Quantum Decoherence and Entanglement Induced by Nonlinear Dissipative Environment

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The analytical description of the dynamics of a nonlinear atom-field system described by a diagonal $f$-deformed Jaynes-Cummings model for a chosen initial state is obtained. The purity loss of the global system, and the atomic and field subsystems are discussed. Information on entanglement between the field and matter is studied by comparing the results for mutual entropy, which is a measure of the total correlation, and the negativity as a measure of the amount of entanglement. It is found that the nonlinearity of the dissipation tends to accelerate the decoherence of the global, field, and atomic states. In addition, we find that the so-called entanglement sudden death can occur when both of the nonlinearity couplings between the cavity field and the dissipation are combined.

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§1. Introduction

The Jaynes-Cummings (JC) model is one of the oldest and paradigmatic nontrivial models in quantum optics. It has been used extensively to describe the quantum features of the interaction of a single two-level atom with a single cavity mode. One of the simplest physical realizations of the JC model is the one-atom maser or micromaser. Recently, cooling techniques have allowed the trapping and manipulation of atoms by optical means. These new technologies have provided an alternative physical realization of the JC model, where cold trapped ions can be excited in such a way that the effective unitary dynamics is described by a nonlinear JC model.

In the above situation, the phonon field corresponding to the vibrational motion of the ion plays the role of an electromagnetic cavity field, and the electronic levels of the ion effectively act as a two-level system. By conveniently choosing the laser excitation, different coupling schemes can be realized. The resulting dynamics is a generalization of the JC model, where the effective two-level system and the vibrational modes can interchange $k$ excitation quanta. There is an increasing interest in the study of this system the main motivation comes from the field of quantum computation and information, where trapped-ion systems are considered as one of the most promising resources for implementing controlled generation and manipulation of quantum states.

One of the major problems to overcome in these areas is decoherence. Much work has been dedicated to the theoretical study of the dissipative JC model by considering cavity loss and atom decay with constant coupling between the cavity field and its environment, and with the coupling being deformed with the atom in a
The interaction of a system with its environment plays a major role in the disappearance of quantum coherence. The monitoring of decoherence, due to the effects of dissipation, in a superposition of two coherent states, was reported in a cavity quantum electrodynamics microwave experiment. Other experiments have also clearly shown the decoherence of the superposition of states due to the interaction of the system with the environment. Also the controlled superposition of states of the vibrational motion of the center of mass of trapped ions has been experimentally realized.

As a promising resource, quantum entanglement plays a key role in quantum information processing such as quantum teleportation, quantum key distribution, and quantum cryptography. However, in the real world, quantum information processing will be inevitably affected by the decoherence that destroys quantum superposition and quantum entanglement. The extent to which decoherence affects quantum entanglement is an interesting problem and based on the basis of various models from the point of view of environment-induced decoherence, many researchers have studied it extensively. The decay of entanglement cannot be restored by local operations and classical communications, which is one of the main obstacles for achieving a quantum computer. Therefore, it becomes an important subject to study the loss of entanglement, where the environment, represented by a thermal reservoir, always exists and affects the system considered. No matter how weak the coupling to such an environment, the evolution of quantum subsystems is eventually affected by nonunitary features such as decoherence, dissipation, and heating.

On the other hand, because of the quantum parallelism and entanglement that arise from the superposition of states in two-level qubit systems, these systems are among the main candidates for implementing quantum information and quantum computation schemes. There are the so called “artificial” atoms that behave like a two-level system and thus become candidates in quantum computers, such as quantum dots, electronic spins and Cooper pair boxes. The simplest model of a qubit-field interaction describing the interaction of an undamped qubit with a single nondecaying electromagnetic field mode is commonly known in the literature as the JC model. These important topics, which were previously discussed, and experimental results on qubits coupled to the environment are motivations for studying the effect of effects of nonlinear coupling in the JC model and also of nonlinearity in the dissipation terms on the properties of quantum coupling, e.g., the purity loss and mutual entropy, which are obtained from the density matrix. In the present study, our main purpose is to study the dynamical behavior of the atom-field system in the framework of a nonlinear JC model. The model presented here is based on the assumption that not only the atom-field interaction but also the coupling between the cavity field and its environment is deformed.

§2. Nonlinear JC model and source of decoherence

The generalized JC model can be described independently of any particular physical situation, in terms of a two-level system interacting with a quantum harmonic oscillator. We denote the two-level states by $|1\rangle_A$ and $|0\rangle_A$, and by $\hat{a}$ and $\hat{a}^\dagger$,
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the annihilation and creation operators corresponding to the harmonic oscillator, respectively. By generalizing the conventional JC model, the interaction is described in the interaction picture by

\[ \hat{H}_{\text{int}} = \hbar \lambda (\hat{a} f(\hat{a}^\dagger \hat{a}) |1_A\rangle \langle 0_A| + f(\hat{a}^\dagger \hat{a})(\hat{a}^\dagger) |0_A\rangle \langle 1_A|), \]  

(2.1)

where \( \lambda \) is the coupling constant between the atom and the field, with \( \hbar = 1 \) and \( f(\hat{a}^\dagger \hat{a}) \) is an operator real function of the mean photon numbers \( \hat{N} = \hat{a}^\dagger \hat{a} \). This Hamiltonian (2.1) describes a nonlinear transition \( |0_A\rangle \leftrightarrow |1_A\rangle \) in the two-level system, where the flip of the lower state \( |0_A\rangle \) to the upper state \( |1_A\rangle \) occurs simultaneously with the absorption of one quanta of the harmonic oscillator. Note that the conventional JC model is recovered using \( f(\hat{a}^\dagger \hat{a}) = \hat{I} \). The nonlinear JC model (2.1) is of considerable interest because of its relevance to the intensity-dependent interaction between a single atom and the radiation field in quantum optics as well as in the study of the quantized motion of a single ion in an anharmonic-oscillator potential trap. The above Hamiltonian describes an intensity-dependent coupling between a single two-level atom and a nondeformed single-mode radiation field in the presence of an additional nonlinear interaction. As a well-known example, if we choose \( f(\hat{N}) = \sqrt{1 + \kappa (\hat{n} - 1)} \), where \( \kappa \) is a positive constant, the model consists of a single two-level atom interacting through an intensity-dependent coupling with a single-mode field surrounded by a nonlinear Kerr-like medium contained inside a lossless cavity. Physically, this model can be realized if the cavity contains two different species of Rydberg atoms, of which one behaves like a two-level atom and the other behaves like an anharmonic oscillator in a single mode field of frequency \( \omega \).

We consider a system of the two-level atoms in which the harmonic oscillator interacts with a zero-temperature reservoir. Recently, when the coupling between the cavity field and its environment was deformed, a master equation has been derived in Ref. 31. Therefore, the evolution of the compound atom-field system in a dispersive \( f \)-deformed JC model and in the presence of the \( f \)-deformed (nonlinear) dissipation at zero temperature can be written in an interaction picture as

\[
\frac{\partial \rho(t)}{\partial t} = -i[\hat{H}_{\text{int}}, \rho] + \gamma ( [\hat{a} f_d(\hat{a}^\dagger \hat{a})\rho, f_d(\hat{a}^\dagger \hat{a})(\hat{a}^\dagger)] \\
+ [\hat{a} f_d(\hat{a}^\dagger \hat{a}), \rho f_d(\hat{a}^\dagger \hat{a})(\hat{a}^\dagger)])],
\]  

(2.2)

where \( \gamma \) is the field damping parameter, but the nonlinearity of the coupling between the cavity field and its environment is described by \( f_d(\hat{a}^\dagger \hat{a}) \), defined as a function of the number operator. The dressed-state basis, i.e., the set of eigenstates of \( \hat{H}_{\text{int}} \), is given by

\[ |i, 0_A\rangle, \ |\psi_\pm_n\rangle = (|n, 1_A\rangle \pm |n + 1, 0_A\rangle)/\sqrt{2}, \]  

(2.3)

where the first set corresponds to the states that decouple from the dynamics described by \( \hat{H}_{\text{int}} \) and the second one satisfies

\[ \hat{H}_{\text{int}} |\psi_\pm_n\rangle = \pm \nu_n |\psi_\pm_n\rangle, \quad \nu_n = \lambda f(n + 1) \sqrt{(n + 1)}. \]  

(2.4)
To obtain the analytic solution of [Eq. (2.2)] in the case of a high-$Q$ cavity limit ($\gamma \ll \lambda$), we use the dressed-state representation.\(^{32}\),\(^{33}\) We express the operators of the cavity field in terms of the dressed states. Next we use the transform

$$
\hat{\beta}(t) = e^{i\hat{H}t} \frac{\partial \hat{\rho}(t)}{\partial t} e^{-i\hat{H}t} + i[\hat{H}, \beta(t)].
$$

(2.5)

This means that $\beta(t) = e^{i\hat{H}t} \rho(t) e^{-i\hat{H}t}$. Therefore, for any initial state, the solution of the master equation in the case of a high-$Q$ cavity limit is given by

$$
\dot{\hat{\rho}}(t) = \rho_0|0, 0_A\rangle\langle 0, 0_A| + \frac{1}{\sqrt{2}} \sum_{m=0}^{\pm} \sum_{\epsilon=\pm} \langle \langle 0, 0_A| \beta(t) |\psi_{m}^{\epsilon}\rangle e^{i\eta t}
$$

$$
\times |0, 0_A\rangle \langle \psi_{m}^{\epsilon}| + \hbar c + \sum_{m, n=0}^{\pm} \langle \langle \psi_{n}^{\epsilon}| \beta(t) |\psi_{m}^{\epsilon}\rangle
$$

$$
\times e^{-i(\omega_{m} - j\eta_{n})t} |\varphi_{m}^{\epsilon}\rangle \langle \varphi_{n}^{\epsilon}|,
$$

(2.6)

where

$$
\rho_0 = \gamma \int_{0}^{t} \left[ \langle \psi_{m}^{\epsilon} | \beta(\tau) |\psi_{n}^{\epsilon}\rangle + \langle \psi_{n}^{\epsilon} | \beta(\tau) |\psi_{n}^{\epsilon}\rangle \right] d\tau
$$

$$
+ \langle 0, 0_A| \beta(0) |0, 0_A\rangle,
$$

(2.7)

$$
\langle \psi_{m}^{\epsilon} | \beta(t) |\psi_{n}^{\epsilon}\rangle = \frac{\gamma}{\lambda} \left( \delta_{m, n} + \frac{\gamma}{\lambda} \right) \langle \psi_{m}^{\epsilon} | \beta(0) |\psi_{n}^{\epsilon}\rangle,
$$

$$
\langle 0, 0_A| \beta(t) |\psi_{m}^{\epsilon}\rangle = \frac{\gamma}{\lambda} \langle 0, 0_A| \beta(0) |\psi_{m}^{\epsilon}\rangle,
$$

$$
\langle \psi_{m}^{\epsilon} | \beta(t) |\psi_{n}^{\epsilon}\rangle = \frac{\gamma}{\lambda} \langle \psi_{m}^{\epsilon} | \beta(0) |\psi_{n}^{\epsilon}\rangle, \quad \forall \ m \neq n,
$$

(2.8)

with

$$
D_{m,n} = (\eta_{m+1}^{2} + \eta_{n+1}^{2} + \eta_{m}^{2} + \eta_{n}^{2})
$$

and $\eta_{n} = f_{d}(n) \sqrt{n}$. But at $m = n$

$$
\langle \psi_{m}^{\epsilon} | \beta(t) |\psi_{n}^{\epsilon}\rangle = \frac{\gamma}{\lambda} \left( \delta_{m, n} + \frac{\gamma}{\lambda} \right) \left( \frac{\gamma}{\lambda} \langle \psi_{m}^{\epsilon} | \beta(0) |\psi_{n}^{\epsilon}\rangle \right)
$$

$$
+ \frac{\gamma}{2} \int_{0}^{t} \frac{\gamma}{\lambda} \left( \delta_{m, n} + \frac{\gamma}{\lambda} \right) \left( \delta_{m, n+1} + \frac{\gamma}{\lambda} \right) \left( \delta_{m, n+1} + \frac{\gamma}{\lambda} \right) \langle \beta(\tau) |\psi_{m}^{\epsilon}\rangle d\tau
$$

$$
+ \frac{\gamma}{\lambda} \langle \psi_{m}^{\epsilon} | \beta(t) |\psi_{n}^{\epsilon}\rangle
$$

(2.9)

where $\delta_{n}^{\pm} = (\eta_{n+2}^{2} + \eta_{n+1})^{2}$ and $|\varphi_{m}^{\epsilon}\rangle \langle \varphi_{n}^{\epsilon}| = \frac{1}{2} \sum_{r, s=0}^{1} \epsilon^{r} \kappa^{s} \langle R_{A}, m + r |S_{A}, n + s\rangle$, with $R_{A} = (1 - r)_{A}$ and $S_{A} = (1 - s)_{A}$. To find the iteration integration of the last term of Eq. (2.9), we assume that there is an upper limit on the number of photons initially present in the system so that $\langle \psi_{n+1}^{\epsilon} | \beta(t) |\psi_{n+1}^{\epsilon}\rangle = 0$, which is always true since the number of photons in the cavity will only decrease.\(^{32}\)

§3. Evolution of uncorrelated initial state

In order to understand the effect of the nonlinearity of the dissipation on the entanglement process in the nonlinear JC model, let us calculate the time evolution
of this initial state: $|\Psi^{AF}(0)\rangle = |1_A\rangle \otimes |\alpha\rangle$. After a pioneering study, different aspects of the problem of decoherence of ‘macroscopic’ quantum superpositions (Schrödinger cat states) were considered in numerous publications. Some theoretical predictions have already been verified in experiments performed by different groups and new experimental proposals have been made recently. Moreover, this subject is still far from being exhausted, because only the decoherence of the simplest superposition of Gaussian packets (coherent or squeezed states) has been studied in detail to date. In particular, one of the most frequently considered models of the Schrödinger cat states is based on the notion of even and odd coherent states introduced in Ref. 40).

These states have the form $|\alpha\rangle_{\pm} = N_{\pm}(\alpha)[D(\alpha) \pm D(-\alpha)]|0\rangle$, where $N_{\pm}(\alpha)$ is the normalization factor, $|0\rangle$ is the vacuum state, and $D(\alpha)$ is the displacement operator. This means that the two-level atom is initially prepared in the excited state, i.e., $\rho^A(0) = |1_A\rangle\langle 1_A|$, while the field is initially in $|\alpha\rangle_{\pm}$. Then the density operator of the system is initially given by $\rho(0) = \hat{\rho}(0) = \hat{\rho}^A(0) \otimes \hat{\rho}^F(0)$.

### 3.1. Purity of global density operator

For a chosen initial state, the purity of the state represented by a density operator $\hat{\rho}(t)$ is conveniently measured using the linear entropy:

$$S = 1 - \text{Tr}[\rho^2(t)].$$

In general, if $\hat{\rho}(t)$ describes a pure state, then $S = 0$, otherwise $S > 0$. We note that the characteristic time of the decay of nondiagonal elements for long times depends on the damping constant $\gamma$, and the effective coupling between atoms and the field $\lambda$, and nondiagonal elements vanish at $t \to \infty$. Consequently, the coherence loss of the atom-field system is complete in a stationary region, i.e., the linear entropy has an upper limit that is characteristic of a complete mixture. Note that the coherence properties of the total system are completely governed by the presence of the reservoir, as determined by the function $C_{oh}(t) = e^{-\gamma D_{m,n} t}$, which is responsible for the prevention of the revival of atomic inversion. In other words, with increasing parameter $\gamma$, a more rapid suppression of quantum coherence is expected, and one can observe the rapid deterioration of the revivals of such inversion.

### 3.2. Purity of reduced density operators

In order to analyze what happens to the atom, we have to calculate the reduced atom density operator $\hat{\rho}_A(t)$ from Eq. (2.6), which is given by $\hat{\rho}_A(t) = \text{Tr}_F\{\hat{\rho}(t)\}$. The atomic coherence loss will be measured using its linear entropy, which is

$$S_A = 1 - \text{Tr}[\rho^2_A(t)].$$

This is because $\rho^2_A(t)$ contains the function $C_{oh}(t)$, which reflects the presence of the reservoir and shows that it affects the coherence properties of an atom even if not directly coupled to it. Also, the function $E_{nt}(t) = \langle \psi_{l}^{\pm}\beta(0)|\psi_{l}^{\mp}\rangle e^{-i(\epsilon_{\nu}-j\nu)t}$ generated during the dynamics has its origin in the entanglement process between the atom and the field. The quantity $S_A$ shows clear evidence of the distinguished roles of what in dissipation and entanglement. However, the reduced field density $\hat{\rho}_F(t)$ is given by $\hat{\rho}_F(t) = \text{Tr}_A\{\rho(t)\}$. The field coherence loss will be measured using $S_F = 1 - \text{Tr}[\rho^2_F(t)]$, which is directly coupled to the reservoir by the function $C_{oh}$. There are two sources of coherence loss in the present system. One of them is the unitary atom-field interaction. This process is usually called entanglement...
and from the point of view of one of the subsystems, purity loss (or coherence loss) will take place. On the other hand, the interaction of the field subsystem with the environment also induces purity loss, and this process is usually called decoherence in the literature. In what follows, we shall reserve the word “decoherence” to mean “coherence loss induced by the environment”.

3.3. Results and discussion

It is well-known that if the field and the atom in the JC model are initially prepared in a pure state, then at \( t > 0 \) the atom-field system evolves into an entangled state in which the field and the atom are separately in mixed states. The entanglement between the atom and the field, as well as the decoherence induced by the cavity of the usual dispersive JC model has been studied\(^{42,43}\) where only the coherence of the atom is affected by the cavity. The coherence of the field remains unchanged by the environment. Also, in large-detuning approximation, it is found that, in the presence of nonlinear quantum dissipation, the amplitude of the entanglement between the field and the atom decreases with time\(^{8}\). However, for the conventional JC model\(^{20}\) the coherence of the field and the atom is affected by the phase damping of the cavity. However, in the nonlinear model we consider here, the entire system as well as the field and atom are also affected by the cavity, and its coherence will be lost owing to the nonlinear dissipation.

**Influence of nonlinear dissipation \( f_d(\hat{N}) \)**

In the absence of damping (\( \gamma = 0 \)), it is shown in Refs. 19) and 20) that \( S_A = S_F \). They have the same time evolution curve, indicating that the field does not return close to its initial state at this time. To determine the effect of nonlinear dissipation on the time evolution of the quantum coherence of the state of a physical system, different graphs are plotted. Figures 1(a)-(c) clearly show the behavior of this effect, where the graphs of \( S, S_A, \) and \( S_F \) as functions of \( \lambda t \), for different forms of \( f_d(\hat{N}) \) with a fixed form of \( f(\hat{N}) = \hat{I} \) and fixed damping parameter \( \gamma = 0.001 \), are shown. The time evolution \( f_d(\hat{N}) = \hat{I} \) is shown in Fig. 1(a). Owing to cavity damping, the state changes from a pure state at \( t = 0 \), to a statistical mixture with time. We note that the total linear entropy \( S \) increases monotonically and is no longer equal to zero. On the other hand, the partial entropies for the atom and the field are no longer equal and hence cannot be used as a measure of entanglement. However, they may be used to study the coherence properties of the atom and field.

Note that there are two sources of coherence loss in the present system. One is the unitary atom-field interaction. This process is usually called entanglement and from the point of view of one of the subsystems, purity loss (or coherence loss) takes place. On the other hand, the interaction of the field subsystem with the environment also induces purity loss. Note that the field attains a higher linear entropy than the atom, so the coherence loss of the field state is faster than the atomic coherence loss. For long times \( \lambda t > 1 \), the coherence loss of the field subsystem is dominated by the function

\[
C_{oh}(t) \bigg|_{f_d=\hat{I}} = e^{-\gamma [(m+1)(n+1)+m+n]t}, \tag{3.1}
\]
which leads to the destruction of the revival and the collapse of the field linear entropy. In this region, the time dependence of the field linear entropy $S_F$ closely
follows that for the global system $S$. Hence, the contribution to the field coherence loss, for long times, is due to the presence of dissipation, and the times of both the field and the full system’s decoherence are the same. On the other hand, for short times ($\lambda t < 1$), the contribution of the field coherence loss is solely due to the unitary atom-field interaction. However, the coherence loss of the atomic state is due to the unitary atom-field interaction and dissipation.

The time evolutions $f_d(\hat{N}) = (\sqrt{\hat{N}}$ and $\hat{N})$ are shown in Figs. 1(b) and (c), respectively. In this case, with a fixed form of $f(\hat{N}) = \hat{I}$, the effect of the unitary atom-field interaction does not change. It is interesting to note that the nonlinearity of the reservoir is represented by the functions

$$C_{oh}(t) \bigg|_{f_d=\sqrt{\hat{N}}} = e^{-\gamma(\lambda n^2+\lambda m^2+\lambda n^2)t} \tag{3.2}$$

and

$$C_{oh}(t) \bigg|_{f_d=\hat{N}} = e^{-\gamma((m+1)^2(n+1)^2+m^2+n^2)t}. \tag{3.3}$$

We note that the order of the power of the exponential function increases. Therefore, the coherence loss of either the global system or the atom or field develops rapidly, since the coherence loss of the global system and field becomes faster than the atomic coherence loss as the intensity function changes from the form $f_d = \sqrt{\hat{N}}$ to the form $f_d = \hat{N}$. Moreover, the coherence loss of the field state for short times is similar to that of the atomic state. As seen from Figs. 1(b) and (c), the contribution to the purity loss of the global system or the field or atomic state results from the presence of the dissipative environment represented in the nonlinearity of the reservoir.

**Combined effect of $f(\hat{N})$ and $f_d(\hat{N})$**

To determine what happens when the effect of the nonlinearity of both the atom field and the system environment on entanglement is combined, the coherence loss of the global, atomic, and field states are shown in Figs. 1 and 2 when $f(\hat{N}) = \hat{I}$, “with $\sqrt{\hat{N}}$” for different forms of $f_d(\hat{N})$. We note that the nonlinearity of $f(\hat{N})$ does not affect the coherence loss induced by the presence of the nonlinear dissipative environment in the long-time region, but its affect such loss resulting from the interaction between the atom and the field in short-time region, i.e., the effect of $f(\hat{N})$ on the purity is almost ineffective in the long-time region.

From Figs. 1 and 2, we note that the graphs of $S_A$ and $S_F$ have the comb structure, which is due to the unitary atom-field interaction. This comb structure clearly appears when the nonlinearity is $f(\hat{N}) = \sqrt{\hat{N}}$. This is due to the change in the form of the function $\nu_n$ from $\nu_n = \lambda\sqrt{n+1}$ to $\nu_n = \lambda(n+1)$. We note that the nonlinearity of the coupling between the field and its reservoir leads to the disappearance of the comb structure (see Figs. 1 and 2). Hence, the coherence properties of the atomic state are affected by the presence of the nonlinearity of $f(\hat{N})$ and $f_d(\hat{N})$, and by the unitary atom-field interaction. It can be easily deduced that the asymptotic value of the atomic entropy tends to $S_A(t \to \infty) = \frac{1}{2}$ this means that the atom loses its purity and finally goes into a mixed state, which is the stationary state for the
atom. Also from Figs. 1 and 2, we see that the stationary global and field states become a statistical mixture and appear faster in the case of \( f_d(\hat{N}) = \hat{N} \) and the envelopes of their time dependencies are the same.

3.4. **Measuring correlations**

The density operator that represents the state of the bipartite system \( A + B \) is described by \( \hat{\rho} \), and the state of each subsystem is described by the reduced density operators \( \hat{\rho}_A = \text{Tr}_B(\hat{\rho}) \) and \( \hat{\rho}_B = \text{Tr}_A(\hat{\rho}) \). However, in general, the global state cannot be determined from the reduced state, i.e., \( \hat{\rho} \neq \rho_A \otimes \rho_B \). Important information on the global state is lost in the partial tracing-out procedure. This information is related to the local (classical) and nonlocal (quantum) correlations between the two subsystems \( A \) and \( B \). We can ask about the “distance” between the global state \( \hat{\rho} \) and the corresponding completely uncorrelated state \( \hat{\rho}_A, \hat{\rho}_B \) as a measure of the total correlations of the state \( \hat{\rho} \). Therefore, we study the total correlations using the mutual information

\[
I(\rho) = \frac{(S_A + S_F - S)}{4}. \tag{3.4}
\]

This quantity means the degree of entanglement. It is a measure of both the quantum and classical correlations residing in a composite system. \( I(\rho) \) is a non-negative quantity; if the field is found in a pure state, we have \( I(\rho) = 0 \), and the global state is characterized by a completely uncorrelated state. The correlation measure defined above does not distinguish between classical and quantum correlations.

In order to evaluate the entanglement of the atom-field system, we chose negativity\(^{45} \) to quantify the amount of entanglement. It has been proved to be a reasonable entanglement measure for mixed states of bipartite systems composed of two-level subsystems. For a system described by the density operator \( \hat{\rho} \), negativity can be defined as\(^{45} \)

\[
N(\rho) = (\| \rho^T \| - 1)/2, \tag{3.5}
\]

where \( \rho^T \) is the matrix obtained by partially transposing the density matrix \( \rho \). Vidal and Werner\(^{45} \) proved that the negativity \( N(\rho) \) is an entanglement monotone and therefore is a good measure of the amount of entanglement of the final state (2.6).

**Effect of nonlinear dissipation** \( f_d(\hat{N}) \)

The graphs of the correlation measure \( I(\rho) \) and the negativity \( N(\rho) \) as functions of time for different forms of \( f_d(\hat{N}) \) with a fixed form for \( f(\hat{N}) = \hat{I} \) and a fixed damping parameter \( \gamma = 0.001 \) are shown in Fig. 3. We can note that, with the function \( f_d \), the measures \( I(\rho) \) and \( N(\rho) \) have a very strong sensitivity to the functional dependence of the nonlinearity. This sensitivity is due to the nonlinearity that appears in the power of the exponential functions (3.1), (3.2), and (3.3). Therefore, with \( f_d(\hat{N}) = \hat{N} \), the asymptotic values of the correlation and the amount of entanglement appear at short time compared with its counterpart \( f_d(\hat{N}) = \hat{I} \) and \( \sqrt{\hat{N}} \). As expected, the asymptotic value of the correlation measure \( I(\rho) \) is not null, since the global system evolves to a classically correlated state. Also, the negativity \( N(\rho) \) vanishes in the asymptotic limit. This means that, in this regime \( (N(\rho) = 0) \),
the entanglement between the atom and the field is completely destroyed. Therefore, the final state (2.6) disentangles completely and abruptly at only a finite time and evolves to a vacuum state and the decay of $N(\rho)$ with $f_d(\hat{N}) = \hat{N}$ is more rapid than...
Combined effect of $f(\hat{N})$ and $f_d(\hat{N})$

From Figs. 3(a) and 4(a), we note that the graphs of $N(\rho)$ and $I(\rho)$ have a comb structure, which is due to the unitary atom-field interaction. This is because of the change in the form of the function $\nu_n$ of the power of the function $E_{nt}(t) = \langle \psi_+^\mp | \beta(0) | \psi_-^\mp \rangle e^{-i(\epsilon \nu_l - j \nu_l) t}$, from $\nu_n = \lambda \sqrt{n + 1}$ to $\nu_n = \lambda (n + 1)$. The comb structure clearly appears when the nonlinearity is $f(\hat{N}) = \sqrt{\hat{N}}$. We note that the nonlinearity of the coupling between the field and its reservoir leads to the disappearance of the comb structure (see Figs. 3(c) and 4(c)).

Recently, some authors have found that entanglement can decrease to zero abruptly and remains zero for a finite time. They call this phenomenon entanglement sudden death,\textsuperscript{46,47} where the entanglement of a two-qubit system decays to zero in a finite time under the influence of pure vacuum noise.\textsuperscript{46} Entanglement sudden death has been experimentally observed in an implementation using twin photons\textsuperscript{48} and atomic ensembles.\textsuperscript{48} When we combine the effect of the nonlinearity of the coupling between the cavity field $f(\hat{N}) = \sqrt{\hat{N}}$ and of the dissipation $f_d(\hat{N}) = \hat{N}$, a novel feature of entanglement sudden death also occurs (see Figs. 3(c) and 4(c)), in which the entanglement decreases to zero abruptly and remains zero for a finite time.\textsuperscript{46} Also, note that the evolution of the negativity with different values for the nonlinearity function leading to entanglement sudden death and asymptotic disentanglement is shown in Figs. 3(c) and 4(c). This means that the decoherence plays a conventional role in destroying the entanglement.

Here, considering entanglement sudden death, which shows that entanglement can decay to zero abruptly in a finite time, one can determine a particular region in which there is no entanglement between the atom and the cavity. It arrives in an asymptotic state at $f_d(\hat{N}) = \hat{N}$, and then remains nearly invariant regardless of the increase in time. Because of these types of nonlinearities, the value of the measure of the entanglement between the atom and the cavity reaches the asymptotic value $N(\rho)(t \to \infty) = 0$, so that the state of the atom cavity becomes disentangled. This is in agreement with the asymptotic behaviour $\rho(t \to \infty) = \rho^A(t \to \infty) \otimes \rho^F(t \to \infty)$. In this asymptotic regime, at the instant the atom and the field become entangled, $I(\rho) \neq 0$ therefore, the stationary state for the global system evolves to a classically correlated state. This stationary state is not affected by the nonlinearity description.

However, the nonlinear dissipation $f_d(\hat{N})$ leads to the death of the entanglement. The death-start points (DSPs) depend on the function $f_d(\hat{N})$. From Figs. 3 and 4, we note that these DSPs can be controlled using the functions $f_d(\hat{N})$ and $f(\hat{N})$. These DSPs can be delayed and occur early by changing these functions at fixed values for $\gamma = 0.001$. They also occur early with a high-order function for $f_d(\hat{N}) = \hat{N}$, while they are delayed with $f_d(\hat{N}) = \hat{N}$.

§4. Conclusions

An analytical description of the dynamics of an atom-field system described by a diagonal $f$-deformed JC model with a nonlinearity of dissipation is considered. We
find that the coherence loss results from the competition of two processes: the unitary interaction between the subsystems and the dissipative dynamics due to the coupling between the system and its environment. The nonlinearity of the coupling between the cavity field and its environment markedly affects the total correlations and the amount of entanglement. A strong sensitivity to the nonlinear coupling between the atom and the field is noted in the short-time regime, although the damping effect is effective in the long-time regime. This study reveals that nonlinearity functions may be used to generate entanglement sudden death.

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