Expanding Lorentzian Wormholes in $R^2$ Gravity

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We present traversable, Lorentzian wormholes in the framework of $R^2$ gravity. These wormholes are obtained by assuming constant Ricci scalar and trace-less equation of state. The metric is asymptotically RW, and locally that of a Lorentzian wormhole. Dynamics of the cosmological background, as well as the expanding wormhole are discussed. Weak energy condition near the wormhole throat is also examined. It is shown that the positive energy density condition can be satisfied all-over a spacelike hypersurface at large enough cosmological times and small enough, positive cosmological constant.

§1. Introduction

The theory of wormholes goes back to 1916, shortly after Einstein published his general theory, when Ludwig Flamm looked at the simplest possible solution of Einstein’s field equation (i.e. the Schwarzschild solution). This solution describes the gravitational field around a spherically-symmetric non-rotating mass. If the mass is sufficiently compact, the solution describes a black hole. Flamm realized that Einstein’s equations allowed a second solution, now known as a white hole, and that the two solutions, describing two different regions of spacetime were connected by a tunnel.\(^1\) Because the theory has nothing to say about where these regions of spacetime might be in the real world, the black hole “entrance” and white hole “exit” could be in different parts of the same universe or in different universes.

In 1935, Einstein and Rosen further explored the possibility of intra- or inter-universe connections in a paper whose actual purpose was to explain fundamental particles, such as electrons, in terms of spacetime tunnels threaded by electric lines of force.\(^2\) Their work gave rise to the formal name Einstein-Rosen bridge. Later, John Wheeler called such a geometrical structure “wormhole”.\(^3\) Wheeler, in his 1955 paper, presented the first diagram of a wormhole as a tunnel connecting two openings in different regions of spacetime. It was proposed that such a hypothetical “tunnel” connecting two different points in spacetime would facilitate a trip between two regions of space which could take much less time than a journey between the same starting and ending points in normal space.

The study of traversable wormholes gained a new motive when Morris and Thorne\(^4\) presented a systematic analysis of the so-called traversable wormholes, which allow a safe, two-way journey across the throat. Such static wormholes, however, suffered from a major drawback: when put in the Einstein equations, they

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needed an exotic kind of matter which violated energy conditions. Modern works on traversable wormholes, concentrate on dynamic ones which provide the possibility of being supported by non-exotic matter (see e.g. 5)–7)).

\( f(R) \) gravity is a type of modified gravity theory proposed as an alternative to Einstein’s general relativity (GR). In \( f(R) \) gravity the gravitational Lagrangian is assumed to be a function of the curvature scalar \( R \). The gravitational field equations are obtained by varying the total action for both the field and matter with respect to the metric tensor and equating this variation to zero. This modification has the potential—in principle—to explain the accelerated expansion of the Universe without adding unknown forms of dark energy.\(^8\)

In the \( f(R) \) gravity models, spacetime reacts differently to the matter in the universe than it does in GR. In GR, gravity is a manifestation of the curvature of space and time, where the source of this curvature is all forms of mass and energy. In the absence of any mass or energy spacetime can become completely flat. What \( f(R) \) gravity does is to allow spacetime to act as a source of its own curvature, so that there can still be some curvature even if spacetime is completely free of mass and energy. Therefore, as the universe expands and the matter density falls, some curvature may remain, capable of driving cosmic acceleration.

Although there are various definitions for a dynamical wormhole (see e.g. 9)–11)), there is still some controversy about this in the literature. In the present paper, by a dynamical wormhole, we mean a constant time, minimum area two-dimensional hypersurface, connecting two asymptotically Robertson-Walker spacetimes.\(^14\) We are interested in obtaining dynamical Lorentzian wormholes in the framework of \( \text{R}^2 \) gravity. To achieve this end, we will first derive expanding wormhole solutions in the framework of \( \text{R}^2 \) gravity by starting with an appropriate metric and imposing the constant Ricci scalar constraint. The structure of the paper is as follows. In the next section, we will prepare the mathematical scene and will present the basic equations. In \( \S 3 \) we will derive the expanding wormhole solutions and categorize them. Section 4 is devoted to the status of energy conditions, and the last section contains summary and conclusions.

\( \S 2 \). Basic equations of \( f(R) \) gravity

The action for \( f(R) \) gravity reads\(^12\)

\[
\mathcal{A} = \int d^4x \sqrt{-g} \left( \frac{f(R)}{8\pi G} + \mathcal{L}_m \right),
\]

in which \( f(R) \) is a function of the Ricci scalar \( R \), and \( \mathcal{L}_m \) is the matter lagrangian density. \( f(R) \) can be expanded as

\[
f(R) = \ldots + \frac{\alpha_2}{R^2} + \frac{\alpha_1}{R} - \Lambda + R + \beta_2 R^2 + \beta_3 R^3 + \ldots,
\]

where \( \alpha_i \) and \( \beta_i \) are constants, and \( \alpha_i = \beta_i = 0 \) corresponds to the Einstein-Hilbert action. The special case \( \beta_2 \neq 0, \beta_i = 0 \ (i \neq 2) \), and \( \alpha_i = 0 \) is known as \( \text{R}^2 \) gravity and so-far it is the most extensively studied gravity beyond the Einstein gravity.
In Eq. (2.2) $A$ represents the cosmological constant. The Euler-lagrange equation corresponding to the action (2.1) reads

$$\mathcal{G}_{\mu\nu} \equiv f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box]f'(R) = 8\pi GT_{\mu\nu}$$

(2.3)

with the trace

$$f'(R)R - 2f(R) + 3\Box f'(R) = 8\pi GT,$$

(2.4)

where $T \equiv T_{\mu\mu}$. $\mathcal{G}_{\mu\nu}$ is the generalized Einstein tensor defined via (2.3), and $f'(R) \equiv \frac{df(R)}{dR}$. In order to investigate wormholes in an expanding cosmological background, we use the ansatz metric:

$$ds^2 = -dt^2 + R^2(t) \left[(1 + a(r))dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

(2.5)

in which $R(t)$ (the cosmological scale factor, not to be confused with the Ricci scalar) and $a(r)$ are unknown functions (to be determined). The above metric, when used in the gravitational field equations, requires an anisotropic fluid as its source. This fluid has the energy density $\rho$, radial pressure $p_r$, and transverse pressure $p_t$. We will show that our assumption of constant Ricci scalar leads to a traceless equation of state.

We do our calculations in the framework of the $R^2$ gravity with cosmological constant:

$$f(R) = R + \beta R^2 - \Lambda,$$

(2.6)

where $\beta(= \beta_2)$ is a constant. We concentrate on this choice of $f(R)$ (the so-called Starobinski model$^{16}$ in the cosmological context), since this is the simplest among $f(R)$ gravities, yet under various interesting investigations in recent years (see e.g. $^{17}$–$^{19}$).

Components of the generalized Einstein tensor for our ansatz metric read:

$$G_t^t = -8\pi G\rho = 3(1 + 2\beta k) \frac{\dot{R}(t)}{R(t)} - \frac{1}{2}(A + k + \beta k^2),$$

$$G_r^r = 8\pi GP_r = (1 + 2\beta k) \frac{\dot{R}(t)}{R(t)} - \frac{1}{2}(A + k + \beta k^2) + 2(1 + 2\beta k) \frac{\dot{R}(t)^2}{R(t)^2}$$

$$+ \frac{(1 + 2\beta k)}{R(t)^2(1 + a(r))^2 r} \frac{a'(r)}{a(r)},$$

$$G_\theta^\theta = G_\phi^\phi = 8\pi GP_\theta = (1 + 2\beta k) \frac{\dot{R}(t)}{R(t)} - \frac{1}{2}(A + k + \beta k^2) + 2(1 + 2\beta k) \frac{\dot{R}(t)^2}{R(t)^2}$$

$$+ \frac{(1 + 2\beta k)}{2R(t)^2(1 + a(r))^2 r} a'(r) + \frac{(1 + 2\beta k)}{R(t)^2r^2(1 + a(r))} a(r).$$

(2.7)

As mentioned earlier, in what follows, Ricci scalar is assumed to be constant ($= k$).
§3. Wormhole solutions and their basic properties

Evolving wormholes have been studied under various assumptions by many authors (see e.g. 15, 20–22) In this section, we obtain wormhole solutions in $\mathbb{R}^2$ gravity by separating $r$-dependent and $t$ dependent variables in the constraint $R = k$, where $R$ is the Ricci scalar. This constraint leads to

$$6\dot{R}^2 + 6\ddot{R} - kR^2 = -2\frac{a'^2 + a^2 + a}{r^2(1 + a)^2} \equiv -2C,$$

(3.1)

where prime and dot denote differentiation with respect to $r$ and $t$, respectively, and $C$ is the separation constant. These lead to two differential equations for the scale factor $R(t)$ and the radial function $a(r)$, as follows:

$$R(t) = \sqrt{-\frac{2C}{k} + \sqrt{3} (C_1 \exp(-\alpha t) - C_2 \exp(\alpha t))} = \sqrt{-\frac{2C}{k} + \sqrt{3} (A \cosh(\alpha t) - B \sinh(\alpha t))},$$

where $\alpha = \sqrt{\frac{k}{3}}$, and $C$ is the constant of separation. In the second equation, $A = C_1 - C_2$ and $B = C_1 + C_2$. We also obtain

$$1 + a(r) = \frac{3}{3 + Cr^2 + 3C_3r^3},$$

(3.2)

where $C_3$ is a constant of integration. It is seen that the scale factor $R(t)$ approaches an exponential behavior at large $t (\gg \alpha^{-1})$, provided $C_2 < 0$. The model does not contain a big-bang singularity ($R = 0$), if

$$\Delta \equiv \left(\frac{2C}{k}\right)^2 + 12C_1C_2 < 0.$$

In order to have traversable, Lorentzian wormholes, the following conditions should be fulfilled:4)

A. $1 + a(r)$ should diverge at a finite $r_0$ (the throat radius), where $r_0$ is a root of $1/(1 + a(r)) = 0$.

B. No event horizon should exist for $r \geq r_0$ (traversability condition),

C. Flare-out condition:

$$\frac{d^2r}{dz^2} \geq 0 \text{ at } r = r_0,$$

(3.3)

where $z$ is an embedding coordinate. Condition A is fulfilled for appropriate ranges of the integration and separation constants $C$ and $C_3$. The metric function $1 + a(r)$
diverges at finite \( r \) for suitable choices of integration constants \( C \) and \( C_3 \). The metric signature remains Lorentzian down to \( r_0 \), where \( r_0 \) is the root of \( 1/(1 + a(r)) = 0 \). Below, \( r_0 \), the signature becomes improper. It can be shown that a suitable coordinate transformation can map the range \((r_0, \infty)\) onto \((0, \infty)\) and the new coordinate can be extended to \((-\infty, 0)\). The surface of a sphere at \( r = r_0 \) remains finite and this is why we recognize the solution as a wormhole. Condition B is guaranteed by the fact that our ansatz metric has vanishing redshift function and the only redshift present is the cosmological one.\(^{23}\) The flare-out condition is also satisfied for \( \frac{d^2 r}{dz^2} = -\frac{a'(r)}{2a(r)} \geq 0 \), so long as \( a(r) \) is a decreasing function of \( r \):

\[
a'(r) = -\frac{3(2C r^3 - 3C_3)}{r^2(3 + Cr^2 + 3C_3 r^{-2})^2} < 0, \tag{3.4}
\]

or

\[
r_0 > \left( \frac{3C_3}{2C} \right)^{1/3}, \tag{3.5}
\]

for \( C > 0 \). The wormhole metric (2.5) is conformally static and the wormhole structure can be easily seen by considering a constant \( t \) slice of the metric:

\[
ds^2 = R^2(t_0) \left[ (1 + a(r))dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right].
\]

The proper surface area of an \( r = r_0 = \text{constant} \) spherical sub-space is given by

\[
A = 4\pi R^2(t_0) r_0^2.
\]

The wormhole throat is defined where \( A \) is minimum and this happens at the minimum possible value of \( r \). Therefore, wormhole solutions exist if \( a(r) \) diverges at finite \( r = r_0 \) and changes sign for \( r < r_0 \). For this kind of metric, it is shown in 6) that there is no event horizon and the wormhole is traversable.

\section{Weak energy condition}

Static wormholes are known to violate energy conditions within the framework of general relativity.\(^5\) Going beyond GR and/or considering dynamical wormholes may change this situation. For example, Maeda et al. have shown that cosmological wormholes can satisfy the null energy condition.\(^{22}\) Here, we investigate the positive energy density condition for the cosmological wormholes derived in the previous sections.

The Ricci = constant condition is consistent with a traceless equation of state in the form

\[-\rho + P_r + 2P_t = 0, \tag{4.1}\]

provided that (see Eq. (2.4)):

\[k + 2\Lambda = 0. \tag{4.2}\]

The weak energy condition (WEC) requires \( T_{\mu\nu}V^\mu V^\nu \geq 0 \) for every non-spacelike \( V^\mu \) which leads to\(^5\)

\[
\rho \geq 0, \quad \rho + P_r \geq 0, \quad \rho + P_t \geq 0.
\]
Fig. 1. The quantity $\rho + p_r$ as a function of $r$ for a constant time hypersurface, for typical values of the parameters and constants of integration ($C_1 = 1$, $C_2 = -1$, $\Lambda = 0.05$). It can be seen that this quantity satisfies WEC everywhere on the hypersurface. The wormhole throat is located at $r_0 = 4.88$.

Fig. 2. The quantity $\rho + p_t$ as a function of $r$ for a constant time hypersurface, for typical values of the parameters and constants of integration ($C_1 = 1$, $C_2 = -1$, $\Lambda = 0.05$). It can be seen that this quantity is positive only beyond a certain radius $r_1$. The wormhole throat is located at $r_0 = 4.88$.

The positive energy-density condition in the constant Ricci case translates into

$$\frac{3ka^2(3\delta C_1^2 e^{-2\alpha t} + 9C_1 C_2 k - 2\sqrt{3}C_1 e^{-\alpha t} + \frac{3k}{2}C_2^2 e^{2\alpha t} - 2\sqrt{3}C_2 e^{\alpha t})}{(\sqrt{3}k - 2)(C_1 e^{-\alpha t} + C_2 e^{\alpha t})^2} \geq k. \quad (4.3)$$

We have kept $k$ on both sides of the inequality, since the sign of $k$ (or $\Lambda$) affects the direction of ultimate inequality. The $\rho + P_r \geq 0$ and $\rho + P_t \geq 0$ conditions, however, become lengthy expressions and it is not illuminating to reproduce them here.
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For large $t$ ($t \gg 1/\alpha$), Eq. (4.3) reduces to

$$\frac{9a^2k^2}{\sqrt{3k^2 - 2}} \geq 2k.$$  \hspace{1cm} (4.4)

Since $k = -2A$ and $A$ is very small, we obtain from (4.4)

$$-\frac{9}{2}k^2a^2 \geq 2k,$$  \hspace{1cm} (4.5)

which is obviously violated for $k > 0$ ($A < 0$). For $k < 0$ ($A > 0$), however, this condition can be satisfied for

$$0 < A \leq \frac{2}{9a^2}.$$  \hspace{1cm} (4.6)

The full expressions for $\rho + p_r$ and $\rho + p_t$ are too lengthy to be reproduced here, but in order to see the typical behavior of these quantities, we have plotted them as a function of $r$ for a constant time hypersurface. As shown in Figs. 1 and 2, $\rho + p_r$ is positive for all values of $r$, while $\rho + p_t$ is positive only beyond a certain radius $r_1$. WEC is therefore fully satisfied, provided that $r_1$ is less than the throat radius $r_0$. Note that $r_0$ is a root of

$$3r + Cr^3 + 3C_3 = 0,$$  \hspace{1cm} (4.7)

which is the denominator of Eq. (3.2) times $r$. For typical choices of the parameters corresponding to Figs. 1 and 2, this equation has three roots at $-5.92$, $1.04$ and $4.88$. The wormhole throat is the last root, i.e. $r_0 = 4.88$.

§5. Conclusions

$R^2$ gravity is a natural generalization of GR and has been under extensive study in past decades. In particular, the cosmological implications of this gravity are known to be interesting.$^{16,19}$ In the present paper, we obtained dynamic wormhole solutions in a cosmological background in the framework of $R^2$ gravity for constant Ricci scalar or traceless equation of state. To achieve this end, we used a cosmological metric, in which the homogeneity assumption is broken in favor of a central object which were shown to be a wormhole. Traversability and flare out conditions were checked for the solutions. The weak energy condition was written for the wormhole solution and a simple relation for the positive energy density condition was obtained for large $t$. Since static wormholes are known to violate energy conditions, one of the main reasons for studying dynamical wormholes is the effect of time-dependence of the metric on the status of energy conditions. Moreover, in modified theories of gravity, it is possible, in principle, that additional curvature terms play the role of exotic matter. In this paper, we learned that within the framework of $R^2$ gravity with cosmological constant, dynamical wormholes in an expanding cosmological background can respect the positive energy density condition all-over a spacelike hypersurface for late enough cosmological times and small enough (positive) cosmological constant. By plotting $\rho + p_r$ and $\rho + p_t$ versus $r$ for a constant time hypersurface, we found that the former is positive everywhere, while the latter is positive only beyond some
radius $r_1$. The WEC is therefore satisfied everywhere, provided that $r_1 < r_0$, otherwise, it is violated inside a spherical shell of outer radius $r_1$ around the wormhole throat.

Although we discussed only the wormhole solutions here, there are other types of solutions, like naked singularities and radial deformations of the RW metric for other choices of integration constants, which are not discussed here.

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**References**

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