Volume of Moduli Space of Vortex Equations and Localization

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We evaluate volume of moduli space of BPS vortices on a compact genus \( h \) Riemann surface \( \Sigma_h \) by using topological field theory and localization technique developed by Moore, Nekrasov and Shatashvili. We apply this technique to Abelian (ANO) vortex and show that the volume of moduli space agrees with the previous results obtained by integrating over the moduli space metric. We extend the evaluation to non-Abelian gauge groups and multi-flavors. We also compare our results with the volume of the Kähler quotient space inspired by the brane configuration.

Subject Index: 134, 135

§1. Introduction

At the critical coupling, static solitons exert no force between them and are called BPS solitons.1) Consequently the solutions of the BPS equations have many parameters such as the position of the soliton, which are called moduli. The moduli space of the BPS solitons plays important roles in understanding dynamics of solitons such as the scattering problem.2),3) In order to study the properties of the solitons, it is desirable to know not only the topologies but also the details of the geometries of the moduli space, including the metric. However, important information on the dynamics of the BPS solitons can often be obtained by just knowing the volume of the moduli space, which is defined by an integral of the volume form made from the metric over the moduli space.

Straightforwardly, the thermodynamical partition function can be obtained from the volume of the moduli space, since the solitons behave as free particles on their moduli space, when they move slowly.4),5) We can evaluate the thermodynamical properties of solitons like the equation of states. There is another non-trivial and important application of the volume of the moduli space, which has been found first in the case of instantons. Nekrasov pointed out that the volume of the moduli space of the instantons, which is the so-called Nekrasov partition function, gives the non-perturbative effective prepotential of \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theory.6) This means that the partition function of the supersymmetric Yang-Mills

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theory essentially evaluates the volume of the BPS solitons which cause the non-perturbative quantum corrections. Thus the volume of the moduli space is very important to understand the thermodynamics and non-perturbative dynamics of the BPS solitons, although it is just one of quantitative properties of the moduli space.

We can define the metric and the volume form on the moduli space from the effective Lagrangian. Integration of the volume form over the whole moduli space gives the volume. This procedure may be called a direct approach, but is difficult in practice. Most of the previous works\(^7\)–\(^{12}\) based on effective Lagrangian considered Abelian gauge theory with single charged scalar field at the critical coupling, allowing the BPS vortices. This vortex is called Abrikosov-Nielsen-Olesen (ANO) vortices,\(^{13}\) or Abelian local vortices. Finite values of the volume of the moduli space can be obtained for vortices on compact base manifolds such as genus \(h\) Riemann surfaces. One of the most interesting results on Riemann surfaces with the finite area \(A\) is the upper bound\(^{14}\)–\(^{16}\) for the number of BPS vortices \(k\). It has also been found that full details of the metric are not needed and that only a certain local structure of the moduli space of the BPS solutions is important to evaluate the total volume of the moduli space.\(^8\) Also in another approach,\(^{12}\) the metric on the whole moduli space is not required. It is replaced by a topological information using the Duistermaat-Heckman localisation formula. More recently, BPS vortices in non-Abelian gauge theories have attracted much attention.\(^{17}\)–\(^{26}\) The asymptotic metric of the moduli space is obtained for well-separated of non-Abelian vortices.\(^{27}\) However, it is not enough to obtain the volume of the moduli space of the non-Abelian vortex.

Another indirect evaluation of the volume of the moduli space is developed by Moore, Nekrasov and Shatashvili.\(^{28},29\) They have not used the metric of the moduli space, but the “localization” technique.\(^{30},31\) The moduli space of the BPS solitons is usually a Kähler manifold. This Kähler structure induces the localization property in the integration of the Kähler form over the moduli space which gives the volumes. The localization means that the integral of the volume form over the moduli space is localized (dominated) at isometry fixed points of the moduli space. This localization simplifies the evaluation of the volume of the moduli space drastically: the volume of the moduli space has been evaluated in this way for the Hitchin equation, which is the two-dimensional vortex system coupled with the Higgs field in the adjoint representation.\(^{28},29\) The localization technique is based on a topological field theory, which can be understood as a twisted version of the supersymmetric theories.\(^{30}\)

In this paper, we apply the localization technique in the evaluation of the moduli space of the BPS vortices on Riemann surfaces.\(^\ast\) We consider \(N_f\) flavors of Higgs fields in the fundamental representation in the Abelian as well as the non-Abelian \(U(N_c)\) gauge theories. For the ANO vortices \((N_c = N_f = 1)\) where volumes were obtained from the metric before,\(^{5},8\) we find that our results by the localization technique completely agree with the previous results for any topology of the Riemann surface. Although it has been difficult to construct the metric of moduli space of

\(^\ast\) Although Ref. 12) also used a localization theorem, it is applied in a quite different context, and the relation to the localization method used in this paper is unclear.
the non-Abelian vortex apart from well-separated local vortices \((N_f = N_c)\),\(^{27}\) we can evaluate the volume of the moduli space of the non-Abelian vortices, since the localization technique does not need the details of the BPS solutions nor the metric. We only need the BPS equations as constraints in the field configuration space. The moduli space of BPS solitons can be regarded as the quotient space of the fields constrained by the BPS equations, in contrast to the usual Kähler and hyper-Kähler quotient spaces which are defined in terms of the spatial coordinates. We have to extend the coordinate integrals to path integrals of the fields. But this extension is straightforward and the localization properties still hold in the path integral. We will show that the integrals, which give the volume of the moduli space in the localization technique, reduce to simple residue integrals. Therefore we can evaluate the volume of the moduli space of the BPS vortices much easier than the explicit construction of the metric from the BPS solutions. We also work out the metric of the moduli space of single Abelian as well as non-Abelian vortices in order to compare it to our result from localization technique.

In §2, we first take up the volume of general Kähler and hyper-Kähler quotient spaces as examples in order to explain an outline of the localization technique by introducing ambient supermanifolds. In §3, we apply the localization technique to the BPS Abelian vortices including the multi-flavor case \((N_f \geq 1)\). We further apply our localization technique to the BPS vortices in the non-Abelian gauge theories in §4. The results of our localization technique are made more explicit for different topologies of the Riemann surfaces in §5. Effective Lagrangians of single vortices are worked out for Abelian as well as non-Abelian gauge theories in §6. Localization technique is applied to the Hanany-Tong moduli space in §7. Relation of our localization technique to supersymmetry is given in §8. Further discussion is given in §9.

\section{The volume of quotient spaces}

In this first section, we review a calculation of the volume of (hyper-)Kähler quotient spaces with a \(U(1)\) isometry by using the localization technique, following Ref. 32). We here consider a quotient space in terms of the usual spatial coordinates, but essentials of the calculations can be applied to field theoretical quotient space of the moduli space in subsequent sections.

\subsection{Kähler quotient space}

We first consider a 2\(n\)-dimensional Kähler manifold \(\mathcal{M}\) whose (real) coordinates are denoted as \(x^i\) \((i = 1, \ldots, 2n)\). The Kähler manifold possesses a Kähler 2-form \(\Omega\) which gives a volume form on \(\mathcal{M}\). Then the volume of the Kähler manifold is given by

\[
\text{Vol}(\mathcal{M}) = \frac{1}{n!} \int_{\mathcal{M}} \Omega^n. \tag{2.1}
\]

Introducing Grassmann variables \(\psi^i\), it can be also written by
\[
\text{Vol}(\mathcal{M}) = \int \prod_{i=1}^{2n} dx^i d\psi^i e^{-\frac{1}{2} \Omega_{ij} \psi^i \psi^j},
\]

where \( \Omega_{ij} \) are anti-symmetric components of \( \Omega \).

The Kähler manifold \( \mathcal{M} \) is now defined as a quotient space in a \( (2n+2) \)-dimensional ambient space \( \hat{M} \)

\[
\mathcal{M} = \frac{\mu^{-1}(0)}{U(1)},
\]

where \( \mu(x^i, y, \theta) \) is a moment map defined in \( \hat{M} \) and \( \mu^{-1}(0) \) represents a space of solutions \( \mu = 0 \). The Kähler quotient space is a space of solutions \( \mu = 0 \) with \( U(1) \) identification, and the coordinate \( y \) represents a transverse coordinate to the surface \( \mu = 0 \) and \( \theta \) is a \( U(1) \) direction in \( \mathcal{M} \). Since \( \mu \) and \( \theta \) are constants on \( \mathcal{M} \) by definition, we can insert identities

\[
\int d\mu \delta(\mu) = \frac{1}{2\pi} \int d\theta = 1
\]

into the volume integral (2.2), then we have

\[
\text{Vol}(\mathcal{M}) = \frac{1}{2\pi} \int \prod_{i=1}^{2n} dx^i d\psi^i d\mu d\theta \delta(\mu) e^{-\frac{1}{2} \Omega_{ij} \psi^i \psi^j}.
\]

We can also write

\[
\delta(\mu) = \frac{1}{2\pi} \int d\phi e^{-i\phi \mu}.
\]

Thus in terms of an integral over the coordinates of \( \hat{M} \), the volume of \( \mathcal{M} \) is given by

\[
\text{Vol}(\mathcal{M}) = \frac{1}{(2\pi)^2} \int d\phi \prod_{i=1}^{2n} dx^i d\psi^i dy d\chi d\tilde{\chi} e^{-S},
\]

where \( S \) is an “action”

\[
S = i\phi \mu + \frac{1}{2} \Omega_{ij} \psi^i \psi^j + \chi \frac{\partial \mu}{\partial y} \tilde{\chi},
\]

and we have introduced Grassmann coordinates \( \chi \) and \( \tilde{\chi} \) to exponentiate a Jacobian \( \frac{\partial \mu}{\partial y} \) of a change of variables from \( \mu \) to \( y \).

We now introduce the \( (2n+2) \) coordinates of \( \hat{M} \) by \( x^\mu = (x^i, y, \theta) \) and \( (2n+2) \) Grassmann coordinates by \( \psi^\mu = (\psi^i, \tilde{\chi}, \chi) \) \( (\mu = 1, \ldots, 2n+2) \). The moment map also generates the \( U(1) \) isometry in \( \mathcal{M} \), that is,

\[
d\mu + i_V \hat{\Omega} = 0,
\]

where \( \hat{\Omega} \) is Kähler 2-form on \( \hat{M} \) and \( i_V \) is an interior product with respect to a vector field \( V \) generated by the \( U(1) \) isometry. In the components, it means

\[
\frac{\partial \mu}{\partial y} + \hat{\Omega}_{\theta y} = 0.
\]
Therefore the action (2.8) can be written in terms of $x^\mu$ and $\psi^\mu$ covariantly

$$S = i\phi \mu(x) + \frac{1}{2} \hat{Q}_{\mu\nu} \psi^\mu \psi^\nu. \quad (2.11)$$

Thus we finally obtain the integral of the volume of the Kähler quotient space $\mathcal{M}$

$$\text{Vol}(\mathcal{M}) = \frac{1}{(2\pi)^2} \int d\phi \prod_{\mu=1}^{2n+2} dx^\mu d\psi^\mu e^{-S(x^\mu, \psi^\mu)}. \quad (2.12)$$

The volume of the Kähler quotient space is now converted into an integral (2.12) over bosonic and fermionic coordinates. The “localization” mechanism owing to a symmetry of the above system will be very useful to evaluate the integral. Indeed if we introduce the following fermionic symmetry (BRST symmetry):

$$Q_x^i = i\psi^i, \quad Q\psi^i = 0,$$

$$Q_y = i\tilde{\chi}, \quad Q\tilde{\chi} = 0,$$

$$Q\theta = i\chi, \quad Q\chi = -\phi,$$

$$Q\phi = 0, \quad (2.13)$$

we find that the action (2.8) is invariant under this symmetry, namely $QS = 0$. Or we can define covariantly in the coordinates of the ambient space (supermanifold),

$$Qx^\mu = i\psi^\mu, \quad Q\psi^\mu = -\phi V^\mu,$$

$$Q\phi = 0, \quad (2.14)$$

where $V^\mu$ is a vector along the isometry direction $\theta$, which satisfies $V^\theta = 1$ and others are zero. If we add a BRST exact action $S' \equiv Q\Xi(x^\mu, \psi^\mu)$ to this system, the volume does not change in the integral (2.12). In other words, the volume integral modified by $S'$

$$\text{Vol}(\mathcal{M}) = \frac{1}{(2\pi)^2} \int d\phi \prod_{\mu=1}^{2n+2} dx^\mu d\psi^\mu e^{-S(x^\mu, \psi^\mu) - \frac{1}{g} Q\Xi(x^\mu, \psi^\mu)} \quad (2.15)$$

is independent of the “coupling” $g$. This fact means that the integral is exact in the WKB approximation $g \to \infty$. If we suitably choose $S'$, we can generally show that supports of the integral (2.12) as well as (2.15) are localized at fixed point of the BRST symmetry (2.14). This localization property is the reason why we can evaluate the volume integral (2.12), and will play an important role in subsequent discussions.

2.2. Hyper-Kähler quotient space

Next we extend the above discussions to hyper-Kähler quotient spaces. The hyper-Kähler quotient space has $SU(2)$ isometry and is defined by zero solutions of three moment maps $\mu_a = 0$ ($a = 1, 2, 3$) with the $U(1)$ identification

$$\mathcal{M} = \frac{\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0)}{U(1)}. \quad (2.16)$$
The three moment maps consist of a triplet of the $SU(2)$ isometry. We introduce coordinates of $\mathcal{M}$ by $x^i (i = 1, \ldots, 2n)$. Similarly to the Kähler quotient case, the volume of $\mathcal{M}$ is calculated by

$$\text{Vol}(\mathcal{M}) = \int_{\mathcal{M}} \Omega^n = \int \prod_{i=1}^{2n} dx^i d\psi^i e^{-\frac{1}{2} \Omega_{ij} \psi^i \psi^j}, \tag{2.17}$$

where $\Omega$ is a Kähler 2-form associated with the third moment map $\mu_3$. Inserting the moment map constraints $\mu_a = 0$ into the integral of volume (2.17), we obtain the integral over the $(2n + 4)$-dimensional ambient space $\hat{\mathcal{M}}$ with the Grassmann coordinates $\psi^i$

$$\text{Vol}(\mathcal{M}) = \frac{1}{2\pi} \int \prod_{i=1}^{2n} (dx^i d\psi^i) \prod_{a=1}^{3} dy^a d\theta \det \left( \frac{\partial \mu_a}{\partial y^b} \right) \prod_{a=1}^{3} \delta(\mu_a) e^{-\frac{1}{2} \Omega_{ij} \psi^i \psi^j}. \tag{2.18}$$

Introducing additional Grassmann coordinates $(\chi_a, \bar{\chi}_a)$ $(a = 1, 2, 3)$ and Lagrange multipliers $(\phi, Y_1, Y_2)$ for the delta functions, the volume integral can be written as

$$\text{Vol}(\mathcal{M}) = \frac{1}{(2\pi)^4} \int d\phi dY_1 dY_2 \prod_{i=1}^{2n} (dx^i d\psi^i) \prod_{a=1}^{3} (dy^a d\chi^a d\bar{\chi}^a) d\theta e^{-S}, \tag{2.19}$$

where the “action” $S$ is defined by

$$S = iY_1 \mu_1 + iY_2 \mu_2 + i\phi \mu_3 + \frac{1}{2} \Omega_{ij} \psi^i \psi^j + \hat{\Omega}_{ia} \psi^i \bar{\chi}^a + \frac{1}{2} \hat{\Omega}_{ab} \bar{\chi}^a \bar{\chi}^b + \sum_{a,b=1}^{3} \chi^a \frac{\partial \mu_a}{\partial y^b} \bar{\chi}^b, \tag{2.20}$$

where extra terms $\hat{\Omega}_{ia} \psi^i \bar{\chi}^a$ and $\frac{1}{2} \hat{\Omega}_{ab} \bar{\chi}^a \bar{\chi}^b$ do not change the fermionic determinants of $\Omega_{ij}$ and $\frac{\partial \mu_a}{\partial y^b}$ since they can be eliminated by shifting of $\psi^i$ and $\bar{\chi}^a$. The third moment map determines the Kähler 2-form on $\hat{\mathcal{M}}$ by

$$\frac{\partial \mu_3}{\partial y^a} + \hat{\Omega}_{3a} = 0. \quad (a = 1, 2, 3) \tag{2.21}$$

If we introduce the $(2n + 4)$-dimensional coordinates of $\hat{\mathcal{M}}$ as $x^\mu = (x^i, y^a, \theta)$ and corresponding Grassmann coordinates $\psi^\mu = (\psi^i, \chi^a, \bar{\chi}^3)$, the action can be written as

$$S = iY_1 \mu_1 + iY_2 \mu_2 + i\phi \mu_3 + \frac{1}{2} \hat{\Omega}_{\mu\nu} \psi^\mu \psi^\nu + \sum_{a=1}^{3} \left\{ \chi^a \frac{\partial \mu_1}{\partial y^a} \bar{\chi}^a + \chi^2 \frac{\partial \mu_2}{\partial y^a} \bar{\chi}^a \right\}. \tag{2.22}$$

Using this action, we finally find an integral expression of the volume of hyper-Kähler quotient space $\mathcal{M}$

$$\text{Vol}(\mathcal{M}) = \frac{1}{(2\pi)^4} \int d\phi dY_1 dY_2 d\chi_2 \prod_{\mu=1}^{2n+4} dx^\mu d\psi^\mu e^{-S}, \tag{2.23}$$
where the integral is taken over the \((2n+4)\)-dimensional ambient supermanifold with the coordinates \((x^\mu, \psi^\mu)\) and Lagrange multipliers \((Y_1, \chi_1), (Y_2, \chi_2)\) and \(\phi\).

We now introduce the following fermionic symmetry (BRST transformations):

\[
\begin{align*}
Q x^i &= i \psi^i, & Q \psi^i &= 0, \\
Q y^a &= i \tilde{\chi}^a, & Q \tilde{\chi}^a &= 0, \\
Q \theta &= i \chi_3, & Q \chi_3 &= -\phi, \\
Q Y_1 &= \phi \chi_2, & Q \chi_1 &= Y_1, \\
Q Y_2 &= -\phi \chi_1, & Q \chi_2 &= Y_2, \\
Q \phi &= 0,
\end{align*}
\tag{2.24}
\]

or

\[
\begin{align*}
Q x^\mu &= i \psi^\mu, & Q \psi^\mu &= -\phi V^\mu, \\
Q Y_1 &= \phi \chi_2, & Q \chi_1 &= Y_1, \\
Q Y_2 &= -\phi \chi_1, & Q \chi_2 &= Y_2, \\
Q \phi &= 0,
\end{align*}
\tag{2.25}
\]

in the ambient supermanifold. Then the action is invariant under this symmetry. Indeed if we write

\[
S = S_1 + S_2,
\tag{2.26}
\]

where

\[
S_1 = i \phi \mu_3 + \frac{1}{2} \tilde{\chi}^a \chi_a \psi^\mu \psi^\nu,
\tag{2.27}
\]

\[
S_2 = i Y_1 \mu_1 + i Y_2 \mu_2 + \sum_{a=1}^3 \left\{ \chi^1 \frac{\partial \mu_1}{\partial y^a} \tilde{\chi}^a + \chi^2 \frac{\partial \mu_2}{\partial y^a} \tilde{\chi}^a \right\},
\tag{2.28}
\]

we can show that \(S_1\) is BRST closed, namely \(QS_1 = 0\), and \(S_2\) can be written simply as a BRST exact form

\[
S_2 = Q \{ i \chi_1 \mu_1 + i \chi_2 \mu_2 \},
\tag{2.29}
\]

if we use \(\frac{\partial \mu_1}{\partial y^a} = 0, \text{ etc.}\) Thus the action is invariant under the BRST symmetry. Because of the BRST symmetry, the integral \((2.23)\) is localized at the fixed points by using similar arguments to the Kähler quotient case.

In this section, we discussed how to evaluate the volume of the Kähler and hyper-Kähler quotient spaces which are spanned by spatial coordinates. We found that the volume integral is represented by an integral \((2.15)\) or \((2.23)\) over the bosonic and fermionic coordinates of the ambient space. In the following, we will treat the moduli space of BPS vortices as a quotient space defined by moment maps imposing BPS constraints on fields instead of spatial coordinates. Therefore we need to extend the integral over spatial coordinates to a path integral over the fields. Apart from replacing the coordinates by the fields, many features of the above arguments in the coordinate space will still be valid in evaluating the volume of the moduli space of BPS vortices.
§3. Abelian vortex

3.1. Localization of path integral in Abelian case

In this section, we consider the volume of the moduli space of BPS vortices in Abelian (\( G = U(1) \)) gauge theory with \( N_f \) Higgs fields on a compact Riemann surface \( \Sigma_h \) of genus \( h \). Introducing complex coordinates \( z = x^1 + ix^2 \) and \( \bar{z} = x^1 - ix^2 \) on \( \Sigma_h \), the conformally flat metric of the Riemann surface is defined as

\[
ds^2 = g_{zz} dzd\bar{z}.
\] (3.1)

We also define the Kähler 2-form \( \omega = \frac{i}{2} g_{zz} dz \wedge d\bar{z} \) from the metric. Then the area of the Riemann surface \( \Sigma_h \) is given by

\[
A = \int_{\Sigma_h} \omega.
\] (3.2)

There exists a \( U(1) \) gauge field on the Riemann surface \( \Sigma_h \). The field strength is defined in terms of the complex coordinates by

\[
F_{zz} = i[D_z, D_{\bar{z}}],
\] (3.3)

where \( D_z \equiv \partial_z - iA_z \) and \( D_{\bar{z}} \equiv \partial_{\bar{z}} - iA_{\bar{z}} \) are covariant derivatives for fields with unit \( U(1) \) charge and \( A_z \) and \( A_{\bar{z}} \) are gauge fields.

We now define the BPS equations of the Abelian vortex

\[
F - \frac{g^2}{2} (c - HH^\dagger) \omega = 0,
\] (3.4)

\[
D_{\bar{z}} H = D_{\bar{z}} H^\dagger = 0,
\] (3.5)

where \( g \) is the \( U(1) \) gauge coupling, \( c \) is a Fayet-Iliopoulos (FI) parameter. The Higgs field \( H(z, \bar{z}) \) has unit charge and is represented by an \( N_f \) component vector. The covariant derivatives of the Higgs fields are defined by \( D_{\bar{z}} H \equiv \partial_{\bar{z}} H - iA_z H \) and \( D_{\bar{z}} H^\dagger \equiv \partial_{\bar{z}} H^\dagger + iA_{\bar{z}} H^\dagger \). The topological charge (vorticity) \( k \) is given by an integral of the 2-form field strength \( F \equiv F_{zz} dz \wedge d\bar{z} \)

\[
k = \frac{1}{2\pi} \int_{\Sigma_h} F.
\] (3.6)

The BPS equations (3.4) and (3.5) define three moment maps

\[
\mu_r \equiv F - \frac{g^2}{2} (c - HH^\dagger) \omega,
\] (3.7)

\[
\mu_{\bar{z}} \equiv D_{\bar{z}} H,
\] (3.8)

\[
\mu_z \equiv D_z H^\dagger.
\] (3.9)

Using these moment maps, the moduli space \( M_k \) of the vortex with the vorticity \( k \) is given by a Kähler quotient space

\[
M_k = \frac{\mu_r^{-1}(0) \cap \mu_{\bar{z}}^{-1}(0) \cap \mu_z^{-1}(0)}{U(1)}.
\] (3.10)
where $\mu_r^{-1}(0)$ stands for the space of solutions which satisfy $\mu_r = 0$ and $\frac{1}{2\pi} \int_{\Sigma_h} F = k$, etc. Precisely speaking, this quotient space is Kähler but not hyper-Kähler since the three moment maps do not form triplets of the $SU(2)$ isometry. However the structure of the three moment maps is very similar to the hyper-Kähler case. Therefore we can utilize the discussions of the hyper-Kähler case in the previous section.

We now introduce fermionic fields $\lambda, \psi$ to define BRST transformations for fields as follows:

$$
\begin{align*}
QA &= i\lambda, \\
Q\lambda &= -d\Phi, \\
QH &= i\psi, \\
Q\psi &= \Phi H, \\
QH^\dagger &= -i\psi^\dagger, \\
Q\psi^\dagger &= \Phi H^\dagger, \\
QY &= \Phi^* \chi, \\
Q\chi &= \lambda, \\
Q\Phi &= 0,
\end{align*}
$$

(3.11)

where we have used form notations for the gauge field $A \equiv A_z dz + A_{\bar{z}} d\bar{z}$, for the bosonic one-form $Y = Y_z dz + Y_{\bar{z}} d\bar{z}$ and fermionic one-form $\chi = \chi_z dz + \chi_{\bar{z}} d\bar{z}$ ($*\chi = i(\chi_z dz - \chi_{\bar{z}} d\bar{z})$). The BRST pair of these auxiliary fields $Y$ and $\chi$ are $N_f$ component vectors similarly to the Higgs field $H$, and will be used as Lagrange multipliers of the moment map constraints. These BRST transformations are nilpotent up to gauge transformations, namely $Q^2 = -i\delta\Phi$, where $\delta\Phi$ is the generator of the gauge transformation with infinitesimal parameter $\Phi$. Thus if we consider gauge invariant operators only, the BRST transformation $Q$ forms a cohomology for those operators, which is called the “equivariant cohomology”. The equivariant cohomology clarifies topological aspects of (topological) field theory considering, and will play an essential but indirect role of the “localization” in the evaluation of the volume.

$\Phi$ is BRST closed itself, and therefore any function of $\Phi$

$$
O_0 \equiv \mathcal{W}(\Phi)
$$

(3.12)

is also BRST closed. In the sense of the BRST symmetry, the 0-form operator becomes a good (physical) observable. The 0-form observable satisfies the so-called descent relation

$$
dO_0 + QO_1 = 0,
$$

(3.13)

where

$$
O_1 \equiv \frac{\partial \mathcal{W}(\Phi)}{\partial \Phi} \lambda.
$$

(3.14)

This fact means that the integral of $O_1$ along a closed circle $\gamma$ on $\Sigma_h$

$$
\int_{\gamma} O_1
$$

(3.15)

is BRST closed and a good cohomological observable. Similarly, $O_1$ satisfies

$$
dO_1 + QO_2 = 0,
$$

(3.16)

where

$$
O_2 \equiv i\frac{\partial \mathcal{W}(\Phi)}{\partial \Phi} F + \frac{1}{2} \frac{\partial^2 \mathcal{W}(\Phi)}{\partial \Phi^2} \lambda \wedge \lambda.
$$

(3.17)
Thus the operator
\[ \int_{\Sigma_h} O_2 \] (3-18)
also becomes BRST closed. If we choose \( W(\Phi) = \frac{1}{2} \Phi^2 \), we find that the integral
\[ \int_{\Sigma_h} \left[ i\Phi F + \frac{1}{2} \lambda \wedge \lambda \right] \] (3-19)
is BRST closed. Furthermore, we can see an integral
\[ \int_{\Sigma_h} \left[ i\Phi H H^\dagger \omega + \psi^\dagger \psi \omega \right] \] (3-20)
is also BRST closed, since the integrand can be written by the BRST exact form \( Q \left[ -iH \psi^\dagger \omega \right] \).

We would like to calculate the volume of the moduli space \( \mathcal{M}_k \), that is the volume of the solutions, of the BPS equations (3-4) and (3-5). The volume of the moduli space of the BPS equations is obtained from the following integral,\(^\ast\) although it looks like a “partition function”
\[ V_k = \int \mathcal{D}\Phi \mathcal{D}^2 Y \mathcal{D}^2 \chi \mathcal{D}^2 A \mathcal{D}^2 \lambda \mathcal{D}^2 H \mathcal{D}^2 \psi e^{-S}, \] (3-21)
where the path integral is taken to satisfy the constraint \( \frac{1}{2\pi} \int F = k \) and the normalization factor \( \frac{1}{2\pi} \) comes from the volume of the unitary group \( U(1) \). In the spirit of the previous section, the “action” \( S \) is defined by
\[ S = S_0 + S_1, \] (3-22)
where
\[ S_0 = \int_{\Sigma_h} \left[ i\Phi \mu_r + \frac{1}{2} \lambda \wedge \lambda + \frac{g^2}{2} \psi^\dagger \psi \omega \right], \] (3-23)
\[ S_1 = t_1 Q \int_{\Sigma_h} i\chi \wedge *\mu_c, \] (3-24)
where \( \mu_c \equiv \mu_z dz + \mu_{\bar{z}} d\bar{z} \). We can show the action is invariant under the BRST symmetry \( QS = 0 \). We expect that this path integral defines the volume of the moduli space \( \mathcal{M}_k \).

If the action includes BRST exact terms with couplings, the path integral does not depend on the couplings. To see this let us consider the following deformation of the integral:
\[ V_k' = \int \mathcal{D}\Phi \mathcal{D}^2 Y \mathcal{D}^2 \chi \mathcal{D}^2 A \mathcal{D}^2 \lambda \mathcal{D}^2 H \mathcal{D}^2 \psi e^{-S - tQ\Xi}. \] (3-25)
\(^\ast\) The normalization of the functional measure usually has an ambiguity, but the BRST symmetry (supersymmetry) can remove most of the ambiguity.
Differentiating it with respect to the coupling $t$, we find

$$\frac{\partial V_k'}{\partial t} = \int D\Phi D^2Y D^2\chi D^2AD^2\lambda D^2HD^2\psi (-Q\Xi)e^{-S-tQ\Xi}$$  \hspace{1cm} (3.26)$$

$$= -\int D\Phi D^2Y D^2\chi D^2AD^2\lambda D^2\psi Q (\Xi e^{-S-tQ\Xi}) = 0,$$  \hspace{1cm} (3.27)

where we have used $QS = 0$ and invariance of the measure under the BRST symmetry. Therefore the integral $V_k'$ is independent of the coupling $t_1$ in (3.24).

Using this coupling independence of the integral, we can add the following BRST exact term to the action $S$ in Eq. (3.22) without changing the value of the integral

$$S_2 = t_2Q \int_{\Sigma_h} i\chi \wedge \ast Y.$$  \hspace{1cm} (3.28)

By exploiting the coupling independence, we can go to a parameter region where the integral can be easily performed: let us take the limit $t_1 \to 0$ and $t_2 \to 1$ of the BRST exact terms. Then the action becomes

$$S' = \int_{\Sigma_h} \left[ i\Phi \mu_r + \frac{1}{2} \lambda \wedge \lambda + \frac{g^2}{2} \psi^\dagger \psi \omega \right] + Q \int_{\Sigma_h} i\chi \wedge \ast Y$$

$$= \int_{\Sigma_h} \left[ i\Phi \left( F - \frac{g^2}{2} (c - HH^\dagger) \omega \right) + \frac{1}{2} \lambda \wedge \lambda + \frac{g^2}{2} \psi^\dagger \psi \omega \right.$$

$$\left. + iY \wedge \ast Y + i\Phi \chi \wedge \chi \right].$$  \hspace{1cm} (3.29)

Thus we can use the integral

$$V_k' = \int D\Phi D^2Y D^2\chi D^2AD^2\lambda D^2HD^2\psi e^{-S'},$$  \hspace{1cm} (3.30)

to evaluate the volume of moduli space $M_k$.

First of all, we wish to integrate out the matter fields $H, \psi, Y$ and $\chi$, whose integrals are Gaussian. Neglecting the possible anomalies coming from the fermionic zero modes of matter fields, we obtain

$$V_k = \int D\Phi D^2AD^2\lambda \left( i\Phi \right)^{N_f(\dim \Omega^1 \otimes L_k - \dim \Omega^0 \otimes L_k)} e^{-\int_{\Sigma_h} \left[ i\Phi (F - \frac{g^2}{2} c) + \frac{1}{2} \lambda \wedge \lambda \right]},$$  \hspace{1cm} (3.32)

where $\dim \Omega^n \otimes L_k (n = 0, 1)$ stands for the number of holomorphic $n$-forms coupled with $U(1)$ gauge field (holomorphic line bundle) of the topological charge $k$. The Hirzebruch-Riemann-Roch theorem says (see e.g. Ref. 36)

$$\dim \Omega^0 \otimes L_k - \dim \Omega^1 \otimes L_k = 1 - h + \frac{1}{2\pi} \int_{\Sigma_h} F = 1 - h + k.$$  \hspace{1cm} (3.33)

Thus we have

$$V_k = \int D\Phi D^2AD^2\lambda \left( i\Phi \right)^{N_f(1 - h + k)} e^{-\int_{\Sigma_h} \left[ i\Phi (F - \frac{g^2}{2} c) + \frac{1}{2} \lambda \wedge \lambda \right]}.$$  \hspace{1cm} (3.34)
By using $2(1 - h) = \frac{1}{4\pi} \int_{\Sigma_h} R^{(2)}$ and $k = \frac{1}{4\pi} \int_{\Sigma_h} F$ in terms of the curvature 2-form $R^{(2)}$ of the Riemann surface and the field strength 2-form $F$, we can exponentiate powers of $\Phi$ in Eq. (3.34) to obtain

$$V_k = \int D\Phi D^2 \Phi D^2 \lambda e^{-S_{\text{eff}}},$$

(3.35)

$$S_{\text{eff}} = S_R + S_F + S_V,$$

(3.36)

where

$$S_R = \frac{1}{8\pi} \int_{\Sigma_h} \log(i\Phi) R^{(2)},$$

(3.37)

$$S_F = \int_{\Sigma_h} \left[ i \left( \Phi + \frac{1}{2\pi i} \log i\Phi \right) F + \frac{1}{2} \lambda \wedge \lambda \right],$$

(3.38)

$$S_V = -i g^2 c \int_{\Sigma_h} \Phi \omega.$$  

(3.39)

However $S_F$ is not invariant under the BRST symmetry (not BRST closed). Since any regularization scheme should preserve the BRST symmetry, this means that we have overlooked contributions from the fermionic zero modes in the integrals of fields $\psi, \chi$. To recover the contributions from the fermionic zero modes, we notice that the BRST closed action must take the form (3.17) given by the descent relation (3.16)

$$S'_{F} = \int_{\Sigma_h} \left[ i \frac{\partial W_{\text{eff}}}{\partial \Phi} F + \frac{1}{2} \frac{\partial^2 W_{\text{eff}}}{\partial \Phi^2} \lambda \wedge \lambda \right],$$

(3.40)

where the tree level term $\frac{1}{2} \Phi^2$ is accompanied by the quantum correction $Nf Nf \Phi (\log i\Phi - 1)$

$$W_{\text{eff}}(\Phi) = \frac{1}{2} \Phi^2 + \frac{Nf}{2\pi i} \Phi (\log i\Phi - 1).$$

(3.41)

Then we obtain

$$S'_{F} = \int_{\Sigma_h} \left[ i \left( \Phi + \frac{1}{2\pi i} \log i\Phi \right) F + \frac{\mu(\Phi)}{2} \lambda \wedge \lambda \right],$$

(3.42)

where

$$\mu(\Phi) = \frac{\partial^2 W_{\text{eff}}}{\partial \Phi^2} = 1 + \frac{Nf}{2\pi i \Phi}.$$  

(3.43)

The only correction due to (previously neglected) anomalies of the fermionic zero modes is changing the coefficient of $\frac{1}{2} \lambda \wedge \lambda$ from unity to $\mu(\Phi)$, which assures the BRST symmetry of the effective action.

To perform the integration over $A$, we decompose the $U(1)$ gauge fields into classical configuration and fluctuations. In terms of the field strength, this means

$$F = F_{cl} + F_q,$$

(3.44)

where $F_{cl} = \frac{2\pi k}{A} \omega$, which satisfies $\frac{1}{2\pi} \int F_{cl} = k$, and $F_q = dA_q$ are quantum fluctuations. Integrating the fluctuations $A_q$, we find a constraint $d\Phi = 0$ on $\Sigma_h$ with
Jacquards from the effective action $\int i \frac{\partial V_{\text{eff}}}{\partial \phi} F$. This means that $\Phi(z, \bar{z})$ takes a constant value $\phi$ everywhere on $\Sigma_h$.

And also, we decompose $\lambda$ into $\lambda = \lambda_0 + \delta \lambda$, where $\lambda_0$ is a harmonic 1-form and $\delta \lambda$ is fluctuation orthogonal to $\lambda_0$. Determinants from integrals of $\delta \lambda$ are completely canceled with the Jacobians induced from the integral of $A_\eta$. Then, integrals of fermionic zero modes $\lambda_0$ as the harmonic 1-form, which has $h$ independent components on $\Sigma_h$, induce a residual contribution $\mu(\phi)^h$ to the integral over the constant mode $\phi$.

Combining these integral pieces and using $\int_{\Sigma_h} \omega = A$ and $\int_{\Sigma_h} F = 2\pi k$, we finally reduce the path integral into the following one-dimensional integral over ordinary real number $\phi$ (the constant mode of the field $\Phi$)

$$
\mathcal{V}_k = \int_{-\infty}^{\infty} d\phi \frac{\mu(\phi)^h}{2\pi} \frac{e^{i\phi(\frac{\pi^2}{2}A-2\pi k)}}{e^{2\pi i\phi B}}, \quad (3.45)
$$

where $\mu(\phi)$ is defined in Eq. (3.43) and

$$
B = g_c^2 \frac{A - k}{4\pi}. \quad (3.46)
$$

Now let us evaluate the above integral. Since the integrand has a pole at $\phi = 0$, we need to look for the correct integration contour to avoid the pole. The term $\int i \Phi HH^\dagger \omega$ in the action (3.30) of the path integral reveals that we need to choose the contour below the real axis in order to assure the convergence of path integral of matter fields $H$. Namely we should avoid the pole at $\phi = 0$ counter-clock-wise below the pole. Expanding the integrand in powers of $\phi$, we can integrate term by term

$$
\mathcal{V}_k = \int_{-\infty - \epsilon i}^{\infty - \epsilon i} d\phi \frac{1 + \frac{N_f}{2\pi} \epsilon_{\phi}^h}{2\pi (i\phi)^{N_f(k+1-h)}} e^{2\pi i\phi B}
$$

$$
= \sum_{j=0}^{h} \frac{h!}{j!(h-j)!} \left( \frac{N_f}{2\pi} \right)^{h-j} (2\pi B)^{d-j} (d-j)! e^{2\pi i\phi B}, \quad \text{for } B \geq 0 \text{ and } d \geq h
$$

$$
= \begin{cases} 
\sum_{j=0}^{h} \frac{h!}{j!(h-j)!} \left( \frac{N_f}{2\pi} \right)^{h-j} (2\pi B)^{d-j} (d-j)! , & \text{for } B \geq 0 \text{ and } 0 \leq d < h \\
0, & \text{for } B < 0 \text{ or } d < 0
\end{cases} \quad (3.47)
$$

where we have defined $d \equiv kN_f + (1-h)(N_f - 1)$ and used the residue integral formula

$$
\int_{-\infty - \epsilon i}^{\infty - \epsilon i} d\phi \frac{1}{2\pi (i\phi)^{n+1}} e^{2\pi i\phi B} = \begin{cases} 
\frac{(2\pi B)^n}{n!}, & B \geq 0 \text{ and } n \in \mathbb{Z}_{\geq 0} \\
0, & B < 0 \text{ or } n \in \mathbb{Z}_{< 0}
\end{cases} \quad (3.48)
$$
Equation (3.47) should give the volume of the moduli space of the Abelian vortex with \( N_f \) Higgs fields. If \( B < 0 \), namely \( A < \frac{4\pi}{g^2_c} k \), the integral vanishes. This means that there is no solution for \( A < \frac{4\pi}{g^2_c} k \). This result is in agreement with the bound found in the case of ANO vortices \( (N_c = N_f = 1) \), which is known as the Bradlow bound. More interestingly, the non-vanishing volume exists only if \( d \geq 0 \), namely \( k \geq \frac{(h-1)N_f-1}{N_f} \). Therefore we can choose any non-negative \( k \) for \( h=0 \), \( h=1 \) or \( N_f=1 \), but \( k \) must be sufficiently large for \( h > 1 \) in the case of \( N_f > 1 \).

In this way, the evaluation of the volume of the vortex moduli space using the path integral reduces finally to the contour integral whose value is determined by the residue at the poles. The positions of the poles correspond to the fixed points of the BRST symmetry. This fact reflects the localization theorem in the topological field theory path integral with the BRST symmetry.

3.2. ANO vortices \( (N_f = N_c = 1) \)

Let us consider the simplest case \( N_f = 1 \), namely the ANO vortices. In this case, Eq. (3.47) reduces to

\[
\mathcal{V}_k = \begin{cases} 
(2\pi)^{k-h} \sum_{j=0}^{h} \frac{h!}{j!(h-j)!} \frac{B^{k-j}}{(k-j)!}, & \text{for } B \geq 0 \text{ and } k \geq h \\
(2\pi)^{k-h} \sum_{j=0}^{k} \frac{h!}{j!(h-j)!} \frac{B^{k-j}}{(k-j)!}, & \text{for } B \geq 0 \text{ and } k < h \\
0, & \text{for } B < 0.
\end{cases}
\]

(3.49)

In our field theoretical evaluation of the volume, the integral of \( k = 0 \) sector, that is the sector of the flat connections represents the volume of the vacuum moduli space. Setting \( k = 0 \), we obtain

\[
\mathcal{V}_0 = \frac{1}{(2\pi)^h}.
\]

(3.50)

The net contribution from the \( k \) vortex sector is obtained by modding out the contribution from the vacuum. Therefore if we define \( \tilde{\mathcal{V}}_k \equiv \mathcal{V}_k / \mathcal{V}_0 \), we find for \( A \geq \frac{4\pi}{g^2_c} k \)

\[
\tilde{\mathcal{V}}_k = (2\pi)^k \sum_{j=0}^{\min(h,k)} \frac{h!}{j!(k-j)!(h-j)!} \left( \frac{g^2_c A}{4\pi} - k \right)^{k-j},
\]

(3.51)

where \( \min(h, k) \) denotes the smaller one of \( h \) or \( k \). We can compare our result to the previous result of the volume of the moduli space obtained from the moduli space metric by Manton and Nasir

\[
\text{Vol}(\mathcal{M}_k) = \pi^k \sum_{j=0}^{\min(h,k)} \frac{(4\pi)^j}{j!(k-j)!(h-j)!} \left( \frac{g^2_c A - 4\pi k}{4\pi} \right)^{k-j} h!.
\]

(3.52)

We find that they are related by just a normalization factor

\[
\tilde{\mathcal{V}}_k = \frac{\text{Vol}(\mathcal{M}_k)}{(2\pi)^k}.
\]

(3.53)
This overall normalization factor can be attributed to an arbitrary normalization scale in defining the moduli space metric. Therefore we find an exact agreement with the previous direct calculation.

Finally, we give a comment on asymptotic behavior in the large area limit $A \to \infty$. Let us define the dimensionless area $\tilde{A}$ in unit of the intrinsic area of single vortex $4\pi/(g^2c)$

$$\tilde{A} = \frac{g^2cA}{4\pi}. \quad (3.54)$$

In the large area limit, the volume of the moduli space behaves

$$V_k \sim (2\pi \tilde{A})^k \frac{(kN_f + N_f - 1)!}{k!}, \quad (3.55)$$

which means that the moduli space of $k$ vortices can be regarded as the moduli space of well-separated and undistinguishable points on $\Sigma_h$ if the area is sufficiently large compared to the intrinsic area $4\pi/(g^2c)$ of each vortex. In other words, the moduli space of $k$ vortices $M_k$ is the symmetric product space of the single vortex moduli space $M_1$, $(M_1)^k/S_k$ in that large area limit.

### 3.3. Abelian semi-local vortices ($N_f > N_c = 1$)

Let us now list the volume of Abelian semi-local vortices for $N_f > N_c = 1$, in the case of the smaller genus: $h = 0$ (sphere $S^2$) or $h = 1$ (torus $T^2$). On the sphere $\Sigma_0 = S^2$, the integral becomes

$$V_k(S^2) = \frac{(2\pi B)^{kN_f + N_f - 1}}{(kN_f + N_f - 1)!}. \quad (3.56)$$

This expression suggests the dimension of $M_k(S^2)$ is $2kN_f + 2(N_f - 1)$. The contribution from the vacuum is obtained by setting $k = 0$

$$V_0(S^2) = \frac{(2\pi)^{N_f - 1}}{(N_f - 1)!} \times \tilde{A}^{N_f - 1}. \quad (3.57)$$

As we will see in §5, the moduli space of single semi-local vortices ($N_f > N_c = 1$) have moduli living on the complex projective space $\mathbb{C}P^{N_f - 1}$, whose squared radius becomes as large as the area $\tilde{A}$. These moduli are usually called non-normalizable modes, since their mode functions become non-normalizable in the limit of infinite area $\tilde{A} \to \infty$. The factor $\frac{(2\pi)^{N_f - 1}}{(N_f - 1)!}$ precisely corresponds to the volume of the complex projective space $\mathbb{C}P^{N_f - 1}$ with a unit radius. It is interesting to note that the volume of the vacuum moduli $V_0(S^2)$ suggests the presence of “non-normalizable” vacuum moduli living on the complex projective space $\mathbb{C}P^{N_f - 1}$ with the radius $\sqrt{\tilde{A}}$ in unit of the vortex size $g\sqrt{c/(4\pi)}$.

If we normalize the integral by modding out the contribution from the vacuum moduli, we obtain the net contribution of $k$-vortex sector

$$\tilde{V}_k(S^2) = \frac{(2\pi)^{kN_f}}{\prod_{j=1}^{kN_f} (j + N_f - 1)} \times \frac{(\tilde{A} - k)^{kN_f + N_f - 1}}{\tilde{A}^{N_f - 1}}. \quad (3.58)$$
In the large area limit $\tilde{A} \to \infty$, the normalized integral behaves
\[
\tilde{V}_k(S^2) \sim \frac{(2\pi)^{kN_f}}{\prod_{j=1}^{kN_f}(j + N_f - 1)} \tilde{A}^{kN_f}.
\]
(3.59)

This behavior suggests the complex dimension of the moduli space of vortex is $kN_f$, which agrees with the analysis from the index theorem in Ref. 17).

On the torus $\Sigma_1 = T^2$, we obtain
\[
V_k(T^2) = \frac{N_f (2\pi)^{kN_f}}{2\pi (kN_f)!} (\tilde{A} - k)^{kN_f - 1}.
\]
(3.60)

In this case, the contribution from the vacuum moduli is an $\tilde{A}$-independent constant $V_0(T^2) = \frac{N_f}{2\pi}$. Therefore we can find that $\dim \mathcal{M}_k(T^2) = kN_f$ from the large area limit.

Finally we give a comment on phenomena just at the saturated point of the Bradlow limit, namely $\tilde{A} = k$ (the dissolving limit). If there is a simple pole in the integrand of the residue integral (3.47), the integral does not vanish even at the Bradlow limit, $B = \tilde{A} - k = 0$. We find that the leading term in the dissolving limit is that with $j = d$ in (3.47)
\[
V_k(\Sigma_h) = \frac{h!}{d!(h - d)!} \left( \frac{N_f}{2\pi} \right)^{h-d} + \mathcal{O}(B),
\]
(3.61)

where $d = kN_f + (1-h)(N_f - 1)$, which must satisfy $0 \leq d \leq h$, that is $(h-1) - \frac{1}{N_f} \leq k \leq (h-1) + \frac{1}{N_f}$. For the case of the ANO vortex ($N_f = 1$), the result agrees with the discussions in Ref. 15).

§4. Non-Abelian vortex

4.1. Localization of path integral in non-Abelian case

In this section, we extend the previous discussions to the non-Abelian gauge group $U(N_c)$ case. The Higgs fields $H$ is now charged as a fundamental representation of $N_f$ flavors, that is, $H$ is represented as an $N_c \times N_f$ matrix.

Three moment maps of the BPS equations are
\[
\begin{align*}
\mu_r &= F - \frac{g^2}{2} (c - HH^\dagger) \omega, \\
\mu_\bar{z} &= D_{\bar{z}} H, \\
\mu_z &= D_z H^\dagger,
\end{align*}
\]
(4.1)

where $\omega$ is the Kähler 2-form on $\Sigma_h$. The moduli space of this non-Abelian vortices is defined by the following hyper-Kähler like quotient space:
\[
\mathcal{M}_k = \frac{\mu_r^{-1}(0) \cap \mu_\bar{z}^{-1}(0) \cap \mu_z^{-1}(0)}{U(N_c)},
\]
(4.4)
where \( \mu_r^{-1}(0) \) stands for a space of solutions to \( \mu_r = 0 \) with a constraint \( \frac{1}{2\pi} \int \text{Tr} \, F = k \), etc.

The BRST transformations are extended as follows:

\[
\begin{align*}
QA &= i\lambda, \\
QH &= i\psi, \\
QH^\dagger &= -i\psi^\dagger, \\
QY_z &= i\Phi \chi_z, \\
QY^\dagger_z &= -i\chi_z \Phi, \\
Q\Phi &= 0, \\
Qg &= 0,
\end{align*}
\]

where \( dA \Phi \equiv d\Phi - i[A, \Phi] \). These BRST transformations also are nilpotent up to gauge transformations and they yield the equivariant cohomology.

The cohomological action for the constraint of \( \mu_r = 0 \) is given by

\[
S_0 = \int_{\Sigma_h} \text{Tr} \left[ i\Phi \left\{ F - \frac{g^2}{2} (c - HH^\dagger) \omega \right\} + \frac{1}{2} \lambda \wedge \lambda + \frac{g^2}{2} \psi^\dagger \psi \omega \right].
\]

The action for the other constraints is given by

\[
S_1 = Q \int_{\Sigma_h} d^2z \text{Tr} \left[ \frac{1}{2} g^{zz'} (\chi_z \mu_{z'} + \mu_z \chi_{z'}) \right],
\]

but we can replace it with the following Gaussian action without changing the value of the path integral

\[
S_2 = Q \int_{\Sigma_h} d^2z \text{Tr} \left[ \frac{1}{2} g^{zz'} (\chi_z Y_{z'} + Y_z \chi_{z'}) \right]
\]

\[
= \int_{\Sigma_h} d^2z \text{Tr} \left[ g^{zz'} (Y_z Y_{z'} + i\Phi \chi_z \chi_{z'}) \right],
\]

because of the BRST exactness of \( S_1 \) and \( S_2 \).

Using these actions, the integral, which gives the volume of the moduli space of the \( k \) non-Abelian vortices on \( \Sigma_h \), is defined by

\[
\mathcal{V}^N_c, N_f (\Sigma_h) = \int \mathcal{D}\Phi \mathcal{D}^2 A \mathcal{D}^2 \lambda \mathcal{D}^2 \psi \mathcal{D}^2 Y \mathcal{D}^2 \chi e^{-S_0 - S_2},
\]

where \( N_c \) and \( N_f \) are the number of colors and flavors, respectively. To integrate out the fields, we choose a gauge which diagonalizes \( \Phi \) as \( \Phi = \text{diag}(\phi_1, \phi_2, \ldots, \phi_{N_c}) \).

After integrating out \( H, \psi, Y, \chi \) and off-diagonal pieces of \( A \) and \( \lambda \) first, the integral reduces to the \( U(1)^{N_c} \) gauge theory

\[
\mathcal{V}^N_c, N_f = \int \prod_{a=1}^{N_c} (\mathcal{D}\phi_a \mathcal{D}^2 A_a \mathcal{D}^2 \lambda_a) \frac{\prod_{a \neq b} (i\phi_a - i\phi_b)^{\dim \Omega^0 \otimes \mathcal{L}_{k_a} \otimes \mathcal{L}_{k_b}^{-1} - \dim \Omega^1 \otimes \mathcal{L}_{k_a} \otimes \mathcal{L}_{k_b}^{-1}}}{\prod_{a=1}^{N_c} (i\phi_a)^{N_f (\dim \Omega^0 \otimes \mathcal{L}_{k_a} - \dim \Omega^1 \otimes \mathcal{L}_{k_a})}}
\]

\[
\times e^{-\sum_{a=1}^{N_c} f_{\Sigma_h} \left[ i\phi_a (\mathcal{F}^{(a)} - \frac{g^2}{2} \omega) + \frac{1}{2} \lambda_a \wedge \lambda_a \right]},
\]

\[
(4.11)
\]
where the diagonal $a$-th $U(1)$ gauge field, field strength and gaugino are denoted as $A_a$, $F^{(a)}$ and $\lambda_a$, and $k_a$’s are diagonal $U(1)$ topological charges $\frac{1}{2\pi} \int F^{(a)} = k_a$, which satisfies the constraint of the total topological charge $k = \sum_{a=1}^{N_c} k_a$. The contribution of the numerator $\prod_{a \neq b} (i \phi_a - i \phi_b)^{\dim \mathcal{O}_a \otimes \mathcal{L}_{k_a} \otimes \mathcal{L}_{k_b}^{-1}}$ is a power of the Vandermonde determinant and comes from the integral of the ghost fields which are needed to fix the diagonal gauge of $\Phi$.

To evaluate the infinite dimensional functional determinants, we can use the Hirzebruch-Riemann-Roch theorem

$$\dim \mathcal{O}_a \otimes \mathcal{L}_{k_a} = 1 - h + k_a,$$

$$\dim \mathcal{O}_a \otimes \mathcal{L}_{k_a} \otimes \mathcal{L}_{k_b}^{-1} - \dim \mathcal{O}_a \otimes \mathcal{L}_{k_a} \otimes \mathcal{L}_{k_b}^{-1} = 1 - h + k_a - k_b,$$

which reduces the infinite dimensional functional determinants to the finite ones. Similarly to the Abelian case, we can exponentiate powers of $\phi$ and the Vandermonde determinant in Eq. (4.11) in terms of the curvature 2-form $R^{(2)}$ of the Riemann surface and the field strength 2-form $F^{(a)}$, and express the volume as a path integral over $\phi_a$, $A_a$, and $\lambda_a$ with the following Abelian effective action:

$$S_{\text{eff}} = S_R + S_F + S_V,$$

where

$$S_R = \frac{1}{8\pi} \int_{\Sigma_h} \left( N_f \sum_{a=1}^{N_c} \log(i \phi_a) \right) R^{(2)},$$

$$S_F = \int_{\Sigma_h} \sum_{a=1}^{N_c} \left[ i \left( \phi_a + \frac{N_f}{2\pi i} \log(i \phi_a) \right) F^{(a)} + \frac{1}{2} \lambda_a \wedge \lambda_a \right],$$

$$S_V = -i g^2 c \frac{2}{\Sigma_h} \sum_{a=1}^{N_c} \phi_a \omega.$$

To maintain the BRST invariance, we should add a fermionic contribution coming from anomaly and obtain

$$S_F' = \int_{\Sigma_h} \left[ \sum_{a=1}^{N_c} \phi_a \partial W_{\text{eff}}(\phi_a) F^{(a)} + \frac{1}{2} \sum_{a,b=1}^{N_c} \frac{\partial^2 W_{\text{eff}}(\phi_a) \lambda_a \wedge \lambda_b}{\partial \phi_a \partial \phi_b} \right],$$

where

$$W_{\text{eff}}(\phi) = \sum_{a=1}^{N_c} \left\{ \frac{1}{2} \phi_a^2 + \frac{N_f}{2\pi i} \phi_a (\log(i \phi_a) - 1) \right\}.$$

Using the same argument as that in the previous section, we find constraints of $d \phi_a = 0$ on $\Sigma_h$, due to the integration of the fluctuations of $A_a$ and $\lambda_a$. Therefore there remains only an integral over constant modes of $\phi_a$. Let us denote the constant zero modes by $\phi_a$ using the same symbol as the field $\phi_a$ itself. We can obtain the
Volume as finite dimensional integrals instead of path integrals of fields

\[ \mathcal{V}_{k}^{N_c, N_f} = \sum_{\sum a k_a = k} \int \prod_{a=1}^{N_c} \frac{d\phi_a}{2\pi} \mu(\phi)^h \prod_{a \neq b} (i\phi_a - i\phi_b)^{1-h+k_a-k_b} e^{2\pi i \sum_{a=1}^{N_c} \phi_a B_a} \]  

(4.20)

\[ = \sum_{\sum a k_a = k} (-1)^{\sigma} \int \prod_{a=1}^{N_c} \frac{d\phi_a}{2\pi} \mu(\phi)^h \prod_{a < b} (i\phi_a - i\phi_b)^{2-2h} e^{2\pi i \sum_{a=1}^{N_c} \phi_a B_a} \]  

(4.21)

where

\[ \mu(\phi) = \det \left| \frac{\partial^2 \mathcal{W}_{\text{eff}}(\phi)}{\partial \phi_a \partial \phi_b} \right| = \prod_{a=1}^{N_c} \left( 1 + \frac{1}{2\pi i \phi_a} \right), \]  

(4.22)

\[ B_a = \bar{A} - k_a, \]  

(4.23)

and \( \sigma = \frac{1}{2} N_c (N_c - 1) (1 - h) - \sum_{a < b} (k_a - k_b) \). The path integral for the non-Abelian gauge theory again reduces to the finite dimensional residue integral similar to the Abelian case. The localization associated with the BRST symmetry causes this reduction and the positions of the poles correspond to the fixed points.

For the non-Abelian case \((N_c > 1)\), the integral for general genus \( h \) is more complicated than the Abelian case due to a power of the Vandermonde determinant \( \det_{a,b} [(i\phi_a)^{b-1}] = \prod_{a < b} (i\phi_a - i\phi_b) \). Therefore let us concentrate the calculation of the integral for the small genus case \((h = 0, 1)\). For \( h = 0 \), we obtain

\[ \mathcal{V}_{k}^{N_c, N_f}(S^2) = \sum_{\sum a k_a = k} (-1)^{\sigma} \int \prod_{a=1}^{N_c} \frac{d\phi_a}{2\pi} \mu(\phi)^h N_c \prod_{a \neq b} (i\phi_a - i\phi_b)^{2} e^{2\pi i \sum_{a=1}^{N_c} \phi_a B_a}. \]  

(4.24)

Let us define coefficients \( a_{l_1, l_2, \ldots, l_{N_c}} \) in expansion of the square of the Vandermonde determinant in powers of \( i\phi_a \)

\[ \prod_{a < b} (i\phi_a - i\phi_b)^{2} = \sum_{\sum l_a = N_c (N_c - 1)} a_{l_1, l_2, \ldots, l_{N_c}} (i\phi_1)^{l_1} (i\phi_2)^{l_2} \cdots (i\phi_{N_c})^{l_{N_c}}. \]  

(4.25)

Then we have

\[ \mathcal{V}_{k}^{N_c, N_f}(S^2) = \sum_{\sum a k_a = k} (-1)^{\sigma} \sum_{\sum l_a = N_c (N_c - 1)} a_{l_1, l_2, \ldots, l_{N_c}} \prod_{a=1}^{N_c} F_{N_f}(k_a, l_a; A), \]  

(4.26)

where

\[ F_{N_f}(k_a, l_a; A) = \int \frac{d\phi_a}{2\pi} \frac{e^{2\pi i \phi_a B_a}}{(i\phi_a)^{N_f(k_a+1)-l_a}} \]  

\[ = \frac{1}{(N_f(k_a+1) - l_a - 1)!} \left(2\pi(A - k_a)\right)^{N_f(k_a+1)-l_a-1}. \]  

(4.27)
The contribution from the vacuum moduli space is obtained from the $k = 0$ case, namely all $k_a = 0$. Then we obtain
\[
\mathcal{V}_0^{N_c,N_f}(S^2) = (-1)^{N_c(N_c-1)/2} (2\pi)^{N_c(N_f-N_c)} \times \sum_{l_a = N_c(N_c-1)} a_{l_1,l_2,\ldots,l_{N_c}} \frac{1}{\prod_{a=1}^{N_c} (N_f - l_a - 1)!} \tilde{A}^{N_c(N_f-N_c)}. \tag{4.28}
\]

It is difficult to evaluate the above formula for general $N_c$. However, results for smaller values of $N_c$ suggest the following formula:
\[
\mathcal{V}_0^{N_c,N_f}(S^2) = N_c! \times \text{Vol}(G_{N_c,N_f}) \tilde{A}^{N_c(N_f-N_c)}, \tag{4.29}
\]
where $\text{Vol}(G_{N_c,N_f})$ is the volume of the Grassmannian $G_{N_c,N_f}$ with a unit radius. (See the Appendix.) We will check this conjecture for the smaller $N_c$ cases in the next section. Recalling the fact that the complex dimension of the Grassmannian is $N_c(N_f-N_c)$, we see $\mathcal{V}_0^{N_c,N_f}(S^2)$ represents the contribution from the vacuum moduli, including non-normalizable modes, in the non-Abelian Yang-Mills-Higgs system.\(^*\)

Using this expression (4.26), we naively expect the large area behavior as
\[
\mathcal{V}_k^{N_c,N_f}(S^2) \sim \sum_{a_k = k} (-1)^{\sigma} \sum_{l_a = N_c(N_c-1)} a_{l_1,l_2,\ldots,l_{N_c}} \frac{1}{\prod_{a=1}^{N_c} (N_f - l_a - 1)!} \times (2\pi \tilde{A})^{kN_f + N_c(N_f-N_c)}. \tag{4.30}
\]

Therefore by modding out the vacuum contribution, we find the volume of the $k$-vortex sector diverges as $kN_f$-th power of $\tilde{A}$. However, we will see this observation is too naive for the case of the local vortex $N_f = N_c$. In this case, the higher powers of $\tilde{A}$ are non-trivially cancelled with each other due to a property of the coefficients $a_{l_1,l_2,\ldots,l_{N_c}}$ in the expansion of the Vandermonde determinant.

For the torus case ($h = 1$), the integral becomes simpler because of the absence of the Vandermonde determinant. We can integrate each $\phi_a$ independently. Thus we obtain
\[
\mathcal{V}_k^{N_c,N_f}(T^2) = \sum_{a_k = k} (-1)^{\sigma} \prod_{a=1}^{N_c} \mathcal{V}_{k_a}^{1,N_f}(T^2), \tag{4.31}
\]
where $\mathcal{V}_{k_a}^{1,N_f}(T^2)$ is the volume of moduli space of the Abelian vortex with $N_f$ flavors, which has vorticity $k_a$, on the torus
\[
\mathcal{V}_{k_a}^{1,N_f}(T^2) = \frac{N_f (2\pi)^{k_aN_f}}{2\pi (k_aN_f)!} \tilde{A}(\tilde{A} - k_a)^{k_aN_f-1}. \tag{4.32}
\]

Substituting (4.32) into (4.31), we find that the general formula
\[
\mathcal{V}_k^{N_c,N_f}(T^2) = (2\pi)^{kN_f} \left( \frac{N_f}{2\pi} \tilde{A} \right)^{N_c} \sum_{a_k = k} (-1)^{\sigma} \prod_{a=1}^{N_c} \frac{(\tilde{A} - k_a)^{k_aN_f-1}}{(k_aN_f)!}. \tag{4.33}
\]

\(^*\) The factor $N_c!$ comes from the Weyl permutation of the Higgs vevs.
In particular, if we set $N_c = N_f = N$ (local vortex), we obtain

$$\mathcal{V}^{N,N}_k(T^2) = (2\pi)^{kN} \left( \frac{N}{2\pi} \hat{A} \right)^N \sum_{\sum a_k = k} (-1)^\sigma \prod_{a=1}^N \frac{(-k_a)k_a^{N-1}}{(k_aN)!}.$$  (4.34)

And in the large area limit $\hat{A} \to \infty$ the volume of the moduli space of the local vortices on the torus behaves

$$\mathcal{V}^{N,N}_k(T^2) \sim (2\pi)^{(k-1)N}N^N \hat{A}^k \sum_{\sum a_k = k} (-1)^\sigma \prod_{a=1}^N \frac{1}{(k_aN)!}.$$  (4.35)

In the following subsections, we evaluate the volume of moduli space of the non-Abelian vortices for some restricted number of $N_c$, $N_f$ and in particular, smaller genus cases of $h = 0$ or $h = 1$ for simplicity, and look in detail at properties of the volume.

4.2. On sphere $S^2$

We first evaluate the spherical case for $N_c = 2$. The integral becomes

$$\mathcal{V}^{2,N_f}_k(S^2) = \sum_{k_1+k_2 = k} (-1)^\sigma \int \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \frac{(i\phi_1 - i\phi_2)^2}{(i\phi_1)^N(k_1+1)(i\phi_2)^N(k_2+1)} e^{2\pi i \phi_1 B_1 + 2\pi i \phi_2 B_2},$$  (4.36)

where $\sigma = k+1$. The square of the Vandermonde determinant has the following expansion:

$$(i\phi_1 - i\phi_2)^2 = (i\phi_1)^2 - 2(i\phi_1)(i\phi_2) + (i\phi_2)^2.$$  (4.37)

Using the formula (4.26), we obtain

$$\mathcal{V}^{2,N_f}_k(S^2) = \sum_{\sum a_k = k} (-1)^\sigma (F_{N_f}(k_1, 2; \mathcal{A})F_{N_f}(k_2, 0; \mathcal{A})$$

$$-2F_{N_f}(k_1, 1; \mathcal{A})F_{N_f}(k_2, 1; \mathcal{A})$$

$$+ F_{N_f}(k_1, 0; \mathcal{A})F_{N_f}(k_2, 2; \mathcal{A})).$$  (4.38)

For $k = 0, 1, 2, 3$, we obtain

$$\mathcal{V}^{2,N_f}_0(S^2) = \frac{2!}{(N_f - 1)!(N_f - 2)!} (2\pi \hat{A})^{2(N_f-2)} \sim O(\hat{A}^{2N_f-4}),$$

$$\mathcal{V}^{2,N_f}_1(S^2) = \frac{(2\pi)^3N_f - 4}{(2N_f - 1)(N_f - 1)!(2N_f - 3)!} \hat{A}^{N_f-3}(\hat{A} - 1)^{2N_f-3}$$

$$\times \left( (N_f - 2)\hat{A}^2 + 2(N_f + 1)\hat{A} + (N_f - 2) \right) \sim O(\hat{A}^{3N_f-4}),$$

$$\mathcal{V}^{2,N_f}_2(S^2) = 2(2\pi)^{4N_f - 4} \left[ \frac{-2}{(N_f - 1)!(3N_f - 1)!} \hat{A}^{N_f-3}(\hat{A} - 2)^{3N_f-3}$$

$$\times \left( (2N_f^2 - 2N_f + 1)\hat{A}^2 + 2(2N_f + 1)(N_f - 1)\hat{A} + 2(N_f - 1)(N_f - 2) \right)$$

$$-2F_{N_f}(k_1, 2; \mathcal{A})F_{N_f}(k_2, 0; \mathcal{A})$$

$$-2F_{N_f}(k_1, 1; \mathcal{A})F_{N_f}(k_2, 1; \mathcal{A})$$

$$+ F_{N_f}(k_1, 0; \mathcal{A})F_{N_f}(k_2, 2; \mathcal{A})).$$  (4.38)
We expect that the integral behaves in general

$$\int_{\mathbb{R}} \frac{1}{(2\pi N_f - 1)!(2\pi N_f - 2)!} (\tilde{A} - 1)^{4N_f - 4} \sim O(\tilde{A}^{4N_f - 4}),$$

where we show the asymptotic powers of $\tilde{A}$. Noting that $\text{Vol}(G_{2,N_f}) = \frac{(2\pi)^{2(N_f - 2)}}{(N_f - 1)!(N_f - 2)!}$, we can write

$$\mathcal{V}_{3,2}^{2,N_f}(S^2) = 2(2\pi)^{5N_f - 4} \left[ \frac{1}{(4N_f - 1)!(N_f - 1)!} \tilde{A}^{N_f - 3}(\tilde{A} - 3)^{4N_f - 3} \times \left( (N_f^2 - 5N_f + 1)\tilde{A}^2 + 6N_f + 1)(N_f - 1)\tilde{A} + 9(N_f - 1)(N_f - 2) \right) \right. \left. + \frac{1}{(3N_f - 1)!(2N_f - 1)!} (\tilde{A} - 2)^{3N_f - 3} \times \left( \frac{793}{60} \tilde{A} - \frac{331}{20} \tilde{A} + \frac{818}{15} \tilde{A}^2 - \frac{2336}{21} \tilde{A} + \frac{18047}{210} \right) \right],$$

$$\sim O(\tilde{A}^{5N_f - 4}),$$

(4.39)

In general, in the large $\tilde{A}$ limit, the integral behaves as $kN_f + 2(N_f - 2)$-th power of $\tilde{A}$, in which $2(N_f - 2)$-th power of $\tilde{A}$ comes from the vacuum contribution $\mathcal{V}_{3,0}^{2,N_f}$. From Eq. (4.39), we conjecture the asymptotic power of $\tilde{A}$ for general $N_f$ as

$$\mathcal{V}_{k}^{2,N_f}(S^2) \propto \tilde{A}^{kN_f + 2(N_f - 2)}.$$  

(4.41)

Thus, in the large area limit $\tilde{A} \to \infty$, the normalized integral behaves

$$\frac{\mathcal{V}_{k}^{2,N_f}(S^2)}{\mathcal{V}_{0}^{2,N_f}(S^2)} \propto \tilde{A}^{kN_f}.$$  

(4.42)

If we specialize to the $N_c = N_f$ case, namely $N_f = 2$, the integral reduces to

$$\mathcal{V}_{0}^{2,2}(S^2) = 2,$$

$$\mathcal{V}_{1}^{2,2}(S^2) = 2 \times (2\pi)^2 (\tilde{A} - 1),$$

$$\mathcal{V}_{2}^{2,2}(S^2) = 2 \times \left( \frac{2\pi}{2!} \right)^4 \left( \tilde{A}^2 - \frac{20}{6} \tilde{A} + \frac{17}{6} \right),$$

$$\mathcal{V}_{3}^{2,2}(S^2) = 2 \times \left( \frac{2\pi}{3!} \right)^6 \left( \tilde{A}^3 - 7\tilde{A}^2 + \frac{331}{20} \tilde{A} - \frac{793}{60} \right),$$

$$\mathcal{V}_{4}^{2,2}(S^2) = 2 \times \left( \frac{2\pi}{4!} \right)^8 \left( \tilde{A}^4 - 12\tilde{A}^3 + \frac{818}{15} \tilde{A}^2 - \frac{2336}{21} \tilde{A} + \frac{18047}{210} \right).$$  

(4.43)

We expect that the integral behaves in general

$$\mathcal{V}_{k}^{2,2}(S^2) = 2 \times \left( \frac{2\pi}{k!} \right)^{2k} \frac{\tilde{A}^k}{k!} + O(\tilde{A}^{k-1}).$$  

(4.44)
Thus, in the large area limit $\tilde{A} \to \infty$, the normalized integral behaves

$$\frac{\mathcal{V}_k^{2,2}(S^2)}{\mathcal{V}_0^{2,2}(S^2)} \sim \frac{(2\pi)^{2k} \tilde{A}^k}{k!}. \quad (4.45)$$

Similarly, we can evaluate for $N_c = 3$. For $k = 0, 1$, we obtain

$$\mathcal{V}_0^{3,N_f}(S^2) = \frac{3!2!}{(N_f - 1)!(N_f - 2)!(N_f - 3)!}(2\pi\tilde{A})^{3(N_f - 3)} \sim \mathcal{O}(\tilde{A}^{3(N_f - 3)})$$

$$\mathcal{V}_1^{3,N_f}(S^2) = \frac{6(2\pi)^{4N_f-9}}{(2N_f - 1)(2N_f - 1)!(2N_f - 3)!}\tilde{A}^{2N_f-8}(\tilde{A} - 1)^2N_f-5$$

$$\times \left( (N_f - 2)(N_f - 3)\tilde{A}^4 + 4(N_f + 1)(N_f - 3)\tilde{A}^3 + 6(N_f^2 - N_f + 4)\tilde{A}^2$$

$$+ 4(N_f + 1)(N_f - 3)\tilde{A} + (N_f - 2)(N_f - 3) \right) \sim \mathcal{O}(\tilde{A}^{N_f + 3(N_f - 3)}). \quad (4.46)$$

Since $\text{Vol}(G_{3,N_f}) = \frac{(2\pi)^{3(N_f - 3)2f}}{(N_f - 1)!(N_f - 2)!(N_f - 3)!}$, we can write

$$\mathcal{V}_0^{3,N_f}(S^2) = 3!\text{Vol}(G_{3,N_f})\tilde{A}^{3(N_f - 3)}. \quad (4.47)$$

Equations (4.40) and (4.47) suggest that the integral of the vacuum sector $k = 0$ for general $N_c$ and $N_f$ is given by

$$\mathcal{V}_0^{N_c,N_f}(S^2) = N_c!\text{Vol}(G_{N_c,N_f})\tilde{A}^{N_c(N_f - N_c)}. \quad (4.48)$$

Equations (4.39) and (4.46) also suggest that the asymptotic power of $\tilde{A}$ for general $N_c$ and $N_f$ is given by

$$\mathcal{V}_k^{N_c,N_f}(S^2) \propto \tilde{A}^{KN_f + N_c(N_f - N_c)}. \quad (4.49)$$

Thus, in the large area limit $\tilde{A} \to \infty$, the normalized integral behaves

$$\frac{\mathcal{V}_k^{N_c,N_f}(S^2)}{\mathcal{V}_0^{N_c,N_f}(S^2)} \propto \tilde{A}^{KN_f}. \quad (4.50)$$

It is interesting to note that the highest power of $\tilde{A}$ is independent of $N_c$.

If we specialize to $N_c = N_f = 3$, we obtain

$$\mathcal{V}_0^{3,3}(S^2) = 3!,$$

$$\mathcal{V}_1^{3,3}(S^2) = 3! \times \frac{(2\pi)^3}{2} (\tilde{A} - 1),$$

$$\mathcal{V}_2^{3,3}(S^2) = 3! \times \frac{(2\pi)^6}{2^22!} \left( \tilde{A}^2 - \frac{46}{15} \tilde{A} + \frac{36}{15} \right),$$

$$\mathcal{V}_3^{3,3}(S^2) = 3! \times \frac{(2\pi)^9}{2^33!} \left( \tilde{A}^3 - \frac{31}{5} \tilde{A}^2 + \frac{3641}{280} \tilde{A} - \frac{23249}{2520} \right). \quad (4.51)$$
Equations (4.44) and (4.51) suggest that the integral of local vortices for general $N_c = N_f = N$ case is asymptotically given by
\[
\mathcal{V}^{N,N}_k(S^2) \sim \frac{N!}{k!} \left( \frac{(2\pi)^N}{\tilde{A}} \right)^k,
\] (4.52)
in the large area limit $\tilde{A} \to \infty$.

4.3. On torus $T^2$

For $N_c = 2$, the integral is
\[
\mathcal{V}^{2,N_f}_k(T^2) = \left( \frac{N_f}{2\pi} \right)^2 \sum_{k_1+k_2=k} \frac{(\tilde{A} - k_1)^{k_1 N_f - 1}}{(k_1 N_f)!} \frac{(\tilde{A} - k_2)^{k_2 N_f - 1}}{(k_2 N_f)!},
\] (4.53)
where we have ignored the overall sign of the integral. Assuming $\tilde{A} > k$, we obtain concretely
\[
\mathcal{V}^{2,N_f}_0(T^2) = \left( \frac{N_f}{2\pi} \right)^2,
\]
\[
\mathcal{V}^{2,N_f}_1(T^2) = \left( \frac{N_f}{2\pi} \right)^2 \times 2(2\pi)^{N_f} \frac{\tilde{A}(\tilde{A} - 1)^{N_f - 1}}{N_f!},
\]
\[
\mathcal{V}^{2,N_f}_2(T^2) = \left( \frac{N_f}{2\pi} \right)^2 \times (2\pi)^{2N_f} \left[ \frac{2 \tilde{A}(\tilde{A} - 2)^{2N_f - 1}}{(2N_f)!} + \left( \frac{\tilde{A}(\tilde{A} - 1)^{N_f - 1}}{N_f!} \right)^2 \right],
\]
\[
\mathcal{V}^{2,N_f}_3(T^2) = \left( \frac{N_f}{2\pi} \right)^2 \times 2(2\pi)^{3N_f} \left[ \frac{\tilde{A}(\tilde{A} - 3)^{3N_f - 1}}{(3N_f)!} + \frac{\tilde{A}(\tilde{A} - 1)^{N_f - 1}}{N_f!} \frac{\tilde{A}(\tilde{A} - 2)^{2N_f - 1}}{(2N_f)!} \right].
\] (4.54)

In the large area limit $\tilde{A} \to \infty$, the integral always behaves as $k N_f$-th power of $\tilde{A}$ for the semi-local and local vortices.

§5. Effective Lagrangian of vortices

In this section we study the effective Lagrangian of vortices on Riemann surfaces to define the moduli space metric of vortices, and compute the volume of moduli space in order to compare with the results of topological field theory. We here will consider only sphere topology for the Riemann surface and use the strong coupling limit $g^2 \to \infty$ to obtain the leading behavior of the moduli space volume for large area $A \to \infty$. To understand the Bradlow limit for small $A$, we also study the case of finite $g^2$ for a limited class of moduli space with single vortex.

5.1. Effective Lagrangian and moduli space metric

Let us consider a $(2 + 1)$-dimensional space-time with the line element
\[
ds^2 = -dt^2 + \sigma [ (dx)^2 + (dy)^2 ] = -dt^2 + g_{zz}dzd\bar{z} = g_{\mu\nu}dx^\mu dx^\nu,
\] (5.1)
where the conformal factor and the complex coordinate are denoted as \( \sigma = g_{z\bar{z}} \) and \( z = x + iy \), respectively. We will denote the time coordinate \( t \) and the spacial coordinates \( x, y \) by 0 and \( i, j = 1, 2 \), respectively, and space-time coordinates by \( \mu, \nu = 0, 1, 2 \). The area \( \mathcal{A} \) of the Riemann surface is given by the conformal factor \( \sigma \) as

\[
\int dx dy \, \sigma = \mathcal{A}. \tag{5.2}
\]

For the sphere, \( \sigma \) can be chosen with \( z = x + iy \) as

\[
\sigma = \frac{\mathcal{A}}{\pi(1 + |z|^2)^2}, \quad z \in \mathbb{C}. \tag{5.3}
\]

We are interested in a \( U(N_c) \) gauge theory in \((2 + 1)\)-dimensional space-time with gauge fields \( A_\mu \) as \( N_c \times N_c \) matrices and \( N_f \) Higgs fields in the fundamental representation of the \( SU(N_c)_C \) as an \( N_c \times N_f \) matrix. The Lagrangian of the theory reads

\[
L = \int dx dy \sqrt{-\det(g_{\mu\nu})} \mathcal{L} = \int dt dx dy \, \sigma \mathcal{L} = T - V, \tag{5.4}
\]

\[
\mathcal{L} = \text{Tr}\left[ -\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + D_\mu H(D^\mu H)^\dagger - \frac{g^2}{4} \left( HH^\dagger - c 1_{N_c} \right)^2 \right], \tag{5.5}
\]

\[
T = \int dx dy \, \text{Tr} \left[ \sigma D_t H(D_t H)^\dagger + \frac{1}{g^2} (F_{ti})^2 \right], \tag{5.6}
\]

\[
V = \int dx dy \, \text{Tr} \left[ \frac{1}{g^2 \sigma} (F_{12})^2 + D_i H(D_i H)^\dagger + \frac{g^2 \sigma}{4} (HH^\dagger - c)^2 \right]. \tag{5.7}
\]

Our notation is \( A_\mu = A_\mu^I t^I \) with the normalization

\[
\text{Tr}(t^I t^J) = \frac{1}{2} \delta^{IJ}. \tag{5.8}
\]

As is well known, this Lagrangian Eq. (5.5) can be embedded into a supersymmetric theory with eight supercharges.

The Bogomol’nyi bound for the energy \( E \) is obtained by completing the square

\[
E = \int dx dy \, \text{Tr} \left[ 4D_z HD_z H^\dagger + \frac{1}{g^2 \sigma} \left( F_{12} - \frac{g^2 \sigma}{2} (c - HH^\dagger) \right)^2 + c F_{12} \right]
\]

\[
\geq c \int dx dy \, \text{Tr}(F_{12}) = 2\pi ck, \tag{5.9}
\]

where the topological charge \( k \) is the vorticity, namely the number of vortices. The bound is saturated if the BPS equations (4.1), (4.2) and (4.3) are satisfied. For the Abelian gauge theory, we should just choose \( N_c = 1 \).

By rewriting the gauge fields in terms of a matrix \( S \in GL(N_c, \mathbb{C}) \)

\[
A_z = -i S^{-1} \partial_z S, \tag{5.10}
\]
we can solve the BPS equations (4.2) and (4.3) for Higgs field $D_z H = 0$ by

$$H = S^{-1} H_0(z),$$  \hspace{1cm} (5.11)

with the holomorphic $N_c \times N_f$ matrix $H_0$ which is called the moduli matrix.\textsuperscript{18,19,21,23,25,27} Also, by defining positive definite Hermitian matrix $\Omega \in GL(N_c, \mathbb{C})$

$$\Omega = S S^\dagger,$$  \hspace{1cm} (5.12)

we can rewrite the remaining BPS equation (4.1) as

$$\partial \bar{z}(\Omega \partial_z \Omega^{-1}) = \frac{g^2}{4} \left( H_0 H_0^\dagger \Omega^{-1} - c 1_N \right) \sigma,$$  \hspace{1cm} (5.13)

which is called the master equation. The master equation is expected to give a unique solution, once the moduli matrix $H_0$ is given.\textsuperscript{20} The vortex number $k$ in Eq. (5.9) can be rewritten in terms of $\Omega$ by using $F_{z \bar{z}} = i S^{-1} \partial \bar{z}(\partial_z \Omega \Omega^{-1}) \Omega S^\dagger$ as

$$2 \pi k = -2i \int dxdy \; \text{Tr}(F_{z \bar{z}}) = 2 \int dxdy \; \partial \bar{z} \partial_z \log \det \Omega
= -i \oint dz \partial_z \log \det \Omega,$$  \hspace{1cm} (5.14)

where the contour integration is counter-clock-wise and $\partial_z \log \det \Omega$ is not single-valued. Equation (5.14) specifies the boundary condition in choosing the moduli matrix $H_0$ and in solving the master equation (5.13). The definitions (5.10) and (5.12) imply that the master equation is covariant under the holomorphic transformation $V(z)$:

$$H_0(z) \rightarrow V(z) H_0(z), \; S \rightarrow V(z) S, \; V(z) \in GL(N_c, \mathbb{C}).$$  \hspace{1cm} (5.15)

Therefore there exists a one-to-one correspondence between the equivalence classes $H_0 \sim VH_0$ and points on the moduli space.

General formula for effective Lagrangian $L_{\text{eff}}$ on vortices has been worked out.\textsuperscript{21,27} Here we generalize the method to a curved compact manifold such as the Riemann surfaces. Assuming slow motion of moduli parameters, effective Lagrangian on a soliton background can be obtained up to the second order in derivative. After subtracting the term in the zero-th order in derivative (static energy $2 \pi c k$ of the soliton multiplied by $-1$), the effective Lagrangian on the curved manifold with the conformal factor $\sigma$ is given by

$$L_{\text{eff}} + 2 \pi c k = \int_{\Sigma_h} dxdy \; \text{Tr} \left[ \sigma \delta_t^\dagger \left\{ \Omega^{-1} \delta_t H_0 H_0^\dagger \right\} \right]$$

$$+ \frac{4}{g^2} \partial \bar{z} \left\{ \partial_z (\Omega^{-1} \delta_t^\dagger \Omega) \Omega^{-1} \delta_t \Omega \right\},$$  \hspace{1cm} (5.16)

where the first term comes from the kinetic term of the Higgs field $\sigma D_t H (D_t H)^\dagger$, and the second term from the electric field $\frac{1}{g^2} (F_{ti})^2$. The time derivative through (anti-)holomorphic moduli $\phi^i$ ($\bar{\phi}^i$) is denoted as

$$\delta_t = \dot{\phi}^i \frac{\partial}{\partial \phi^i}, \; \delta_t^\dagger = \dot{\bar{\phi}}^i \frac{\partial}{\partial \bar{\phi}^i}.$$  \hspace{1cm} (5.17)
In the strong coupling limit \( g^2 \to \infty \), the master equation (5.13) can be solved algebraically

\[
\Omega = \frac{H_0 H_0^\dagger}{c}, \quad g^2 \to \infty.
\]

(5.18)

Therefore the effective Lagrangian (5.16) is reduced to a simple formula for the Kähler potential \( K \)

\[
L_{\text{eff}} + 2\pi c k = \delta_{\gamma}^\dagger \delta_{\gamma} K,
\]

(5.19)

where

\[
K = c \int_{\Sigma_h} dx dy \, \sigma \log \det(H_0 H_0^\dagger).
\]

(5.20)

5.2. Moduli space metric of Abelian semi-local vortices

Let us first work out the effective Lagrangian of Abelian semi-local vortices \((N_f > N_c = 1)\) on the sphere, using the strong coupling \( g^2 \to \infty \).

Moduli matrices should be chosen to satisfy the boundary condition (5.14). If we consider the large enough area \( A \to \infty \), the solution \( \Omega \) of the master equation should approach \( \Omega \to H_0 H_0^\dagger/c \) asymptotically \( z \to \infty \). Therefore the boundary condition (5.14) is satisfied by requiring that at least one of components of moduli matrix to be a polynomial of order \( k \), and all other components to be at most of order \( k \)

\[
H_0^{(k)}(z) = \sqrt{c} \left( \sum_{j=0}^k a_j^{(1)} z^j, \cdots, \sum_{j=0}^k a_j^{(N_f)} z^j \right),
\]

(5.21)

where at least one of the coefficients of the \( k \)-th power is non-vanishing: \( a_k^{(j)} \neq 0 \). If we adiabatically deform the base Riemann surface, this should still be the appropriate moduli matrix, until possible critical value is reached where the solution of the master equation ceases to exist. We expect that the Bradlow bound\(^{14}\) gives such a critical value of the area.

Compared to the usual noncompact plane, we emphasize two new features of the vortex moduli on the compact Riemann surfaces which are realized in the moduli matrix (5.21). Firstly we allow the leading power of \( z \) to be in any components. If we use global \( SU(N_f) \) rotations combined with the \( V \)-transformations, it is possible to place the leading power to be in a particular component, say in the first component. This form is the usual choice for the moduli matrix on noncompact flat plane.\(^{22}\)

These new \( N_f - 1 \) moduli may be regarded as an orientation of the vacuum at infinity and are nonnormalizable on noncompact plane: \((a_k^{(1)}, \cdots, a_k^{(N_f)})/a_k^{(1)}\) after dividing out by \( V \)-transformations. These \( N_f - 1 \) extra complex moduli parameters are present even in the case of vacuum \((k = 0)\) on compact Riemann surfaces. Secondly the additional \( N_f - 1 \) “size” moduli are retained on compact Riemann surfaces, since they become normalizable and are dynamical variables in the effective Lagrangian. More specifically, the standard moduli matrix on noncompact plane contains up to
only \((k - 1)\)-th power of \(z\) except in the first component. The \(N_f - 1\) coefficients of these \((k - 1)\)-th power represent “size” of vortices, are nonnormalizable, and have to be fixed by the boundary condition on noncompact plane. Both of the “vacuum” and the “size” moduli become normalizable and provide additional \(2N_f - 2\) complex moduli on compact Riemann surfaces. From now on, we take Eq. (5.21) as the general moduli matrix for the \(k\)-vortex sector.

By inserting the moduli matrix (5.21) and the conformal factor (5.3) into Eq. (5.20), we obtain the Kähler potential on the sphere

\[
K^{(k)} = \mathcal{A} c \int_{S^2} dx dy \frac{1}{\pi(1 + |z|^2)} \log \left( \sum_{i=1}^{N_f} \left| \sum_{j=0}^{k} a^{(i)}_j z^j \right|^2 \right). \tag{5.22}
\]

We find that the integral is convergent and is proportional to \(\mathcal{A}\), indicating that all the moduli parameters take values of order \(\mathcal{A}\). Equations (5.19) and (5.22) give the effective Lagrangian of \(k\) semi-local vortices on the sphere. From the effective Lagrangian, we can define the metric in the moduli space

\[
dx_{\text{mod}}^2(S^2, k) = N \mathcal{A} c \sum_{i=1}^{N_f} \sum_{j=0}^{k} \sum_{i'=1}^{N_f} \sum_{j'=0}^{k} da^{(i)}_j d\bar{a}^{(i')}_j \frac{\partial}{\partial a^{(i)}_j} \frac{\partial}{\partial \bar{a}^{(i')}_j} K^{(k)}. \tag{5.23}
\]

Therefore we find that each complex moduli gives a power of \(\mathcal{A}\). Since there are \(kN_f + N_f - 1\) complex moduli, we obtain the volume of the moduli space asymptotically \(\mathcal{A} \to \infty\) to be proportional to the \(kN_f + N_f - 1\) power of \(\mathcal{A}\)

\[
\mathcal{V}_k^{1,N_f}(S^2) = (N c \pi \mathcal{A})^{kN_f+N_f-1} \times \int \prod_{i=1}^{N_f} \prod_{j=0}^{k} da^{(i)}_j \prod_{i'=1}^{N_f} \prod_{j'=0}^{k} d\bar{a}^{(i')}_j \frac{\partial}{\partial a^{(i)}_j} \frac{\partial}{\partial \bar{a}^{(i')}_j} \left( \frac{K^{(k)}}{c \pi \mathcal{A}} \right), \tag{5.24}
\]

where the coefficient of \((N c \pi \mathcal{A})^{kN_f+N_f-1}\) is given by an integral representation for \(K^{(k)}\). This asymptotic power agrees with the result (3.56) of the topological field theory.

### 5.3. Moduli space metric of non-Abelian vortices

Let us next work out the effective Lagrangian of non-Abelian semi-local vortices \((N_f > N_c > 1)\) on the sphere, using the strong coupling \(g^2 \to \infty\).

In the same spirit as the Abelian vortices, we should choose the moduli matrices to satisfy the boundary condition (5.14). The maximal degree of all possible \(N_c \times N_c\) minor determinants of \(N_c \times N_f\) matrix \(H_0\) should be \(k\). Following the method of Kähler quotient,\(^{37}\) we parametrize the moduli matrix as

\[
H^{(k)}_0(z) = \sqrt{c} (\mathbb{D}(z), \mathbb{Q}(z)), \tag{5.25}
\]

where \(\mathbb{D}\) is an \(N_c \times N_c\) matrix and \(\mathbb{Q}\) is an \(N_c \times N_f\) matrix with \(\bar{N}_c = N_f - N_c\). Let us define

\[
P(z) = \det \mathbb{D}(z). \tag{5.26}
\]
Contrary to the usual parametrization\cite{22}, we require the degree of determinants of all possible $N_c \times \tilde{N}_c$ minor matrices of $H_0(z)$ to be at most $k$, instead of the usual $k - 1$. If the base space is a flat non-compact plane, these extra moduli represent orientation of “vacuum” at infinity, and are non-normalizable and discarded. On compact Riemannn surfaces, however, we should retain these “vacuum” moduli, since they become normalizable. Moreover, we find that the above moduli matrix also contains the so-called “size” moduli in the usual parametrization of moduli matrix, which are nonnormalizable on noncompact plane. We find that not only the “vacuum” moduli, but also these “size” moduli become normalizable, and should be retained on compact Riemann surfaces, similarly to the case of Abelian gauge theories.

Following Ref. 22, we define the following $N_c \times \tilde{N}_c$ matrix $F$
\[
F(z) = P(z)D^{-1}Q(z) .
\] (5.27)
We also define the following $N_c \times k$ matrices $\Phi$ and $J$
\[
D(z)\Phi(z) = JP(z) = 0, \mod P(z) .
\] (5.28)
Multiplication of $\Phi(z)$ by $z$ can be divided by a polynomial $P(z)$ and gives constant $k \times k$ matrix $Z$ and $N_c \times k$ matrix $\Psi$
\[
z\Phi(z) = \Phi(z)Z + \Psi P(z) .
\] (5.29)
The matrices $Z$ and $\Psi$ give moduli parameters. From the definition, we obtain the relation
\[
D(z)F(z) = P(z)Q(z) .
\] (5.30)
The matrix $F$ now has a degree $k$ instead of the usual $k - 1$, it is a linear combination of column vectors of $\Phi$ after a division by $P(z)$
\[
F(z) = AP(z) + \Phi(z)\tilde{\Psi} ,
\] (5.31)
where constant $N_c \times \tilde{N}_c$ and $k \times \tilde{N}_c$ matrices $A$ and $\tilde{\Psi}$ are the remaining moduli parameters. In particular, the matrix $A$ is a new moduli parameters which are discarded as non-normalizable if the base space is non-compact. Summarizing the moduli parameters in the moduli matrix (5.25), we find $k \times k$ matrix $Z$, $N_c \times k$ matrix $\Psi$, $k \times \tilde{N}_c$ matrix $\tilde{\Psi}$, and $N_c \times \tilde{N}_c$ matrix $A$ as complex moduli parameters. Since the points in the moduli space are one-to-one correspondence with the $GL(k, \mathbb{C})$ equivalence classes,\cite{22} we find that the complex dimension of the moduli space of $k$-vortex sector is given by $kN_f + N_c(N_f - N_c)$.

By inserting the moduli matrix (5.25) and the conformal factor (5.3) into Eq. (5.20), we obtain the Kähler potential on the sphere
\[
K^{(k)} = \mathcal{A}c \int_{S^2} dxdy \frac{1}{\pi(1 + |z|^2)} \log \det \left( |P(z)|^2 + F(z)F^\dagger(z) \right) .
\] (5.32)
We find that the integral is convergent and is proportional to $\mathcal{A}$, indicating that all the moduli parameters take values of order $\mathcal{A}$. Equations (5.19) and (5.32) give the
effective Lagrangian of \( k \) non-Abelian semi-local vortices on the sphere. Since there are \( kN_f + N_c(N_f - N_c) \) complex moduli, we obtain the volume of the moduli space asymptotically \( \mathcal{A} \to \infty \) to be proportional to the \( kN_f + N_c(N_f - N_c) \) power of \( \mathcal{A} \), in agreement with the result (4.49) of the topological field theory.

If we let \( N_f = N_c = N \), we obtain the so-called non-Abelian local vortices. In this case, the moduli matrix (5.25) does not have a \( \mathbb{Q} \) piece. For single vortex \( k = 1 \), the metric has been obtained explicitly\(^\text{27}\) and only the position moduli can be of order \( \sqrt{\mathcal{A}} \), whereas other orientational moduli consists of \( \mathbb{C}P^{N_f-1} \) with the radius of order \( 1/g\sqrt{c} \). Moduli space of multi-vortices \( k > 1 \) is symmetric product of \( k \) moduli spaces of each single vortex except for separations of order smaller than the vortex size \( 1/g\sqrt{c} \). These facts imply that the orientational moduli can only give a finite volume unrelated to \( \mathcal{A} \), whereas the vortex position can be of order \( \sqrt{\mathcal{A}} \). Therefore the volume of the moduli space for \( k \) local non-Abelian vortices is proportional to \( \mathcal{A}^k \), which agrees with our result (4.52) of the topological field theory.

5.4. Moduli space metric at finite couplings

In this subsection, we consider the effective Lagrangian at finite gauge couplings. For simplicity, we take Abelian semi-local vortices. Let us first take the vacuum sector \( k = 0 \) with the moduli matrix

\[
H_0^{(0)}(z) = \sqrt{c}(a^{(1)}, \ldots, a^{(N_f)}).
\]

The corresponding solution of the master equation (5.13) is given by \( \Omega = H_0 H_0^\dagger/c \), and we find \( \partial_z(\Omega^{-1}\delta_t^\dagger\Omega) = 0 \). By inserting this moduli matrix to Eq. (5.16), we obtain

\[
L_{\text{eff}}^{(k=0)}(S^2) = g \int_{S^2} dx dy \sigma \delta_t^\dagger \left[ \frac{\delta_t \left( \sum_{j=1}^{N_f} |a^{(j)}|^2 \right)}{\sum_{l=1}^{N_f} |a^{(l)}|^2} \right] \delta_t \log \left( \sum_{j=1}^{N_f} |a^{(j)}|^2 \right).
\]

This is precisely the nonlinear sigma model with the target space \( \mathbb{C}P^{N_f-1} \) of area \( \mathcal{A} \). The moduli \( a^{(j)} \) are homogeneous coordinates of the Kähler manifold \( \mathbb{C}P^{N_f-1} \) with the Kähler potential \( \log \left( \sum_{j=1}^{N_f} |a^{(j)}|^2 \right) \).

Interpreting the effective Lagrangian in Eq. (5.34), we can define the line element \( ds_{\text{mod}}^{(k=0)}(S^2) \) of the moduli space for the vacuum sector on the sphere as

\[
ds^2_{\text{mod}}(S^2, k = 0) = N c \mathcal{A} \left( \frac{da^{(i)}db^{(i)}}{\sum_{l=1}^{N_f} |b^{(l)}|^2} - \frac{db^{(i)}b^{(j)}db^{(j)}}{\sum_{l=1}^{N_f} |b^{(l)}|^2} \right),
\]

where we have put an arbitrary normalization factor \( N \) in defining the moduli space metric from the effective Lagrangian.

The volume of the moduli space of the vacuum sector \( k = 0 \) in the case of the...
sphere is given by

\[ \hat{V}^{1,N_f}_{k=0}(S^2) = (Nc\pi A)^{N_f-1} \left( \frac{1}{(N_f - 1)!} \right). \]  (5.36)

Let us next evaluate the moduli space metric of single vortex \((k = 1)\) on a sphere. It is not possible to obtain the metric analytically for generic point of the moduli space, even for the usual flat non-compact base space.\(^{22}\) To understand the behavior of the moduli of a single vortex on a sphere near the Bradlow limit, we will first restrict ourselves to the moduli matrix where zeros of all components coincide

\[ H^{(k=1)}_0(z) = \sqrt{c}(z - z_0)(a_1^{(1)}, \cdots, a_1^{N_f}), \]  (5.37)

where the complex moduli parameters \((a_1^{(1)}, \cdots, a_1^{N_f})\) are again inhomogeneous coordinates of \(\mathbb{C}P^{N_f-1}\).

For Abelian gauge theories, the master equation can be written in terms of \(\psi \equiv \log \Omega\). The master equation (5.13) on the sphere reads

\[ \partial_z \partial_{\bar{z}} \psi = \frac{g^2c}{4} \left( 1 - e^{-\psi} \left( \sum_{j=1}^{N_f} |a_j|^2 \right) |z - z_0|^2 \right) \sigma. \]  (5.38)

By defining a new function \(\psi^{(1)}\)

\[ \psi^{(1)} = \psi - \log(1 + |\vec{b}|^2), \]  (5.39)

we can rewrite the master equation (5.38)

\[ \partial_z \partial_{\bar{z}} \psi^{(1)} = \frac{g^2c}{4} \left( 1 - e^{-\psi^{(1)}} |z - z_0|^2 \right) \sigma, \]  (5.40)

since \((1 + |\vec{b}|^2)\) is independent of the world-volume coordinates \(z, \bar{z}\). This is precisely the same master equation for the single ANO vortex \((N_c = N_f = 1)\) on the Riemann surface. Moreover, \(\psi\) in the boundary condition (5.14) can be replaced by \(\psi^{(1)}\), since \((1 + |\vec{b}|^2)\) is independent of \(z, \bar{z}\).

\[ 2\pi k = -i \oint d\bar{z} \partial_z \psi = -i \oint d\bar{z} \partial_z \psi^{(1)}. \]  (5.41)

From the geometric reason, \(\psi^{(1)}\) should have a logarithmic singularity which depends on the geodesic distance between \(z\) and \(z_0\). This implies that the variational derivative with respect to the antiholomorphic moduli \(\bar{z}_0\) can depend on \(\bar{z}\), but not on \(z\). Terefore we obtain

\[ \partial_z (\Omega^{-1} \delta_t^\dagger \Omega) = \partial_z \delta_t^\dagger \psi = \partial_z \delta_t^\dagger (\psi^{(1)} + \log(1 + |\vec{b}|^2)) = \partial_z \bar{z}_0 \frac{\partial \psi^{(1)}}{\partial \bar{z}_0} = 0. \]  (5.42)

The effective Lagrangian (5.16) with the moduli matrix (5.37) becomes

\[ L_{\text{eff}} + 2\pi c = c \int_{S^2} dx dy \sigma \delta_t^\dagger \left[ e^{-\psi} \delta_t \left\{ \sum_{j=1}^{N_f} |a_j|^2 |z - z_0|^2 \right\} \right]. \]
where we replaced the integration region by a region $D_0$ with an infinitesimal hole around the vortex position $z_0$, since we can safely ignore the contribution from the hole because of the smooth integrand without a singularity. By using the master equation (5.38), we obtain

$$L_{\text{eff}} + 2\pi c = c \int_{D_0} dxdy \delta_t^i \left[ \left( \sigma - \frac{4}{g^2 c} \partial_z \partial_{\bar{z}} \psi \right) \left( \frac{\delta_t (\sum_{j=1}^{N_f} |a_j^1|^2)}{\sum_{j=1}^{N_f} |a_j^1|^2} + \frac{\delta_t (z - z_0)}{\bar{z} - \bar{z}_0} \right) \right],$$

(5.43)

where $D_0$ is a region around $z_0$.

Since there is no singularity in the integrand of $L_1$, we can take the limit of the vanishing size of the hole. By using the definition of area (5.2) and vortex number (5.14) with $k=1$, we readily find

$$L_1 = \delta_t^i \left[ \frac{\delta_t (\sum_{j=1}^{N_f} |a_j^1|^2)}{\sum_{j=1}^{N_f} |a_j^1|^2} \int dxdy \left( c\sigma - \frac{4}{g^2} \partial_z \partial_{\bar{z}} \psi \right) \right] = \left( cA - \frac{4\pi}{g^2} \right) \left( \frac{\bar{a}_1^1 \bar{a}_1^1 \bar{a}_1^1}{\sum_{j=1}^{N_f} |a_j^1|^2} - \frac{\bar{a}_1^1 \bar{a}_1^1 \bar{a}_1^1}{\sum_{j=1}^{N_f} |a_j^1|^2} \right).$$

(5.45)

This is precisely the standard metric of $CP^{N_f-1}$ orientational moduli $\bar{b}$ with the radius $\sqrt{cA - \frac{4\pi}{g^2}}$.

On the other hand, the integrand of $L_2$ can be put into a total derivative

$$L_2 = -\frac{4}{g^2} \int_{D_0} dxdy \partial_{\bar{z}} \left[ \delta_t (z - z_0) \psi \right] = \frac{4}{g^2} \int d\bar{z} \frac{\partial_t (z - z_0) \delta_t}{z - z_0} \partial_{\bar{z}} \psi = -\frac{4\pi}{g^2} \bar{z}_0 \left[ \delta_t^i \partial_{\bar{z}} \psi \right]_{z=z_0},$$

(5.46)

where the contour integration is around $z = z_0$ and is counter-clock-wise, since the sphere has no boundary and there is only a hole around $z = z_0$. We can replace $\psi$ in the effective Lagrangian (5.46) by $\psi^{(1)}$, since $(1 + |\bar{b}|^2)$ is independent of $z, \bar{z}$

$$L_2 = -\frac{4\pi}{g^2} \bar{z}_0 \left[ \delta_t^i \partial_{\bar{z}} \psi^{(1)} \right]_{z=z_0}.$$ 

Therefore we find that $L_2$ is identical to the effective Lagrangian for the single ANO vortex ($N_c = N_f = 1$) on the Riemann surface.
We can evaluate the residue \([\delta^i_0 \partial_z \psi^{(1)}]_{z=z_0}\) by expanding around \(z = z_0\) the solution of the master equation \(\psi^{(1)}\) for the ANO vortex

\[
\psi^{(1)}(z, \bar{z}) = -\log |z - z_0|^2 - a - \frac{\bar{b}}{2}(z - z_0) - \frac{b}{2}(\bar{z} - \bar{z}_0) + \frac{g^2 c}{4}\sigma(z_0, \bar{z}_0)|z - z_0|^2 + \cdots,
\]

(5.48)

where the coefficient of \(|z - z_0|^2\) is determined by the master equation. We need to obtain the moduli dependence of the coefficient\(^3\)

\[
\left[\delta^i_0 \partial_z \psi^{(1)}\right]_{z=z_0} = \frac{\partial \sigma}{\partial z}(z_0, \bar{z}_0) - \frac{1}{2} \frac{\partial \bar{b}}{\partial \bar{z}_0}.
\]

(5.49)

Since the geodesic distance is the same as the chordal distance \(2|z - z_0|/\sqrt{(1 + |z|^2)(1 + |z_0|^2)}\) near \(z \approx z_0\) up to the order that we are interested in, we obtain the moduli dependence of the expansion coefficients \(b, \bar{b}\) as

\[
\psi^{(1)}(z, \bar{z}) = -\log \left(\frac{|z - z_0|^2(1 + |z_0|^2)}{1 + |z|^2}\right)
\]

\[
= -\log |z - z_0|^2 + \frac{z_0}{1 + |z_0|^2}(z - z_0) + \frac{z_0}{1 + |z_0|^2}(\bar{z} - \bar{z}_0) + \cdots,
\]

(5.50)

By comparing the expansion with Eqs. (5.48) and (5.50), we find

\[
\bar{b} = -2 \frac{\bar{z}_0}{1 + |z_0|^2}, \quad b = -2 \frac{z_0}{1 + |z_0|^2}.
\]

(5.51)

Thus we find \(L_2\) on sphere as

\[
L_2(S^2) = c \left(\frac{A - \frac{4\pi}{g^2c}}{1 + |z_0|^2}\right) \frac{|\dot{z}_0|^2}{(1 + |z_0|^2)^2}.
\]

(5.52)

Combining this with Eq. (5.45), we can now define the line element \(ds_{\text{mod}}\) in the case of vortex on the sphere as

\[
ds_{\text{mod}}^2(S^2, k = 1) =
\]

\[
Nc \left(\frac{A - \frac{4\pi}{g^2c}}{(1 + |z_0|^2)^2}\right) + \left(\frac{db^i db^j}{1 + |\bar{b}|^2} - \frac{db^i db^j b^k db^k}{(1 + |\bar{b}|^2)^2}\right).
\]

(5.53)

Although the moduli matrix covers only a part of moduli space, the result shows that both the vortex position \(z_0\) and orientational moduli \(\vec{a}_1\) are meaningful only for \(A \geq 4\pi/(g^2c)\), satisfying the Bradlow bound. It is interesting to note that the same \(\mathbb{C}P^{N - 1}\) orientational moduli as the vacuum moduli emerges except that their radius is changed from \(\sqrt{A}\) to \(\sqrt{A - 4\pi/(g^2c)}\) satisfying the the Bradlow bound.
So far, we have discussed the volume of the moduli space of the BPS equations for the vortex. According to Ref. 17), the string theoretical realization of the BPS vortex admits an alternative description of the vortex moduli space. Hanany and Tong have defined the ADHM-like quotient space from effective theory on the brane configuration for vortex in superstring theory and discussed the properties of the moduli space of the vortex. The volume of the moduli space corresponding to the Hanany-Tong (HT) quotient space has already evaluated by the localization method.\(^{38)}-44)\) In this section, we extend these evaluations to the non-Abelian case and compare with the results in the previous sections.

Using the brane configuration of the supersymmetric gauge theory, the 2 + 1-dimensional gauge theory is realized on the D3-branes stretched between two NS5-branes. The world-volumes of the D3-branes and NS5-branes are along \((x^0, x^1, x^2, x^6)\) and \((x^0, x^1, x^2, x^3, x^4, x^5)\) directions, respectively. The rank of the gauge theory corresponds to the number of the D3-branes and the inverse of the square of the gauge coupling \(1/g^2\) is proportional to the distance of two NS5-branes. We can also add the matter fields (hypermultiplets) in the fundamental representation by introducing semi-infinite D3-branes from one side of the NS5-branes. The number of the semi-infinite D3-branes is the number of the flavors \(N_f\).

In the Higgs phase, the D3-branes between the NS5-branes connects with the semi-infinite D3-branes. Then two NS5-branes can move relatively along \((x^7, x^8, x^9)\)-directions. The distance of the NS5-branes along these directions relates to the FI-parameters of the gauge theory. Turning on the FI-parameters, the vortex is admitted in this system. The \(k\) vortices are described by \(k\) D1-branes stretching between the NS5-brane and the connected D3-branes. (See Fig. 1.)

Hanany and Tong have considered that the moduli space of the vortex is realized
by the vacuum equations of the effective theory on the D1-branes. The effective theory on the $k$ D1-branes is a $U(k)$ gauge theory with $N_c$ chiral superfields in the fundamental $k$ representation and $\tilde{N}_c \equiv N_f - N_c$ chiral superfields in the anti-fundamental $\tilde{k}$ representation. From the brane configuration, we can see that the role is exchanged between the gauge coupling and the FI-parameter, when we consider the effective $U(k)$ gauge theory on the vortex instead of the $U(N_c)$ gauge theory, and vice versa. Therefore the FI-parameter $r$ of the effective $U(k)$ theory is proportional to $1/g^2$. Strictly speaking, the effective theory on the D1-branes is a 1 + 1-dimensional gauge theory. However we reduce the effective theory to a 0-dimensional theory (matrix model), since we are here interested in (vacuum) solutions of the effective theory that are independent of the world-volume coordinates on the D1-branes.

Considering the vacuum equation of the reduced effective theory on the D1-branes, the HT moduli space is defined by the Kähler quotient of the following moment map:

$$\mu_{\text{HT}} = [Z, Z^\dagger] + II^\dagger - J^\dagger J - r,$$

where $Z$, $I$ and $J$ are $k \times k$, $k \times N_c$, and $\tilde{N}_c \times k$ matrices, respectively. The quotient space is

$$\mathcal{M}^\text{HT}_k = \frac{\mu_{\text{HT}}^{-1}(0)}{U(k)}.$$

Introducing fermionic partners of $Z$, $I$, $J$ as $\lambda, \psi_I, \psi_J$, we define the BRST transformations for these matrices

$$QZ = i\lambda, \quad Q\lambda = -[\Phi, Z],$$

$$QZ^\dagger = -i\lambda^\dagger, \quad Q\lambda^\dagger = [\Phi, Z^\dagger],$$

$$QI = i\psi_I, \quad Q\psi_I = \Phi I - IM,$$

$$QI^\dagger = -i\psi_I^\dagger, \quad Q\psi_I^\dagger = I^\dagger \Phi - MI^\dagger,$$

$$QJ = i\psi_J, \quad Q\psi_J = J\Phi - \tilde{M}J,$$

$$QJ^\dagger = -i\psi_J^\dagger, \quad Q\psi_J^\dagger = J^\dagger \Phi - J^\dagger \tilde{M},$$

$$Q\Phi = 0,$$

where $\Phi$ is now $k \times k$ matrix. Note that $\Phi$ differs from that in the previous sections. We assume for a while the masses for the hypermultiplets $I$ and $J$, $M = \text{diag}(m_1, m_2, \ldots, m_{N_c})$ and $\tilde{M} = \text{diag}(m_1, m_2, \ldots, m_{\tilde{N}_c})$, are not degenerate for simplicity of the calculations while so far we have treated the degenerate cases.

Following the same argument in the previous sections, the volume of the HT moduli space $V_{k, N_c, N_f}$ is given by the following matrix integral:

$$V_{k, N_c, N_f}(\vec{m}, \vec{\tilde{m}}, r) = \int D\Phi D^2 Z D^2 \lambda D^2 I D^2 J D^2 \psi_I D^2 \psi_J e^{-S},$$

where $S$ is a matrix model “action”, which gives the constraint $\mu_{\text{HT}} = 0$, defined by

$$S = \text{Tr} \left[ i\Phi([Z, Z^\dagger] + II^\dagger - J^\dagger J - r) - iIMI^\dagger + iJ^\dagger \tilde{M}J + \lambda\lambda^\dagger - \psi_I^\dagger \psi_I - \psi_J^\dagger \psi_J \right].$$

We can see the action is BRST closed, $QS = 0$. 

The solution $\mu_{HT}$ of the moment map (6.1) contains the flat directions along the commuting elements of $Z$. Therefore the HT moduli space is non-compact and its volume $V_{N_c,N_f}^{N_c,N_f}$ diverges in general. To regularize the volume of the HT moduli space, we introduce the so-called $\Omega$-background.\(^5\),\(^28\) The $\Omega$-background deforms the BRST transformations as

$$
\begin{align*}
Q_e Z &= i\lambda, \\
Q_e Z^\dagger &= -i\lambda^\dagger, \\
Q_e I &= i\psi_I, \\
Q_e I^\dagger &= -i\psi_J^\dagger, \\
Q_e J &= i\psi_J, \\
Q_e J^\dagger &= -i\psi_J^\dagger, \\
Q_e \Phi &= 0,
\end{align*}
$$

where $\epsilon$ is a parameter of the $\Omega$-background. Note that the hypermultiplet masses $M$ and $\tilde{M}$ are independent of $\epsilon$ in comparison with the instanton counting (ADHM equations), where there exist extra hyper-Kähler constraints.

The action is also deformed to

$$
S_\epsilon = \text{Tr} \left[ i\Phi([Z, Z^\dagger] + II^\dagger - J^\dagger J - r) + i\epsilon ZZ^\dagger - iIMI^\dagger + iJ^\dagger(\tilde{M} - \epsilon)J \\
+ \lambda\lambda^\dagger - \psi_I\psi_I^\dagger - \psi_J^\dagger\psi_J \right].
$$

The parameter $\epsilon$ appears as the mass term for $Z$ which lifts the flat directions of the constraint $\mu_{HT} = 0$.

The regularized volume of the HT moduli space is

$$
V_{N_c,N_f}^{N_c,N_f}(\vec{m}, \vec{m}, r; \epsilon) = \int D\Phi D^2 Z D^2 \lambda D^2 I D^2 J D^2 \psi_I D^2 \psi_J e^{-S_\epsilon}.
$$

After integrating all matrices except for $\Phi$, we obtain

$$
V_{N_c,N_f}^{N_c,N_f}(\vec{m}, \vec{m}, r; \epsilon) = \int D\Phi \frac{1}{P(\Phi)Q(\Phi)} \frac{1}{\text{det}(-i[\Phi, \cdot] + i\epsilon)} e^{ir\text{Tr}\Phi},
$$

where $P(\Phi) = \text{det}(i\Phi \otimes 1 - i1 \otimes M)$ and $Q(\Phi) = \text{det}(i1 \otimes \tilde{M} - i\Phi \otimes 1 - i\epsilon 1 \otimes 1)$. Then, diagonalizing $\Phi = \text{diag}(\phi_1, \phi_2, \ldots, \phi_k)$, we obtain

$$
V_{N_c,N_f}^{N_c,N_f}(\vec{m}, \vec{m}, r; \epsilon) = \frac{1}{ik^{N_f}e^k} \int \prod_{a=1}^k \frac{d\phi_a}{2\pi i} \prod_{a=1}^k \prod_{i=1}^{N_c}(\phi_a - m_i) \prod_{a=1}^k \prod_{i=1}^{N_c}(\vec{m}_i - \phi_a - \epsilon)
$$

$$
\times \prod_{a<b} \frac{(\phi_a - \phi_b)^2}{(\phi_a - \phi_b)^2 - \epsilon^2} e^{ir\sum_{a=1}^k \phi_a}.
$$

This integral is essentially the residue integral and localized at the fixed points of the BRST transformations. The fixed points is classified by $N_f$ partitions of $k$, and the eigenvalues of $\Phi$ at the fixed points are given by
\[
\Phi_0 = \left( \begin{array}{c}
    m_1 1_{k_1} + \mathcal{J}^{(k_1)} \\
    \vdots \\
    m_{N_c} 1_{k_{N_c}} + \mathcal{J}^{(k_{N_c})} \\
    \tilde{m}_1 1_{\tilde{k}_1} + \mathcal{J}^{(\tilde{k}_1)} \\
    \vdots \\
    \tilde{m}_{\tilde{N}_c} 1_{\tilde{k}_{\tilde{N}_c}} + \mathcal{J}^{(\tilde{k}_{\tilde{N}_c})}
\end{array} \right),
\]

(6.11)

where \( \sum_{l=1}^{N_c} k_l + \sum_{l'=1}^{\tilde{N}_c} \tilde{k}_{l'} = k \) and \( \mathcal{J}^{(k)} \) is a \( k \times k \) diagonal matrix which has the form

\[
\mathcal{J}^{(k)} = \left( \begin{array}{cccc}
    0 & \epsilon & & \\
    & 2\epsilon & & \\
    & & \ddots & \\
    & & & (k-1)\epsilon
\end{array} \right).
\]

(6.12)

Using this fixed point data, we can evaluate the integral (6.10). To evaluate the integral, it is useful to introduce the "character" \( \chi_{N_c, N_f}^{N_c, N_f}(t) = (t - 1) V \times V^* + W \times \tilde{W}^* + t V \times \tilde{W}^* \),

(6.13)

where \( t = e^{-\beta \epsilon}, V = \text{Tr} e^{-\beta \Phi_0}, V^* = \text{Tr} e^{\beta \Phi_0}, W = \text{Tr} e^{-\beta M} \) and \( \tilde{W}^* = \text{Tr} e^{\beta \tilde{M}} \). This character is defined from the integrand (measure) of (6.10) by a K-theoretic extension. The term \( (t - 1) V \times V^* \) corresponds to the Vandermonde difference products in (6.10) and the terms \( W \times V^* \) and \( V \times \tilde{W}^* \) come from the poles after integrating the hypermultiplets. The volume is obtained in the following procedure. Once we obtain the character in polynomial of \( t \)

\[
\chi_{N_c, N_f}^{N_c, N_f}(t) = \sum_{i=1}^{N} a_i(m, \tilde{m}; \beta) t^{n_i},
\]

(6.14)

the volume is obtained in the limit of the extra parameter \( \beta \)

\[
V_{N_c, N_f}^{N_c, N_f}(\tilde{m}, \tilde{m}, r; \epsilon) = \lim_{\beta \to 0} \beta^d \prod_{i=1}^{N} \frac{1}{1 - a_i(m, \tilde{m}; \beta) t^{n_i}} e^{i r \text{Tr} \Phi_0},
\]

(6.15)

where \( d \) is the complex dimension of the moduli space.

The character replaces the products in the integrand of the partition function (6.10) with the polynomials of \( t \). This evaluation of the character regularizes the residue integral and we can evaluate the partition function efficiently.
ANO vortex

Let us first consider the simplest ANO vortex case, namely \( k \) vortices of \( N_c = N_f = 1 \). In this case, there is only one mass parameter \( m \) and a single partition of \( k \). \( \text{Tr} \Phi_0 \) is rather simple

\[
\text{Tr} \Phi_0 = \sum_{l=1}^{k} (m + (l-1)\epsilon) = km + \epsilon \frac{k(k-1)}{2},
\]

(6.16)

and \( V \) and \( V^* \) are given by

\[
V = e^{-\beta m} \frac{1-t^k}{1-t}, \quad V^* = e^{\beta m} \frac{1-t^{-k}}{1-t^{-1}}.
\]

(6.17)

If we evaluate the character, the mass dependence disappears

\[
\chi_{1,1}^k(t) = (t-1)e^{-\beta m} \frac{1-t^k}{1-t} \times e^{\beta m} \frac{1-t^{-k}}{1-t^{-1}} + e^{-\beta m} \times e^{\beta m} \frac{1-t^{-k}}{1-t^{-1}}
\]

\[
=t \frac{1-t^k}{1-t} = t + t^2 + t^3 + \cdots + t^k.
\]

(6.18)

From this character polynomial in \( t \), we find the partition function (regularized volume)

\[
V_{1,1}^k(m, r; \epsilon) = \lim_{\beta \to 0} \frac{\beta^k}{t^k} \prod_{i=1}^{k} \frac{1}{1-t} e^{i\epsilon (mk+\epsilon k(k-1)/2)}
\]

\[
= \frac{1}{(i\epsilon)^k k!} e^{i\epsilon (km+\epsilon k(k-1)/2)}.
\]

(6.19)

To compare with our previous result, we must set \( m = 0 \). Then we obtain

\[
V_{1,1}^k(r; \epsilon) = \frac{1}{(i\epsilon)^k k!} e^{i\epsilon k(k-1)/2}.
\]

(6.20)

This volume diverges in the \( \epsilon \to 0 \) limit as expected. The divergence should come from the position moduli of the vortices since the commuting parts of the matrix \( Z \) of the HT moment map (6.1) correspond to the positions. In the \( \epsilon \to 0 \) limit, the volume behaves as

\[
V_{N_c,N_f}^k(r; \epsilon) \sim \frac{1}{(i\epsilon)^k k!}.
\]

(6.21)

On the other hand, our previous result (3.55) also behaves as

\[
V_{N_c,N_f}^k \sim \frac{(2\pi \tilde{A})^k}{k!}.
\]

(6.22)

In this observation, we notice that the asymptotic behavior agrees with each other by identifying \( \tilde{A} \sim \frac{1}{i\epsilon} \). Of course, the relation between the area of the Riemann surface \( A \) and the \( \Omega \)-background parameter \( \epsilon \) is unclear, but both parameters regularize the divergences coming from the non-normalizable modes of the moduli. Therefore
we naively expect that the asymptotic (divergent) behaviors of the volume of the HT moduli space and the moduli space of the vortices on the compact Riemann surface coincide with each other, although the whole structures of the both moduli spaces may differ. To understand precise relation between two moduli spaces, we need further investigation of the details of the moduli space, but we will check some examples to compare with the divergent behavior of the volume of the HT moduli space in the following:

Local vortex

Next we consider the case of \( N_c = N_f = N > 1 \), where the solution is called the local vortex. In this case, the matrix \( J \) disappears since \( \tilde{N}_c = 0 \). Then the HT moment map (6.1) becomes simpler

\[
\mu_{HT} = [Z, Z^\dagger] + II^\dagger - r. \tag{6.23}
\]

The poles in (6.10) coming from the contribution of \( J \) also disappears and the localization fixed points are classified by an \( N \) partition of \( k \), namely \( \vec{k} = (k_1, k_2, \ldots, k_N) \). Using the fixed point data, we find

\[
\begin{align*}
\text{Tr } \Phi_0 &= \sum_{l=1}^{N} \left[ k_lm_l + \epsilon \frac{k_l(k_l - 1)}{2} \right], \\
V &= \sum_{l=1}^{N} e^{-\beta m_l} \frac{1 - t^{k_l}}{1 - t}, \\
V^* &= \sum_{l'=1}^{N} e^{\beta m_{l'}} \frac{1 - t^{-k_{l'}}}{1 - t^{-1}}.
\end{align*} \tag{6.24-6.25}
\]

Using those, the character is given by

\[
\chi_k^{N,N}(t) = \sum_{l,l'=1}^{N} e^{-\beta(m_l-m_{l'})} \left( (t-1)^{1-t^{k_l}} \frac{1 - t^{-k_{l'}}}{1 - t^{-1}} \right) \nonumber
\]

\[
= \sum_{l,l'=1}^{N} e^{-\beta(m_l-m_{l'})} \sum_{j=1}^{k_{l'}} t^{k_l-j+1}. \tag{6.26}
\]

The volume of the HT moduli space becomes

\[
V_k^{N,N}(\vec{m}, r; \epsilon) = \sum_{\sum_{l=1}^{N} k_l=k} \frac{1}{\prod_{l,l'=1}^{N} k_{l'}^{m_{l'}} \prod_{j=1}^{N} (m_l - m_{l'} + \epsilon(k_l - j + 1))} \times e^{ir} \frac{1}{\sum_{l=1}^{N} (k_l m_l + \epsilon k_l(k_l - 1)/2)}. \tag{6.27}
\]

To compare with our previous results, we have to take the limit \( m_l \to 0 \), but it is difficult to evaluate this limit for the non-Abelian local vortex in general. However, for the case of \( k = 1 \), we can perform the integral (6.10) directly. Using the residue integral formula (3.48),

\[
V_1^{N,N}(0, r; \epsilon) = \frac{1}{i\epsilon} \int_{-\infty}^{\infty} d\phi \frac{1}{(i\phi)^N} e^{ir\phi} = \frac{1}{i\epsilon (N-1)!} r^{N-1}. \tag{6.28}
\]
For general $k$, it is difficult to calculate the integral for fixed values of $\epsilon r$ in the limit of degenerate masses $m_l \to 0$. However if we instead fix $m_l$ and take the leading term of (6.27) at the $\epsilon \to 0$ limit which corresponds to the large area limit, we find the volume $V_k^{N,N}$ is proportional to $1/(i\epsilon)^k$

$$
\lim_{\epsilon \to 0} V_k^{N,N}(\vec{m}, r; \epsilon) = \frac{1}{i k N (i\epsilon)^k} \sum_{\sum_{l=1}^{N} k_l = k} \left( \prod_{l=1}^{N} \frac{1}{k_l!} \right) \prod_{l \neq \ell} \frac{1}{(m_l - m_\ell)^k} e^{i r \sum_l k_l m_l}.
$$

If we next take the degenerate mass limit $m_l \to 0$, the coefficient of $1/(i\epsilon)^k$ in (6.29) should be proportional to $r^N$ from dimensional reasons. The divergent behavior (divergent power) looks like the asymptotic volume of the non-Abelian local vortex on the sphere (4.52) in the large area limit, but the coefficients depending on $N$ and $k_l$ does not agree with the asymptotic behavior neither on the sphere (4.52) nor on the torus (4.35) in §4. Since our treatment of Hanany-Tong moduli space does not carry information of topology of base space, it is not clear at present how to relate the results of Hanany-Tong moduli space and those in previous sections.

**Semi-local vortex**

The semi-local vortex case ($N_f > N_c$) is more complicated. At least, we can see from the integral (6.10) for the $k = 1$ case

$$
\hat{V}_1^{N_c,N_f}(\vec{0}, \vec{0}, r; \epsilon) = \frac{1}{i \epsilon} \int_{-\infty}^{\infty} \frac{d\phi}{2\pi} \frac{1}{(i\phi)^{N_c}(-i\phi - i\epsilon)^{N_c}} e^{i r \phi},
$$

where we have set $m_i = \vec{m}_i = 0$. Using the residue formula, we find that the volume of the HT moduli space behaves in general

$$
\hat{V}_1^{N_c,N_f}(\vec{0}, \vec{0}, r; \epsilon) = \frac{1}{(-1)^{N_c} (i\epsilon)^{N_c}} \left[ \sum_{l=1}^{N_c} \frac{(-1)^{N_c-l} (N_f - l - 1)!}{(N_c - 1)! (l - 1)! (N_c - l)!} \frac{r^{l-1}}{(i\epsilon)^{N_c-l}} + \sum_{l=1}^{\tilde{N}_c} \frac{(-1)^{\tilde{N}_c-l} (N_f - l - 1)!}{(N_c - 1)! (l - 1)! (N_c - l)!} \frac{r^{l-1}}{(i\epsilon)^{N_f-l} e^{-i\epsilon r}} \right].
$$

In the limit of $\epsilon \to 0$, the divergent power of $1/(i\epsilon)$ is $N_f$, which agrees with the dimension of the moduli space of the vortices including the non-normalizable modes.

For general $k$, we expect that the volume of the HT moduli space behaves

$$
\hat{V}_k^{N_c,N_f}(\vec{0}, \vec{0}, r; \epsilon) \sim \frac{1}{(i\epsilon)^{k N_f}}
$$
in the $\epsilon \to 0$ limit, but it is difficult to prove it. We leave in the future deeper investigation of the volume of the HT moduli space to understand the relation to our results.

**§7. Conclusion and discussion**

In this paper, we have investigated the volume of the moduli space of the Abelian/non-Abelian vortex on the compact Riemann surface by using the path integral in topological field theory. We can obtain the properties of the volume of
the moduli space without any knowledge on the metric of the moduli space. This advantage of our field theoretical method makes easy to evaluate the volume for any number of the colors or flavors, and for any genus of the Riemann surface. The volume obtained by our method gives us much information on the moduli space, for example, dimension of the normalizable or non-normalizable modes, criterion of existence of the vortices, behavior near the Bradlow limit, and so on.

The metric on the moduli space can be defined from the effective Lagrangian on vortices. We have obtained the effective Lagrangian on the Riemann surface of sphere topology in the strong coupling limit $g^2 \to \infty$, or for a restricted class of vortex number, $k = 0, 1$ at finite couplings. By integrating the metric, we obtain the volume which agrees with our results of the topological field theory. Although more detailed information than the volume alone can be obtained from the effective Lagrangian, it is generally difficult to evaluate it explicitly. For instance, we have not understood the difference of the effective Lagrangian due to the topology of the Riemann surface, such as torus, or higher genus. We stress that our method of topological field theory is much more versatile than the method of the effective Lagrangian, and still can give valuable information such as the volume of the moduli space. Even without the knowledge of the metric, the volume from the field theoretical method may lead to a novel knowledge on the moduli space, which cannot be captured from the effective Lagrangian.

The topological filed theory, which we have utilized for the evaluation of the volume, might be embedded in a supersymmetric gauge theory. Indeed, the BRST algebra in §§3 and 4 could be constructed from the supercharges in two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theory by topological twisting. The cohomological operators correspond to the supersymmetric (SUSY invariant) operators, and the “action” is the essential part of the action of the supersymmetric gauge theory, in the sense of the localization. Thus the volume evaluated in the path integral can be understood as the vacuum expectation value (vev) of the supersymmetric operator. This strongly suggests that our exact evaluation of the volume gives an exact evaluation of the vev including non-perturbative corrections in the supersymmetric gauge theory. Therefore we can expect that the volume has another side view as the non-perturbative corrections in the supersymmetric gauge theory.

We have concentrated in the case of vortex to evaluate the volume of the moduli space. Our method of topological field theory should be extendable to various problems in solitons and (supersymmetric) gauge theories. Our localization method can be applicable to other types of solitons like domain-walls and monopoles, which have codimension one and three, respectively. Once we have the BPS equations for these solitons, we should be able to construct an appropriate “action” and BRST symmetry from the constraints. The localization of the path integral will give the volume. We can obtain the thermodynamical properties of these solitons from the volume. These BPS solitons also play a very important role in supersymmetric gauge theories. We expect that the evaluation of the volume shed light on the non-perturbative aspects of the supersymmetric gauge theories like dualities and confinement.
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Appendix A

Volume of Grassmannian

The complex Grassmann manifold (Grassmannian) $G_{N_c,N_f}$ is defined by a homogeneous coset space of the unitary groups

\[ G_{N_c,N_f} = \frac{U(N_f)}{U(N_c) 	imes U(N_f - N_c)}, \]  
\[ (A.1) \]

The volume of $G_{N_c,N_f}$ is expressed by the volume of each unitary groups

\[ \text{Vol}(G_{N_c,N_f}) = \frac{\text{Vol}(U(N_f))}{\text{Vol}(U(N_c)) \times \text{Vol}(U(N_f - N_c))}. \]  
\[ (A.2) \]

The volume of $U(N)$ group is given by

\[ \text{Vol}(U(N)) = \frac{(2\pi)^{N(N+1)/2}}{G(N+1)}, \]  
\[ (A.3) \]

where $G(z)$ is the Barnes G-function which satisfies

\[ G(z + 1) = \Gamma(z)G(z), \]  
\[ (A.4) \]

that is

\[ G(N + 1) = \prod_{i=1}^{N} \Gamma(i) = \prod_{i=1}^{N} (i - 1)! \quad \text{for} \ N \in \mathbb{Z}_{>0}. \]  
\[ (A.5) \]

Using this formula, we find the volume of the Grassmannian $G_{N_c,N_f}$

\[ \text{Vol}(G_{N_c,N_f}) = (2\pi)^{N_c(N_f-N_c)} \prod_{i=1}^{N_c} (i - 1)! \times \prod_{i=1}^{N_f-N_c} (i - 1)! \] 
\[ = (2\pi)^{N_c(N_f-N_c)} \prod_{i=1}^{N_f-N_c} \frac{(i - 1)!}{(N_f - i)!}. \]  
\[ (A.6) \]

For example, the volume of complex projective space $\mathbb{C}P^{N_f-1} \simeq G_{1,N_f}$ is given by

\[ \text{Vol}(\mathbb{C}P^{N_f-1}) = \frac{(2\pi)^{N_f-1}}{(N_f-1)!} \]  
\[ (A.7) \]

in this normalization.
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