Note on a New Quaternionic Approach to Relativity and the Dirac Theory

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It is shown that there exists an extra imaginary unit commuting with Hamilton’s imaginary units of quaternions. The proof is based upon the fact that the Dirac representation of the Lorentz group can be obtained not only over the complex field but also over the field of quaternions.

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§1. Introduction

The purpose of this note is to provide a convincing argument that the quaternionic spinor analysis\(^1\) leads to the existence of an extra imaginary unit commuting with Hamilton’s imaginary units of quaternions so that the conventional quaternionic approach\(^2\) to relativity, which makes use of complex quaternions, is derived from our new quaternionic approach\(^1\) defined over the field of quaternions, \(\mathbb{H}\). The proof is based on the fact that the Dirac representation is the only fundamental representation of the quaternionic spinor group \(\text{Spin}(2, \mathbb{H}) \subset \text{SL}(2, \mathbb{H})\) but is defined as a direct sum of two fundamental representations (two inequivalent Weyl representations) of \(\text{SL}(2, \mathbb{C})\). It is explicitly shown that 2-component quaternionic Dirac spinor when transformed into 2-component complex quaternionic Dirac spinor via complexification is decomposed into 4-component Dirac spinor and its Dirac adjoint in the Weyl representation of Dirac matrices linearly.

Let us first summarize the quaternionic spinor analysis\(^1\) in which we associate an arbitrary space-time point \(x^\mu\) with an hermitian quaternionic matrix

\[
X = x^\mu \Gamma_\mu = x^0 \Gamma_0 + \mathbf{x} \cdot \mathbf{\Gamma} = \begin{pmatrix} -x^0 & -\mathbf{x} \cdot \mathbf{e} \\ \mathbf{x} \cdot \mathbf{e} & -x^0 \end{pmatrix},
\]

\[
(1.1)
\]

where \(\mathbf{e} = (i, j, k) \equiv (e_1, e_2, e_3)\) are Hamilton’s imaginary units of quaternions satisfying \(e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -e_0\) with \(e_0 \equiv 1\) being the unit element of quaternions. Greek indices take the values 0, 1, 2, 3 and Einstein summation convention is used throughout this note. Defining the matrices \(\tilde{\Gamma}_0 = \Gamma_0\) and \(\tilde{\mathbf{\Gamma}} = -\mathbf{\Gamma}\) it can be shown from the multiplication rules of quaternions that

\[
\Gamma_\mu \tilde{\Gamma}_\nu + \Gamma_\nu \tilde{\Gamma}_\mu = -2\eta_{\mu\nu}, \quad \tilde{\Gamma}_\mu \Gamma_\nu + \tilde{\Gamma}_\nu \Gamma_\mu = -2\eta_{\mu\nu},
\]

\[
(1.2)
\]

where \(\eta_{\mu\nu}\) is the Lorentz metric tensor with \(\eta = (\eta_{\mu\nu}) = \text{diag}(-1, +1, +1, +1)\). The restricted Lorentz group \(L^\dagger_+\) can be represented by the quaternionic spinor group
Spin(2, ℋ) as

\[ X \rightarrow X' = AXA^\dagger, \quad x'^\mu = \Lambda^\mu_\nu x^\nu, \quad \Lambda = (\Lambda^\mu_\nu) \in L^\dagger_+, \quad A \in \text{Spin}(2, ℋ), \]  

(1.3)

where

\[ \text{Spin}(2, ℋ) = \left\{ A = \left( \begin{array}{cc} P & -Q \\ Q & P \end{array} \right) \in \text{SL}(2, ℋ); |Q|^2 - |P|^2 = 1, \, \overline{TFQ} + \overline{QFP} = 0 \right\}, \]  

(1.4)

is a subgroup of SL(2, ℋ).\textsuperscript{1)} We denote the quaternionic conjugation of a quaternion \( Q = e_0 Q_0 + e \cdot Q \) by \( \overline{Q} = \bar{e}_\mu Q_\mu = e_0 Q_0 - e \cdot \bar{Q} \) so that \( \overline{QF} = \overline{FP} \), and its squared norm by \( |Q|^2 = \overline{QQ} = Q_0^2 + Q^2 \). Recall that \( \text{Spin}(2, ℋ) \) is defined through the relations

\[ A\sigma_1 A^\dagger = \sigma_1, \quad A\sigma_3 A^\dagger = \sigma_3, \]  

(1.5)

for general \( A \in \text{SL}(2, ℋ) \), where \( \sigma_{1,3} \) are the real Pauli matrices to be regarded as \( 2 \times 2 \) quaternionic matrices, \( \sigma_3 \) being assumed to play a role of ‘parity’ transformation of the quaternionic Dirac spinor belonging to the representation \( \Lambda \rightarrow A \), while pure imaginary matrix \( \sigma_2 \) is generalized to \( \Gamma \) in (1.1). Namely, any matrix \( A \in \text{Spin}(2, ℋ) \) satisfies (1.5), implying that \( A^{-1} = \overline{A} \). The representation \( \Lambda \rightarrow A \) is then equivalent to the one \( \Lambda \rightarrow A^\dagger = A^T \) (note that \( (A_1 A_2)^T = A_2^T A_1^T \)) but \( A^T = A^\dagger \) and \( \overline{A} \) are no longer the representations of \( L^\dagger_+ \) because, in general, \( (A_1 A_2)^\dagger = \overline{A}_1 \overline{A}_2 \neq \overline{A}_1 A_2^\dagger \), and \( \overline{A}_1 \overline{A}_2 \neq \overline{A}_2 \overline{A}_1 \neq \overline{A}_1 \overline{A}_2 \). Consequently, one obtains only one 2-dimensional spinor representation in the quaternionic spinor analysis, which is shown\textsuperscript{1)} to be the quaternionic Dirac representation.\textsuperscript{1)} If \( A_1 X A_1^\dagger = A_2 X A_2^\dagger \) for any hermitian quaternionic matrix \( X \), then \( A_1 = \pm A_2 \). The proof will be given in Appendix. Hence we have the isomorphism\textsuperscript{1)}

\[ \text{Spin}(2, ℋ)/Z_2 \cong L^\dagger_+ \]  

(1.6)

in a way completely parallel to the well-known one

\[ \text{SL}(2, ℂ)/Z_2 \cong L^\dagger_+. \]  

(1.7)

Needless to say the proof of (1.7) is based on the spinor analysis over the complex field using the association

\[ x^\mu \rightarrow X = x^0 \sigma_0 + x \cdot \sigma \equiv \left( \begin{array}{ccc} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{array} \right). \]  

(1.8)

Equation (1.1) is obtained from (1.8) by simply replacing the Pauli matrix \((\sigma_0, \sigma)\) by \((\Gamma_0, \Gamma)\). The restricted Lorentz transformation is then represented by \( A \in \text{SL}(2, ℂ) \) through \( X \rightarrow X' = AXA^\dagger \), which leaves the determinant invariant, i.e. \( \det X' = \det X \), namely, \( -\eta_{\mu\nu} x'^\mu x'^\nu = -\eta_{\mu\nu} x^\mu x^\nu \). This leads us to two inequivalent Weyl representations, since the representation \( \Lambda \rightarrow A \) is not equivalent to the one \( \Lambda \rightarrow A^\dagger = A^T \) and there are no other irreducible representations of 2 dimensions. Furthermore it is well-known that the Dirac representation is a direct sum of two inequivalent Weyl representations. A natural problem then occurs how to decompose the quaternionic Dirac representation into two irreducible Weyl representations. We discuss this interesting question in what follows.

\textsuperscript{1)} On the contrary, \( \text{SL}(2, ℋ) \) has two fundamental representations just as \( \text{SL}(2, ℂ) \) has two fundamental representations.
§2. Diagonalization of Spin(2, ℍ) matrices

According to our usual wisdom that the Dirac representation is a direct sum of two inequivalent Weyl representations, any matrix belonging to Spin(2, ℍ) should be diagonalizable with different diagonal elements. This diagonalization is carried through the eigenvalue problem

\[ A x = \lambda x \rightarrow \begin{pmatrix} Q & -P \\ P & Q \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{with} \quad |Q|^2 - |P|^2 = 1, \quad \overline{PQ} + \overline{QP} = 0. \]  

(2.1)

In this note we employ the Study determinant of a quaternionic matrix

\[ \text{Sdet} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \text{Sdet} \left( \begin{array}{cc} a - bd^{-1}c & bd^{-1} \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ c & d \end{array} \right) = |a - bd^{-1}c|^2|d|^2, \quad a, b, c, d \in \mathbb{H}. \]  

(2.2)

Thus the eigenvalues are determined by

\[ \text{Sdet} \left( Q - \lambda \begin{pmatrix} -P \\ P \end{pmatrix} \begin{pmatrix} Q - \lambda \end{pmatrix} \right) = 0. \]  

(2.3a)

Since the set of matrices A with P = 0 forms SO(3) not containing Lorentz transformations, we consider exclusively the case P ≠ 0. Thus we can assume \( Q - \lambda \neq 0 \). The reason as follows. Put \( Q - \lambda = \epsilon \). Then

\[ \lim_{Q \to \lambda} \text{Sdet} \left( \begin{array}{cc} Q - \lambda & -P \\ P & Q - \lambda \end{array} \right) = \lim_{\epsilon \to 0} |(Q - \lambda)(Q - \lambda + P(Q - \lambda)^{-1}P)|^2 \]

\[ = \lim_{\epsilon \to 0} |\epsilon(\epsilon + P\epsilon^{-1}P)|^2 = \lim_{\epsilon \to 0} |\epsilon P\epsilon^{-1}P|^2 \]

\[ = \lim_{\epsilon \to 0} |\epsilon \mu \epsilon \nu P\epsilon \mu \epsilon \nu |^2 = (1/16)|\epsilon \mu P\epsilon \mu P|^2 = |P_0 P|^2 \neq 0 \]  

(2.3b)

shows that \( \lambda = Q \) is not a solution and, hence, the inverse \( (Q - \lambda)^{-1} \) exists. Here we have used the relation \( \lim_{\epsilon \to 0} \epsilon \mu \epsilon \nu = (1/4)|\epsilon|^2 \delta_{\mu \nu} \). We thereby obtain

\[ \text{Sdet} \left( \begin{array}{cc} Q - \lambda & -P \\ P & Q - \lambda \end{array} \right) = |(Q - \lambda)(Q - \lambda + P(Q - \lambda)^{-1}P)|^2 = 0, \]  

(2.3c)

which leads to

\[ Q - \lambda + P(Q - \lambda)^{-1}P = 0. \]  

(2.3d)

Consequently, we have

\[ u \equiv P^{-1}(Q - \lambda) = -(Q - \lambda)^{-1}P = -u^{-1}. \]  

(2.3e)

This implies that \( u^2 = -1 \), whence \( |u| = 1 \) and \( \bar{u} = -u \). It is easy to see that Eq. (2.3e) is satisfied by

\[ Pu = Q - \lambda = \pm uP. \]  

(2.3f)
However, if \( P = P_0 \) is real, it obviously commutes with any quaternion (including a unit pure quaternion \( u \)), hence only the case \( Pu = uP \) is allowed. It follows that

\[
Pu = Q - \lambda = uP. \tag{2.3g}
\]

Since we can change the sign of \( u \) in (2.3e), there is another solution with opposite sign of \( u \),

\[
-Pu = Q - \lambda = -uP. \tag{2.3h}
\]

Consequently, there are just two different eigenvalues

\[
\lambda = Q \pm uP \quad \text{with} \quad uP = Pu. \tag{2.3i}
\]

There is another proof of (2.3i). Putting \( \varrho = Q - \lambda \) Eq. (2.3d) reads \( P^{-1} \varrho = -(P^{-1} \varrho)^{-1} \). There are four solutions to this equation, \( \varrho = \pm Pu, \pm uP \) provided \( u = -u^{-1} \). Disregarding degenerate eigenvalues, there are six pairs of eigenvalues in no particular order within each pair, \( (Q + Pu, Q + uP), (Q - Pu, Q - uP), (Q + Pu, Q - uP), (Q - Pu, Q + uP) \). Since we are considering \( 2 \times 2 \) matrix, there must be only two independent eigenvalues. Therefore, these six pairs should be fused into a single one. This is possible only if \( Pu = \pm uP \) holds true, leaving only the pair \( (Q - uP, Q + uP) \) (we should discard four pairs which give degenerate eigenvalues). However, as said before only the case \( Pu = uP \) is allowed. The eigenvalues are then given by (2.3i).

In this way we conclude that there must be a number \( u \) which commutes with any quaternion and is squared to be \( -1 \). It is beyond the field of quaternions but must satisfy \( \bar{u} = -u \) and \( |u| = 1 \). Hence it may be identified with the familiar complex imaginary unit \( u = i \) which must commute with Hamilton’s imaginary units, putting \( \bar{i} = i^* \) so that \( \bar{i} = -i \) and \( |i| = 1 \). We thus write the eigenvalues

\[
\lambda = Q \pm iP, \tag{2.4a}
\]

which are complex quaternions and whose quaternionic conjugation is given by

\[
\bar{\lambda} = \bar{Q} \pm iP. \tag{2.4b}
\]

That is, the eigenvalue problem (2.1) has no solution within the field of quaternions but has a solution if we extend \( \mathbb{H} \) to complex quaternions \( \mathbb{H}^c \). Correspondingly, the eigenvectors are also defined over \( \mathbb{H}^c \).

For the eigenvalue \( \lambda = Q - iP \equiv U \) the corresponding normalized eigenvector is given by

\[
\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix},
\]

while the normalized eigenvector corresponding to the eigenvalue \( \lambda = Q + iP \equiv U^* \) turns out to be

\[
\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}.
\]
Consequently, the matrix $A$ is diagonalized by a unitary matrix with the elements being complex quaternions

$$T = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad \text{Sdet } T = 1,$$

$$T^{-1} \begin{pmatrix} Q & -P \\ P & Q \end{pmatrix} T = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}. \quad (2.5)$$

Using the relation

$$U \overline{U} = (Q - iP)(\overline{Q} - i\overline{P}) = Q\overline{Q} - P\overline{P} - i(Q\overline{P} + P\overline{Q}) = 1, \quad (2.6)$$

which means that $U$ belongs to $SL(1, \mathbb{H}^c) = \{ q \in \mathbb{H}^c; \ |q| = 1 \}$, we see that

$$\text{Sdet } A = \text{Sdet } \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} = 1.$$

The diagonalization (2.5) as well as that of the matrices $\Gamma$ and $\tilde{\Gamma}$ were first pointed out in Ref. 3) rather intuitively and carried through in Ref. 1) after concluding the necessity of $i$ from the Pauli-Gürsey symmetry of the quaternionic Dirac theory in the massless limit. In this note we have succeeded in proving a natural appearance of the complex imaginary unit commuting with Hamilton’s imaginary units of quaternions at the level of the representation theory.

We now proceed to diagonalize $\Gamma$ and $\tilde{\Gamma}$ by means of the matrix $T$ as follows:

$$T^{-1}\Gamma^\mu T = \begin{pmatrix} -b^\mu & 0 \\ 0 & -\overline{b}^\mu \end{pmatrix}, \quad T^{-1}\tilde{\Gamma}^\mu T = \begin{pmatrix} -\overline{b}^\mu & 0 \\ 0 & -b^\mu \end{pmatrix}. \quad (2.7a)$$

Here

$$b_0 \equiv e_0, \quad b_1 \equiv ie_1, \quad b_2 \equiv ie_2, \quad b_3 \equiv ie_3, \quad b^\mu = \eta^{\mu\nu} b_\nu, \quad (2.7b)$$

are the basis of complex quaternions satisfying the relations

$$b_\mu \overline{b}_\nu + b_\nu \overline{b}_\mu = -2\eta_{\mu\nu}, \quad \overline{b}_\mu b_\nu + \overline{b}_\nu b_\mu = -2\eta_{\mu\nu}, \quad (2.7c)$$

with $\overline{b}_0 = b_0$ and $\overline{b}_i = -b_i$ ($i = 1, 2, 3$). Equation (2.7c) is obtained from (1.2) and (2.7a). Any complex quaternion

$$q = b_0 q^0 + b_1 q^1 + b_2 q^2 + b_3 q^3 \equiv b_\mu q^\mu, \quad q^\mu \in \mathbb{C}, \quad (2.8)$$

with the quaternionic conjugation $\overline{q} = b_0 q^0 - b_1 q^1 - b_2 q^2 - b_3 q^3 = \overline{b}_\mu q^{\mu}$ and the complex conjugation $q^* = b_0 (q^0)^* - b_1 (q^1)^* - b_2 (q^2)^* - b_3 (q^3)^*$ has an indefinite norm squared

$$|q|^2 = q\overline{q} = \overline{q}q = -\eta_{\mu\nu} q^\mu q^\nu \quad (2.9)$$

so that the group $SL(1, \mathbb{H}^c) = \{ q \in \mathbb{H}^c; \ |q| = 1 \}$ is noncompact. In fact, it can be shown that $SL(1, \mathbb{H}^c) \cong SL(2, \mathbb{C})$. 

What about the Lorentz transformation (1.3)? We find

\[ T^{-1}X = \begin{pmatrix} -\bar{x} & 0 \\ 0 & -\bar{x} \end{pmatrix}, \quad x = b_\mu x^\mu, \quad (2.10a) \]

\[ T^{-1}X' = \begin{pmatrix} -U\bar{x}U^\dagger & 0 \\ 0 & -U^\dagger\bar{x}U^\ast \end{pmatrix} = \begin{pmatrix} -x' & 0 \\ 0 & \bar{x}' \end{pmatrix}, \quad x' = b_\mu x'^\mu, \quad (2.10b) \]

with \(|x|^2 = -\eta_{\mu\nu}x^\mu x^\nu\) and \(|x'|^2 = -\eta_{\mu\nu}x'^\mu x'^\nu\), where hermitian conjugation is defined by

\[ q^\dagger = b_0(q^0)^\dagger + b_1(q^1)^\dagger + b_2(q^2)^\dagger + b_3(q^3)^\dagger. \quad (2.11) \]

Note that the (space-time) coordinate quaternion \(x = b_\mu x^\mu\) is hermitian, \(x^\dagger = x\), since \(x^\mu \in \mathbb{R}\). If \(q^\mu\) are complex, \(q^\mu = q^\mu^\ast\), then \(q^\ast = \bar{q}\). Equation (2.10) leads to (2.14a) below.

Next we see what happens in the representation space of the quaternionic spinor representation \(\Lambda \mapsto \tilde{A}\), which is spanned by 2-component quaternionic Dirac spinor

\[ \Psi = \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}, \quad \zeta^{1,2} \in \mathbb{H}, \quad (2.12) \]

such that \(\Psi \rightarrow \tilde{A}(\Lambda)\Psi\) under the restricted Lorentz transformation \(\Lambda \in L_+^{1}\). It is transformed into a complex quaternionic Dirac spinor \(\psi\) and its complex conjugate \(\psi^\ast\) via

\[ T^{-1}\Psi = \begin{pmatrix} \psi \\ \psi^\ast \end{pmatrix}, \quad \begin{cases} \psi = \frac{1}{\sqrt{2}}(\zeta^1 - i\zeta^2) \equiv b_\mu\psi^\mu, \\ \psi^\ast = \frac{1}{\sqrt{2}}(\zeta^1 + i\zeta^2) \equiv (b_\mu)^\ast\psi^\mu = \bar{\psi}\dagger, \end{cases} \quad (2.13a) \]

\[ T^{-1}\tilde{A}\Psi = \begin{pmatrix} U\psi \\ U^\ast\psi^\ast \end{pmatrix} = \begin{pmatrix} \psi' \\ \psi'^\ast \end{pmatrix}, \quad \begin{cases} \psi' = U\psi, \\ \psi'^\ast = U^\ast\psi^\ast. \end{cases} \quad (2.13b) \]

What we have found is that the Lorentz transformations of the space-time coordinate and the quaternionic Dirac spinor defined over the field of real quaternions are reduced to

\[ x' = UxU^\dagger, \quad |x|^2 = |x'|^2, \quad (2.14a) \]

\[ \psi' = U\psi, \quad U \in SL(1, \mathbb{H}^c), \quad (2.14b) \]

which are nothing but the fundamental equations in the conventional quaternionic approach to relativity and the Dirac equation.2) From our point of view, however, it is not necessary to assume complex quaternions from the outset but sufficient to simply replace \((\sigma_0, \sigma)\) by \((\Gamma_0, \Gamma)\) to represent the space-time coordinate. It results in the automatic antisymmetrization of bilinear covariants of the Dirac theory obtained from the quaternionic Dirac theory. This is essentially rooted in the fact that the charge conjugation transformation of the quaternionic Dirac spinor \(\Psi\) or the complex
quaternionic Dirac spinor \( \psi \) is linear already in the classical level. This is in sharp contrast to the 4-component formalism in which the charge conjugation transformation of the 4-component Dirac spinor is antilinear in the classical level. We do not dwell upon this aspect of our new quaternionic approach in this note, since it was fully discussed in Ref. 1).

§3. Comparison with Klein’s formula of Lorentz transformation using complex quaternions

It should be noted that (2-14a) is essentially equivalent to what Klein first observed already in 1911 that any Lorentz transformation is elegantly written by complex quaternion. We recapitulate Klein’s formula using Klein’s original notations for comparison. Let \( i \) be the ordinary complex imaginary unit, while \( i_1, i_2, i_3 \) are Hamilton’s imaginary units. Suppose \( A, A', D, D' \) be the 8 real parameters subject to \( AA' + BB' + CC' + DD' = 0 \) and \( A^2 + B^2 + C^2 + D^2 > A'^2 + B'^2 + C'^2 + D'^2 \). Klein claims without explicit calculation (at least in Ref. 4)) that the (homogeneous) Lorentz transformation is given by

\[
i_1 x' + i_2 y' + i_3 z' + ic t' = \left[ \begin{array}{l}
\left( i_1 (A + iA') + i_2 (B + iB') + i_3 (C + iC') + (D + iD') \right) \\
\cdot (i_1 x + i_2 y + i_3 z + ic t) \\
\left( i_1 (A - iA') + i_2 (B - iB') + i_3 (C - iC') - (D - iD') \right)
\end{array} \right] \\
\div \left[ (A^2 + B^2 + C'^2 + D'^2) - (A'^2 + B'^2 + C^2 + D^2) \right].
\]

(3.1)

We now relate Klein’s parameters to ours. Construct two real quaternions from them

\[
Q = i_1 A + i_2 B + i_3 C + D, \quad Q' = i_1 A' + i_2 B' + i_3 C' + D'
\]

(3.2)

and rewrite the two conditions imposed by Klein as \( \overline{QQ'} + Q'Q = 0 \) and \( (Q + iQ') (Q + iQ') = \overline{QQ} - \overline{QQ'} > 0 \). Then (3.1) turns out to be

\[
i_1 x' + i_2 y' + i_3 z' + ic t' = \frac{(Q + iQ') (i_1 x + i_2 y + i_3 z + ic t) (Q - iQ')}{\overline{QQ} - \overline{QQ'}}.
\]

(3.3)

Put \( U^* = (Q + iQ')/\sqrt{\overline{QQ}' - \overline{QQ}} \). It is apparent that \( U \) belongs to \( SL(1, \mathbb{H}^c) \). Multiplying both sides of (3.3) by \( -i \) and putting \( x = x^1, y = x^2, z = x^3 \) and \( ct = x^0 \) as well as \( i_1 = e_1, i_2 = e_2, i_3 = e_3 \), we precisely recover the quaternionic conjugation of (2-14a). It has been rederived independently by many authors in different notations.

§4. Pauli representation of quaternions and the quaternionic chiral decomposition

It is well known that the isomorphism

\[
SL(1, \mathbb{H}^c) \cong SL(2, \mathbb{C})
\]

(4.1)
is proved by the Pauli representation of quaternions

\[ H \ni q = e_\mu q_\mu \mapsto f(q) = \begin{pmatrix} q_0 + q_3 e_3 & -q_2 + q_1 e_3 \\ q_2 + q_1 e_3 & q_0 - q_3 e_3 \end{pmatrix} \tag{4.2} \]
with \( C \equiv \mathbb{C}(1,i) \cong \mathbb{C}(1,e_3) \). This representation is applicable to both real and complex quaternions. Put

\[ \rho(q) = f(q)|_{e_3 = -i} = \begin{pmatrix} q_0 - iq_3 & -q_2 - iq_1 \\ q_2 - iq_1 & q_0 + iq_3 \end{pmatrix}, \tag{4.3a} \]

\[ \bar{\rho}(q) = f(q)|_{e_3 = i} = \begin{pmatrix} q_0 + iq_3 & -q_2 + iq_1 \\ q_2 + iq_1 & q_0 - iq_3 \end{pmatrix}. \tag{4.3b} \]

Then

\[ \bar{\rho}(q) = \omega^{-1} \rho(q) \omega, \quad \omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.3c} \]

Consequently, the representations \( q \mapsto \rho(q) \) and \( q \mapsto \bar{\rho}(q) \) are equivalent so that the real part of quaternion \( q \) is obtained by the matrix trace of either representation

\[ \text{Re}[q] = \frac{1}{2} \text{tr} \rho(q) = \frac{1}{2} \text{tr} \bar{\rho}(q) = q_0. \]

The Pauli representation of the basis of complex quaternions turns out to be

\[ \rho(b_\mu) = \sigma_\mu, \quad \bar{\rho}(\bar{b}_\mu) = \bar{\sigma}_\mu, \quad \bar{\rho}(b_\mu) = \sigma_\mu, \quad \bar{\sigma}_\mu = \omega \sigma_\mu \omega^{-1}. \tag{4.4} \]

Consequently, the Pauli representation of (2·14a) is given by \( \rho(x') = \rho(U)\rho(x)\rho(U^\dagger) \)
which is equivalent to

\[ X' = AXA^\dagger, \quad A \equiv \rho(U) = (\sigma_\mu U^\mu) \in SL(2,\mathbb{C}), \tag{4.5} \]

since \( \rho(x) = X \) and \( UU^\dagger = 1 \). That is, the hypercomplex number system underlying the association (1.8) is complex quaternions, a well-known fact.

The next problem is to find two Weyl spinors, \( \xi \) and \( \bar{\eta} \), from the complex quaternionic Dirac spinor \( \psi \). In the 4-component formalism the Dirac spinor in the Weyl representation transforms like

\[ \psi = \begin{pmatrix} \xi \\ \bar{\eta} \end{pmatrix} \rightarrow \psi' = \begin{pmatrix} \xi' \\ \bar{\eta}' \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^\dagger -1 \end{pmatrix} \begin{pmatrix} \xi \\ \bar{\eta} \end{pmatrix}, \tag{4.6} \]

under the restricted Lorentz transformation. Please do not confuse our notation in which the same letter \( \psi \) is used both in the usual 4-component formalism and in our new quaternionic approach. It turns out that, in contrast, our \( \psi \) contains \( \xi \) and \( \eta \) linearly.

To see this remember the restricted Lorentz transformation of the complex quaternionic spinor \( \psi \), (2·14b).\(^1\) It reads \( \psi' = U\psi = (Ub_\mu)\psi^\mu = (b_\nu U^\nu)b_\mu \psi^\mu = b_\mu \psi'^\mu \), where

\[ \psi'^\mu = (\Omega U^\rho)_{\nu}^\mu \psi^\nu. \tag{4.7a} \]
Here $\Omega_\mu$ is the 4-dimensional representation of the basis of the complex quaternions defined by

$$Ub_\mu = (\Omega_\rho U^\rho)^\mu b_\nu, \quad U = b_\rho U^\rho, \quad U^\rho \in \mathbb{C},$$

so that

$$\omega(b_0) \equiv \Omega_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \omega(b_1) \equiv \Omega_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

$$\omega(b_2) \equiv \Omega_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \omega(b_3) \equiv \Omega_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (4.7c)$$

This representation is reducible.

$$V \omega(q) V^{-1} = \begin{pmatrix} \rho(q) & 0 \\ 0 & \bar{\rho}(q) \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & i & 0 \end{pmatrix}. \quad (4.7d)$$

Consequently, (4.7a) takes the form

$$\left( V \psi \right)^{\prime \mu} = \left( \begin{array}{c} \rho(U) \\ 0 \end{array} \right) \left( V \psi \right)^{\nu}. \quad (4.7e)$$

This suggests to put

$$\xi^1 = (V \psi)^0 = \psi^0 + \psi^3, \quad \xi^2 = (V \psi)^1 = \psi^1 + i\psi^2,$$

$$\eta_1 = (V \psi)^2 = \psi^0 - \psi^3, \quad \eta_2 = (V \psi)^3 = -\psi^1 + i\psi^2, \quad (4.8a)$$

obtaining the well-known transformation property from (4.7e)

$$\xi' = A \xi, \quad A \equiv \rho(U), \quad (4.8b)$$

$$\eta' = A^{-1} \eta, \quad A^{-1} = \rho^{-1}(U) = \bar{\rho}(U). \quad (4.8c)$$

Defining the split basis of quaternions by

$$v_1 = (bV^{-1})_0 = \frac{1}{2}(b_0 + b_3), \quad v_2 = (bV^{-1})_1 = \frac{1}{2}(b_1 - ib_2),$$

$$v^1 = (bV^{-1})_2 = \frac{1}{2}(b_0 - b_3), \quad v^2 = (bV^{-1})_3 = -\frac{1}{2}(b_1 + ib_2), \quad (4.9)$$

we obtain the decomposition

$$\psi = b_\mu \psi^\mu = (bV^{-1})_\mu (V \psi)^\mu = v_\alpha \xi^\alpha + v^\alpha \eta_\alpha, \quad (\alpha = 1, 2) \quad (4.10a)$$
which may be called the quaternionic chiral decomposition through the projection operators $E_{\pm} = (b_0 \pm b_3)/2$,

$$\psi = \psi (E_+ + E_-) \equiv \psi_L + \psi_R,$$

$$\psi_L \equiv \psi E_+ = v_\alpha \xi^\alpha, \quad \psi_R \equiv \psi E_- = v^\alpha \eta_\alpha,$$  \hspace{1cm} (4.10b)

since $v_1 = b_0 E_+ = b_3 E_+, v_2 = b_1 E_+ = -ib_2 E_+, v^1 = b_0 E_- = -b_3 E_-, v^2 = -b_1 E_- = -ib_2 E_-$. The 4-component Dirac spinor and its Dirac conjugate in the Weyl representation appear in the decomposition

$$\begin{pmatrix} \psi \\ \psi^* \end{pmatrix} = \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} (E_+ + E_-) \equiv \nu_\alpha \begin{pmatrix} \xi^\alpha \\ \bar{\eta}_\dot{\alpha} \end{pmatrix} + \nu^{\dot{\alpha}} \begin{pmatrix} \eta_\alpha \\ \bar{\xi}^\dot{\alpha} \end{pmatrix},$$ \hspace{1cm} (4.10c)

where the summation runs over $\alpha = 1, 2$ which correspond, respectively, to $\dot{\alpha} = \dot{1}, \dot{2}$.

In this way we recover Weyl spinors $\xi$ and $\eta$ from our complex quaternionic spinor $\psi$. For those readers who are not familiar with quaternion calculus it might be a bit lengthy calculation to see that the quaternionic Dirac spinor $\psi$ is decomposed into Weyl spinors $\xi$ and $\eta$ linearly. It is this aspect, however, that the bilinear covariants of the 4-component Dirac spinor are automatically antisymmetrized.\(^1\)

§5. Brief summary

When applying quaternions to relativity, it is necessary to assume complex quaternions from the outset. We showed in this note that this assumption is totally unnecessary but the extra complex imaginary unit commuting with Hamilton’s imaginary units is required by the eigenvalue problem which also helps define two inequivalent Weyl representations from the quaternionic Dirac representation. We are inclined to believe that the division algebra of quaternions is closely connected with our 4-dimensional Minkowski space-time with indefinite metric, since the quaternionic spinor group $Spin(2, \mathbb{H})$ plays almost the same role as the more familiar special linear group $SL(2, \mathbb{C})$ except for one important point and it leads to the natural appearance of complex quaternions and the automatic antisymmetrization of bilinear Dirac covariants. In particular, a natural appearance of $i$ commuting with Hamilton’s imaginary units of quaternions originates from the reducibility of the Dirac representation being a direct sum of inequivalent Weyl representations would be significant.

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We prove in this appendix that, if
\[ A_1 X A_1^\dagger = A_2 X A_2^\dagger, \quad A_1 \sigma_{1,3} A_1^\dagger = A_2 \sigma_{1,3} A_2^\dagger, \]  
(A.1)
then \( A_2 = \pm A_1 \). It is easy to see that (A.1) implies that \( B = A_2^{-1} A_1 \) satisfies \( B X B^\dagger = X \), hence \( B \) is unitary by putting \( X = 1_2 \) (unit quaternionic matrix). Consequently, it commutes with any hermitian quaternionic matrix
\[ B X = X B. \]  
(A.2)

Here \( X \) is not necessarily restricted to be the quaternionic space-time matrix (1.1) but is regarded as an arbitrary hermitian quaternionic matrix considering the second equations of (A.1)
\[ X = \begin{pmatrix} \alpha & q \\ \bar{q} & \beta \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}, \quad q \in \mathbb{H}. \]  
(A.3)

Putting
\[ B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{H}, \]  
(A.4)
we have
\[ X B - B X = \begin{pmatrix} (\alpha a + c q) - (a \alpha + b \bar{q}) & (\alpha b + q d) - (a q + b \beta) \\ (\bar{q} a + \beta c) - (c \alpha + d \bar{q}) & (\bar{q} b + \beta d) - (c q + d \beta) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \]  
(A.5)
since \( \alpha \) and \( \beta \) are real. If we put \( q = 0 \) in this equation, we get \( b(\alpha - \beta) = c(\alpha - \beta) = 0 \), leading to \( b = c = 0 \) because \( \alpha \) and \( \beta \) are arbitrary real numbers. Then we obtain
\[ q d - a q = \bar{q} a - d \bar{q} = 0 \]
which implies, by putting \( q = q_0 \in \mathbb{R}^*, \ a = d \). Consequently, \( q a = a q \). Since \( q \) is an arbitrary quaternion, \( a \) must be real (we are considering real quaternions). However, since \( B \) is unitary,
\[ B B^\dagger = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]  
(A.6)
hence \( a = \pm 1 \). Namely, we have proved the relation \( A_2 = \pm A_1 \) if they satisfy (A.1). This is what we promised to prove at the end of §1.
References


The spinor analysis using the quaternionic spinor group $\text{Spin}(2,\mathbb{H})$ is simply called the quaternionic spinor analysis in this note. It is different from the quaternionic spinor analysis using $\text{SL}(2,\mathbb{H})$.

A pedagogical and more extensive explanation of the background of this paper can be found in K. Morita, *Quaternions · Octonions and the Dirac Theory* (Nippon-Hyoron-sha, 2011), in particular, §4.5 (in Japanese).

2) See, for instance, C. Lanczos, Z. Phys. **57** (1929), 447. English Translation of this German paper is available at physics/0508002.


A complete list of papers on quaternions is given by A. Gsponer and J.-P. Hurni, math-ph/0510059; math-ph/0511092.
