar’s strong magnetic field ranges from $10^{12}$–$10^{15}$ Gauss. Pulsar magnetosphere is an excellent natural laboratory for studying plasma physics embedded in a very strong magnetic field. There now exists an extensive literature on the subject using different velocity distribution. Suvorov and Chugonov2) investigated the dispersion relations in a one dimensional plasma with a power law distribution of the particles. Kaplan and Tsytovitch3) used the same technique by considering an infinitely strong magnetic field with vacuum polarizability neglected. Melrose and Stoneham4) included quantum effects and evaluated the dispersion relations in low density limit. However, the treatment of the important features of wave dispersion in a pulsar plasma requires the introduction of a relativistic distribution function. Arons and Bernard5) used this approach by choosing “water bag” distribution which is not sufficiently general to include all the important effects in the application to pulsar plasmas. To describe the wave propagation properly, Gedalin, Melrose and Gruman6) used a distribution function with continuous first and second derivatives. Melrose and Gedalin7) chose Juttner (relativistic thermal) distribution in their calculations. More recently, Zaheer and Murtaza8) employed relativistic Maxwellian distribution function to study some wave propagation phenomenon.

The objective of the present paper is to determine the dispersion relation for a strongly magnetized plasma, using an ultra-relativistic Maxwellian distribution...
function. For that purpose, we first derive the conductivity tensor components in compact and simplified form which in turn facilitate the study of wave propagation and its damping for high frequency modes (specifically O-mode and X-mode). Collisionless damping occurs when wave transfers energy to the plasma through resonant interaction with the particles. The interaction between particles and wave that leads to resonance depends upon the electric field. X-mode cyclotron damping occurs due to the components of the electric field perpendicular to the ambient magnetic field while O-mode cyclotron damping may occur due to “effective transverse electric field” arising from the transverse perturbed magnetic field and the parallel component of velocity. However, the latter effect is weaker than the former.\(^{18}\)

The plasma we have considered can also exist in Jupiter’s magnetosphere. Since Jupiter has a magnetic field of \(10^4\) Gauss\(^9,10\) which is stronger than the Sun and the Earth, the planet’s magnetosphere fills a vast volume of space 1,000 times the volume of the Sun and a million times the volume of the Earth’s magnetosphere. Measurements confirmed the existence of electrons at energies up to 20 MeV in Jupiter’s high-energy electron-radiation belts. More recent observations of 13.8 GHz synchrotron emission confirm the presence of electrons with energies up to 50 MeV.\(^{11,12}\)

The paper is organized as follows. In §2 we start with the relativistic Vlasov equation and present an efficient method for studying the strongly magnetized electron plasma. We use it in §3 to derive the components of the dyadic, by using ultra-relativistic Maxwellian distribution function. From there we calculate the dispersion relations for O-mode and X-mode. Finally in §4 we discuss the graphs of phase velocity squared \(\omega^2/k^2\) versus frequency \(\omega_r\) and the imaginary part \(\gamma\) versus \(ck\) for O-mode and X-mode and summarize their important feature.

\section{2. General formalism}

On linearizing Maxwell’s equations, we get

\[ \omega^2 E - k^2 E + k(k \cdot E) = -4\pi i\omega J. \]

Expressing the current density \(J\) in terms of conductivity tensor, we get\(^{15,16}\)

\[ J = \sigma \cdot E \]

or

\[ J_\alpha = \sigma_{\alpha\beta} E_\beta, \]

where \(\sigma_{\alpha\beta}\) is the conductivity tensor. Summation over dummy index is understood everywhere. Combining the above equations, we obtain

\[ [(\omega^2 - k^2)\delta_{\alpha\beta} + k_\alpha k_\beta + 4\pi i\omega \sigma_{\alpha\beta}] E_\beta = 0. \]

We may rewrite the above relation as

\[ R_{\alpha\beta} E_\beta = 0, \tag{2.1} \]
where

\[ R_{\alpha\beta} = (\omega^2 - k^2)\delta_{\alpha\beta} + k_\alpha k_\beta + 4\pi i\omega\sigma_{\alpha\beta} \]  

(2.2)

is a dyadic which may be written as a 3 by 3 matrix and \( \sigma_{\alpha\beta} \) the conductivity tensor is obtained from the linearized Vlasov equation and is given below:

\[
\sigma_{\alpha\beta} = \frac{e^2}{\omega} \int_0^\infty \int_0^{2\pi} \int_0^{\pi} p^2 \sin \theta dp d\theta d\varphi \frac{v_\alpha}{\Omega} \int_{-\infty}^{\varphi} d\varphi' \exp \left[ \frac{1}{\Omega} \int_{\varphi}^{\varphi'} (-i\omega + ik \cdot \mathbf{v}''')d\varphi'' \right] \\
\left[ (\omega - \mathbf{k} \cdot \mathbf{v}') \delta_{\beta l} + v_{\beta l}' k_l \right] \frac{\partial f_0}{\partial p_l}.
\]  

(2.3)

Here \( \Omega = \frac{eB_0}{\gamma mc} \) is the relativistic cyclotron frequency, the ambient magnetic field is chosen along z-axis. We have used spherical polar coordinates so that the velocity vector \( \mathbf{v} \) is

\[ \mathbf{v} = (v \sin \theta \cos \varphi, v \sin \theta \sin \varphi, v \cos \theta). \]

\( \Omega \) and \( \mathbf{v} \) may be expressed in terms of momentum as

\[ \Omega = \frac{eB_0}{\sqrt{p^2 + m^2c^2}} \quad \text{and} \quad \mathbf{v} = \frac{\mathbf{p}}{\sqrt{p^2 + m^2c^2}}. \]  

(2.4)

In Eq. (2.3), we first perform \( \varphi'' \)-integration and then introduce Bessel functions. Executing integration over the Bessel function, we obtain after some straightforward but lengthy algebra, various components of the conductivity tensor, assuming \( f_0(p) \) to be isotropic

\[
\sigma_{xx} = -2\pi i e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi \sin \theta \sum_n \left( \frac{n^2 \Omega^2}{v k_x^2} \right) J_n^2(\xi) \frac{d\theta}{(\omega - k_z v \cos \theta - n\Omega)},
\]

\[
\sigma_{xy} = 2\pi e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi \sin^2 \theta \sum_n \left( \frac{n \Omega}{vk_x} \right) J_n(\xi) J_n'(\xi) \frac{d\theta}{(\omega - k_z v \cos \theta - n\Omega)},
\]

\[
\sigma_{xz} = -2\pi i e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi \sin \theta \cos \theta \sum_n \left( \frac{n \Omega}{k_x} \right) J_n^2(\xi) \frac{d\theta}{(\omega - k_z v \cos \theta - n\Omega)},
\]

\[
\sigma_{yx} = -2\pi e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi \sin^2 \theta \sum_n \left( \frac{n \Omega}{k_x} \right) J_n(\xi) J_n'(\xi) \frac{d\theta}{(\omega - k_z v \cos \theta - n\Omega)},
\]

\[
\sigma_{yy} = -2\pi i e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi v \sin^2 \theta \sum_n J_n^2(\xi) \frac{d\theta}{(\omega - k_z v \cos \theta - n\Omega)},
\]
\[
\sigma_{yz} = -2\pi e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi v \sin^2 \theta \cos \theta \sum_n J_n (\xi') J'_n (\xi) \frac{d\theta}{(\omega - k_z v \cos \theta - n\Omega)},
\]

\[
\sigma_{zx} = -2\pi i e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi \sin \theta \cos \theta \sum_n \left( \frac{n\Omega}{k_x} \right) J_n^2 (\xi) \frac{d\theta}{(\omega - k_z v \cos \theta - n\Omega)},
\]

\[
\sigma_{zy} = 2\pi e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi \sin^2 \theta \cos \theta \sum_n J_n (\xi) J'_n (\xi) \frac{d\theta}{(\omega - k_z v \cos \theta - n\Omega)},
\]

\[
\sigma_{zz} = -2\pi i e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi \sin \theta \cos^2 \theta \sum_n J_n^2 (\xi) \frac{d\theta}{(\omega - k_z v \cos \theta - n\Omega)},
\]

where \( \xi = \frac{k_z v \sin \theta}{\Omega} \). For \( \xi \ll 1 \), the asymptotic value of the Bessel function becomes

\[
J_n (\xi) \to \frac{1}{\Gamma(n + 1)} \left( \frac{\xi}{2} \right)^n.
\]

Here Gamma function \( \Gamma(n) \) is general which reduces to a simple factorial for positive integers i.e., \( \Gamma(n) = (n - 1)! \). To perform \( \theta \)-integration, we use the following results:

\[
\int_0^\pi (\sin \theta)^{2n+1} d\theta = \frac{\sqrt{\pi} \Gamma(n + 1)}{\Gamma(n + \frac{3}{2})},
\]

\[
\int_0^\pi (\sin \theta)^{2n+1} \cos \theta d\theta = 0,
\]

\[
\int_0^\pi (\sin \theta)^{2n+1} \cos^2 \theta d\theta = \Gamma(n + 1) \left( -\frac{\sqrt{\pi}}{\Gamma(n + \frac{3}{2})} + \frac{2^{n+3} \Gamma(n + 3)}{\Gamma(2n + 4)} \right).
\]

So the resulting components of the conductivity tensor for perpendicular propagation \( (k_z = 0) \) take the form

\[
\sigma_{xx} = -4\pi e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \sum_{n=1}^{\infty} \left( \frac{k_x}{\Omega} \right)^{2n-2} \frac{v^{2n-1} n^2}{\Gamma(2n + 2)} \left\{ \frac{1}{\omega - n\Omega} + \frac{1}{\omega + n\Omega} \right\}
\]

\[
= \sigma,
\]

\[
\sigma_{zz} = -2\pi i e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} \left[ \frac{2 v}{3 \omega} \right. \]

\[
+ \sum_{n=1}^{\infty} \left( \frac{k_x}{\Omega} \right)^{2n} \frac{v^{2n+1} n^2 (n + 1)}{\Gamma(2n + 4)} \left\{ \frac{1}{\omega - n\Omega} + \frac{1}{\omega + n\Omega} \right\} \right] dp.
\]
All other components vanish and

$$\sigma_{yy} = \sigma, \quad \sigma_{xy} = i\sigma = -\sigma_{yx}. \quad (2.8)$$

It may be mentioned here that we have separately added terms for $n = 0$ and for negative integers so that the net summation is now over positive integers only. This is necessary to execute $\theta$-integration. An earlier work had this flaw which showed up in their final result.

§3. Ultra-relativistic electron plasma

When the plasma under consideration is ultra-relativistic, the particle thermal momentum must be much larger than the rest mass momentum i.e., $p \gg m_0c$ so that from Eq. (2.4) we get

$$|v| = c \quad \text{and} \quad \Omega = \frac{eB_0}{p} \quad (3.1)$$

We shall assume the distribution function to be isotropic though the ambient magnetic field is strong\(^{13},^{14}\) Thus the ultra-relativistic Maxwellian distribution function for electron plasma is chosen to be\(^{16}\)

$$f_0 = \frac{N_e c^3}{8\pi T^3} \exp\left(-\frac{cp}{T}\right), \quad (3.2)$$

so that

$$\frac{\partial f_0}{\partial p} = -\frac{N_e c^4}{8\pi T^4} \exp\left(-\frac{cp}{T}\right).$$

3.1. O-mode

Using Eq. (3.1) and Eq. (3.2) in Eq. (2.7), we get

$$\sigma_{zz} = \frac{iN_e c^2 c^4}{4T^4} \int_0^\infty p^2 \exp\left(-\frac{cp}{T}\right) \left[ \frac{2c}{3\omega} + \sum_{n=1}^{\infty} \frac{4(n+1)c}{\Gamma(2n+4)} \left( \frac{ck_x p}{eB_0} \right)^{2n} \left\{ \frac{1}{\omega + \frac{neB_0}{p}} + \frac{1}{\omega - \frac{neB_0}{p}} \right\} \right] dp$$

$$= \frac{iN_e c^2 c^5}{4\omega T^4} \left[ \frac{2T^2}{3c^2} \int_0^{\infty} \frac{c^2 p^2}{T^2} \exp\left(-\frac{cp}{T}\right) dp + \sum_{n=1}^{\infty} \frac{4(n+1)T^{2n+2}}{\Gamma(2n+4)c^{2n+2}} \left( \frac{ck_x}{eB_0} \right)^{2n} \right]$$

$$\times \left\{ \int_0^{\frac{cp}{T} + \frac{neB_0c}{\omega T}} \frac{cp}{T} \exp\left(-\frac{cp}{T}\right) dp + \int_{\frac{cp}{T} - \frac{neB_0c}{\omega T}}^{\infty} \frac{cp}{T} \exp\left(-\frac{cp}{T}\right) dp \right\}. \quad (3.3)$$

On executing integration, we obtain

$$\sigma_{zz} = \frac{iN_e c^2 c^2}{4\omega T} \left[ \frac{4}{3} + \sum_{n=1}^{\infty} \frac{4(n+1)}{\Gamma(2n+4)} \left( \frac{ck_x T}{ceB_0} \right)^{2n} \right]. \quad (3.4)$$
In the above calculation, we have used the following integrals:

\[
\int_0^\infty \frac{x^{2n+3} \exp(-x)}{(x-nb)} dx = \exp[-nb] \left\{ E_{2n+4} (-nb) \Gamma(2n+4) + i\pi (nb)^{2n+3} \right\}
\]

and

\[
\int_0^\infty \frac{x^{2n+3} \exp(-x)}{(x+nb)} dx = \exp[nb] \left\{ E_{2n+4} [nb] \Gamma(2n+4) + i\pi (nb)^{2n+3} \right\},
\]

where \( E_n [z] \) defined as

\[
E_n [z] = \int_1^\infty \frac{\exp[-zt]}{t^n} dt
\]

is the standard exponential integral function. As evident, we have used the usual Landau contour integration formula to give the principal part and the imaginary part. We have not applied any approximation to solve the momentum integration. So this result is more general in that sense. Equation (2.2) now becomes

\[
R_{zz} = \omega^2 - c^2 k_x^2 + 4\pi i\omega \sigma_{zz}
\]
or

\[
R_{zz} = \omega^2 - c^2 k_x^2 - \omega_p^2 - 3\omega_p^2 \sum_{n=1}^\infty (n+1) \left( \frac{ck_x}{\omega_c} \right)^{2n} \left\{ \exp \left( \frac{n\omega_c}{\omega} \right) E_{2n+4} \left( \frac{n\omega_c}{\omega} \right) \right\}
\]

\[
+ \exp \left( -\frac{n\omega_c}{\omega} \right) E_{2n+4} \left( -\frac{n\omega_c}{\omega} \right) - \frac{i\pi}{\Gamma(2n+4)} \exp \left( -\frac{n\omega_c}{\omega} \right) \left( \frac{n\omega_c}{\omega} \right)^{2n+3} \right\},
\]

where

\[
\omega_p^2 = \frac{4\pi N_e e^2 c^2}{3T} \quad \text{and} \quad \omega_c = \frac{ce B_0}{T}
\]

are respectively the plasma and gyro frequencies for the ultra-relativistic plasma. This equation yields the real frequency and the damping rate as

\[
\omega_r^2 = c^2 k_x^2 + \omega_p^2 + 3\omega_p^2 \sum_{n=1}^\infty (n+1) \left( \frac{ck_x}{\omega_c} \right)^{2n} \left\{ \exp \left( \frac{n\omega_c}{\omega_r} \right) E_{2n+4} \left( \frac{n\omega_c}{\omega_r} \right) \right\}
\]

\[
+ \exp \left( -\frac{n\omega_c}{\omega_r} \right) E_{2n+4} \left( -\frac{n\omega_c}{\omega_r} \right) \right\}
\]

(3.3)
and
\[ \gamma_O = -\frac{3 \omega_p^2}{2 \omega_r} \sum_{n=1}^{\infty} (n+1) \left( \frac{ck_x}{\omega_c} \right)^{2n} \frac{\pi}{\Gamma(2n+4)} \exp \left( - \frac{n \omega_c}{\omega_r} \right) \left( \frac{n \omega_c}{\omega_r} \right)^{2n+3}. \] (3.4)

It may be recalled that for the non-relativistic case, the imaginary part does not occur but for the ultra-relativistic case, the imaginary part does occur since the relativistic cyclotron frequency is now momentum dependent.

3.2. X-mode

The X-mode is determined by the following equation:
\[
\left| \begin{array}{cc}
R_{xx} & R_{xy} \\
R_{yx} & R_{yy}
\end{array} \right| = 0,
\] (3.5)

where
\[
R_{xx} = \omega^2 - \chi,
R_{yy} = \omega^2 - c^2 k_x^2 - \chi,
R_{xy} = -i\chi = -R_{yx}
\]
and \( \chi = -4\pi i \omega \sigma \). Therefore the X-mode dispersion relation is given by
\[
(\omega^2 - \chi)(\omega^2 - c^2 k_x^2 - \chi) - \chi^2 = 0
\]
or
\[
\frac{\omega^2}{c^2 k_x^2} = \frac{\omega^2 - \chi}{\omega^2 - 2\chi}.
\]

We can write the real and imaginary parts as
\[
\omega_r^4 - c^2 k_x^2(\omega_r - \text{Re} \chi) - 2\omega_r^2 \text{Re} \chi = 0
\] (3.6)
and
\[
\gamma_X = \frac{(c^2 k_x^2 - 2\omega_r^2)}{2\omega_r (c^2 k_x^2 - 2\omega_r^2 + 2 \text{Re} \chi)} \text{Im} \chi.
\] (3.7)

By using the values from Eq. (3.1) and Eq. (3.2) in Eq. (2.6), we get
\[
\sigma = \frac{4\pi i N_e e^2 c^4}{8\pi T^4} \sum_{n=1}^{\infty} \frac{n^2 c^{2n-1}}{\Gamma(2n+2)} \left( \frac{k_x}{e B_0} \right)^{2n-2}
\]
\[
\int_0^p 2n^2 \exp \left( -\frac{cp}{T} \right) \left\{ \frac{1}{\omega - neB_0/p} + \frac{1}{\omega + neB_0/p} \right\} dp
\]
\[
= \frac{i N_e e^2 c^5}{2\omega T^4} \sum_{n=1}^{\infty} \frac{n^2 T^{2n}}{\Gamma(2n+2)c^{2n}} \left( \frac{ck_x}{e B_0} \right)^{2n-2} \int_0^\infty \frac{(cp/T)^{2n+1}}{cp - n e c B_0/\omega T} dp.
\]
Performing the integration, the conductivity tensor $\sigma$ simplifies to

$$
\sigma = \frac{iN_c e^2 c^5}{2\omega T^4} \sum_{n=1}^{\infty} \frac{n^2 T^{2n+1}}{\Gamma(2n+2)c^{2n+1}} \left( \frac{ck_x}{c B_0} \right)^{2n-2} \\
\quad \left\{ \exp \left( -\frac{n c e B_0}{\omega T} \right) E_{2n+2} \left( -\frac{n c e B_0}{\omega T} \right) \Gamma(2n+2) \right. \\
\quad + \exp \left( \frac{n c e B_0}{\omega T} \right) E_{2n+2} \left( \frac{n c e B_0}{\omega T} \right) \Gamma(2n+2) \\
\quad + i\pi \exp \left( -\frac{n c e B_0}{\omega T} \right) \left( \frac{n c e B_0}{\omega T} \right)^{2n+1} \right\}
$$

or

$$
\chi = -4\pi i\omega \sigma \\
= \frac{3}{2} \omega_p^2 \sum_{n=1}^{\infty} n^2 \left( \frac{ck_x}{\omega c} \right)^{2n-2} \left\{ \exp \left( -\frac{n \omega_c}{\omega} \right) E_{2n+2} \left( -\frac{n \omega_c}{\omega} \right) \\
\quad + \exp \left( \frac{n \omega_c}{\omega} \right) E_{2n+2} \left( \frac{n \omega_c}{\omega} \right) \right\} + i\pi \exp \left( -\frac{n \omega_c}{\omega} \right) \left( \frac{n \omega_c}{\omega} \right)^{2n+1} \right\}.
$$

Using the value of $\chi$, Eqs. (3.6) and (3.7) yields the real frequency and the damping rate as under

$$
\omega_r^4 = \frac{c^2 k_x^2}{\omega_r^2} \left[ \omega_r^2 + \frac{3}{2} \omega_p^2 \sum_{n=1}^{\infty} n^2 \left( \frac{ck_x}{\omega c} \right)^{2n-2} \left\{ \exp \left( -\frac{n \omega_c}{\omega} \right) E_{2n+2} \left( -\frac{n \omega_c}{\omega} \right) \\
\quad + \exp \left( \frac{n \omega_c}{\omega} \right) E_{2n+2} \left( \frac{n \omega_c}{\omega} \right) \right\} \right] \\
\quad - 3\omega_r^2 \omega_p^2 \sum_{n=1}^{\infty} n^2 \left( \frac{ck_x}{\omega c} \right)^{2n-2} \left\{ \exp \left( -\frac{n \omega_c}{\omega} \right) E_{2n+2} \left( -\frac{n \omega_c}{\omega} \right) \\
\quad + \exp \left( \frac{n \omega_c}{\omega} \right) E_{2n+2} \left( \frac{n \omega_c}{\omega} \right) \right\},
$$

(3.8)

$$
\gamma_X = \frac{\left( c^2 k_x^2 - 2\omega_r^2 \right)}{2\omega_r} \left[ \frac{3}{2} \omega_p^2 \sum_{n=1}^{\infty} n^2 \left( \frac{ck_x}{\omega c} \right)^{2n-2} \frac{\pi}{\Gamma(2n+2)} \exp \left( -\frac{n \omega_c}{\omega} \right) \left( \frac{n \omega_c}{\omega} \right)^{2n+1} \right] \\
\quad \left\{ \exp \left( -\frac{n \omega_c}{\omega} \right) E_{2n+2} \left( -\frac{n \omega_c}{\omega} \right) \\
\quad + \exp \left( \frac{n \omega_c}{\omega} \right) E_{2n+2} \left( \frac{n \omega_c}{\omega} \right) \right\},
$$

(3.9)
For wave numbers such that $c^2 k_x^2 \gg \omega_p^2$, the dispersion relation for X-mode reduces to

\[ R_{yy} = \omega^2 - c^2 k_x^2 - \frac{3}{2} \omega_p^2 \sum_{n=1}^{\infty} n^2 \left( \frac{ck_x}{\omega_c} \right)^{2n-2} \left\{ \exp \left( -\frac{n \omega_c}{\omega} \right) E_{2n+2} \left( -\frac{n \omega_c}{\omega} \right) \right. \]

\[ + \exp \left( \frac{n \omega_c}{\omega} \right) E_{2n+2} \left( \frac{n \omega_c}{\omega} \right) \left. \right\} + \frac{i \pi}{\Gamma(2n+2)} \exp \left( -\frac{n \omega_c}{\omega} \right) \frac{(n \omega_c)^{2n+1}}{\omega^2}. \]

Thus the real dispersion relation becomes

\[ \omega_r^2 = c^2 k_x^2 + \frac{3}{2} \omega_p^2 \sum_{n=1}^{\infty} n^2 \left( \frac{ck_x}{\omega_c} \right)^{2n-2} \left\{ \exp \left( -\frac{n \omega_c}{\omega_r} \right) E_{2n+2} \left( -\frac{n \omega_c}{\omega_r} \right) \right. \]

\[ + \exp \left( \frac{n \omega_c}{\omega_r} \right) E_{2n+2} \left( \frac{n \omega_c}{\omega_r} \right) \left. \right\}, \quad (3.10) \]

while the imaginary part simplifies to

\[ \gamma_X = \frac{3}{4} \frac{\omega_p^2}{\omega_r} \sum_{n=1}^{\infty} n^2 \left( \frac{ck_x}{\omega_c} \right)^{2n-2} \frac{\pi}{\Gamma(2n+2)} \exp \left( -\frac{n \omega_c}{\omega_r} \right) \left( \frac{n \omega_c}{\omega_r} \right)^{2n+1}. \quad (3.11) \]

These results are similar to those obtained for O-mode since the X-mode determined by the component $R_{yy}$ is reduced to a pure transverse mode.$^{15}$

\section*{§4. Results and discussion}

By solving the relativistic Boltzmann-Vlasov equation along with the Maxwells equations we have obtained the real dispersion relations and the damping rates for

![Graph for O-mode](https://academic.oup.com/ptp/article-abstract/124/6/1083/1851070)
O-mode and X-mode. For non-relativistic case, damping rate for these modes does not occur since the cyclotron frequency is independent of momentum. In order to understand the analytical results, graphical representation is given choosing specific values of $\omega_c$, $c_k$, $\omega_p$ so that the real dispersion relation and the imaginary part of the expressions for O-mode and X-mode are simplified. We plot $\frac{\omega^2}{c^2 k^2}$ versus $\frac{\omega}{\omega_p}$ for the real part of dispersion relation and $\gamma$ versus $\frac{c_k}{\omega_p}$ for the imaginary part. Instead of plotting for some particular values of $n$, we retain the summation on $n$.

Figure 1 shows the plot of phase velocity squared versus frequency for O-mode.
This graph is very much similar to the ones given in the standard text books for non-relativistic cases. However the imaginary part which is absent in the non-relativistic case has a definite behaviour for the ultra-relativistic case as shown in Fig. 2.

To plot the imaginary part we use the leading term from the real dispersion relation i.e., $\omega_r = ck_x$, which on substitution in Eqs. (3·4), (3·9) and (3·11) makes the dependence of damping rate on wave number explicit. For large wave numbers i.e., $c^2k_x^2 \gg \omega_p^2$ the damping rate reduces and eventually vanishes. So the damping is prominent only for large wavelengths. We also observe that if we increase the
value of $\omega_c$ or in other words the strength of the magnetic field, the damping is reduced. In Fig. 3, we have plotted the real dispersion relation for X-mode as given by Eq. (3.8). This graph is the same as that of the non-relativistic case except that the first resonance point is not visible in this graph. We can get that point by reducing the range of frequency in the graph.

Figure 4 shows the graph for the case where we reduce the X-mode to pure transverse mode. So we get the same graph as that for O-mode with a different resonance point. Figure 5 shows the plot for the imaginary part of the dispersion relation given by Eq. (3.9). The graph is similar to that shown in 8), where damping increases with the decrease in the wave number.

Finally in Fig. 6, the damping rate we get from Eq. (3.11) is plotted which shows similar behavior as that for O-mode. Thus we may conclude that the perpendicularly propagating modes in the ultra-relativistic plasmas have similar behaviour for the real dispersion relation as that for non-relativistic case. For the ordinary mode and pure transverse X-mode, the damping does not increase indefinitely for small wave numbers and is constrained in the presence of strong magnetic field. For the mixed X-mode, the damping rate has the same behavior as that shown in 8) but there it was for the pure transverse X-mode, which is not right.

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**References**

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