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Uncertainty and sensitivity analysis of water transport modelling in a layered soil profile using fuzzy set theory

Karsten Schulz and Bernd Huwe

ABSTRACT

A methodology based on fuzzy set theory is presented to express imprecision of input data in a non-probabilitic sense. Imprecision may originate from indirect measurements, estimation routines, subjective interpretation, and expert judgement of available information. A numerical finite difference solution scheme was chosen to solve one-dimensional steady-state water flow in the unsaturated zone of a layered soil profile. To extend the solution algorithm to operate with fuzzy soil-hydraulic properties and boundary conditions, it is necessary to incorporate the scheme into a nonlinear optimisation routine from where resulting membership functions of soil water pressures with depths can be calculated. By subsequently considering the different imprecise parameters in the calculations and analysing their impact, it is concluded that resulting imprecision not only depends on the degree of imprecision and the number of uncertain parameters but also very much on the system context (e.g. boundary conditions and spatial distribution). The comparison with a closed form solution to solve the fuzzy water flow problem show the potential of that method to be extended towards transient, two- and three-dimensional process descriptions.

Key words | water transport, fuzzy set theory, unsaturated zone, uncertainty analysis, finite difference

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INTRODUCTION

In this paper, a method based on fuzzy set theory is presented to utilise imprecise parameters in the modelling of water transport of the unsaturated zone as well as to analyse their impact on the predictions for a layered soil profile. Calculation of water transport requires the knowledge of various soil hydraulic parameters such as saturated/unsaturated hydraulic conductivity and water retention curves, as well as the determination of boundary conditions like precipitation or evapotranspiration rates and soil water pressures at certain depths.

However, these parameters are subject to different kinds of uncertainties, which are usually set in a mathematical framework by using a statistical approach and regarding the relevant parameters as random variables with well-known probability distributions. Therefore the resulting flow descriptions are also of a stochastic nature and, for example, Monte Carlo simulation techniques (Freeze 1975) can be applied to derive the statistics of the output parameters (e.g. soil water pressures). Examples for the application of this concept are given by Tang & Pinder (1979), Russo & Bressler (1981), Ünlü *et al*. (1989), Russo & Bouton (1992), among many others, where the effects of spatial variability and uncertainties of soil hydraulic properties on water and solute transport are investigated.

The probabilistic approach supposes all parameter uncertainties to be random. This asumes the outcome of a specific parameter value from an experiment to be a realisation of a stochastic process and a matter of chance, which can objectively be quantified and then described by a probability distribution or probability density function. However, randomness is only one kind of uncertainty. Imprecise information such as may arise from incomplete data, vague descriptions or subjective interpretations of expert judgements often play a major role in data acquisition for the study of complex environmental systems. In soil physics and hydrology vague information may arise when (a) for example, soil hydraulic properties (which are expensive and time consuming to determine) are estimated via pedo-transfer functions (Arya & Paris 1981; Ahuja *et al*. 1985; Tietje & Tapkenhinrichs 1993) from other soil properties (e.g. porosity, grain size distributions, organic matter content) or are even simply derived from soil maps, or (b) boundary conditions (e.g. groundwater level or vegetation parametrisations in evapotranspiration predictions) are derived by rough estimates or expert appraisals. The treatment of such data imprecision in a frequentistic view as probability distributions (e.g. by using mean values, variances and covariances) is a very crucial step and is not always appropriate for the available information (Bardossy & Duckstein 1995; Schulz & Huwe 1997).

Recently, fuzzy set theory has been increasingly used for imprecise information in a non-probabilistic sense, and allows integration of information of different quality into the modelling and evaluation process. Fuzzy sets describing imprecision or vagueness were first introduced by Zadeh (1965) and have been applied in different fields such as decision making and controls (Dubois & Prade 1980). Incorporation of imprecise (fuzzy) information in the geostatistic field was first introduced by Bardossy *et al*. (1988). Fuzzy kriging (Bardossy *et al*. 1990*a*) using fuzzy variogram parameters and fuzzy regression techniques (Bardossy *et al*. 1990*b*) are helpful tools for dealing with the problem of an insufficient number of measurements, which results in fuzzy estimates (kriged values) for each unsampled locations and fuzzy-dependent parameters in the regression procedure.

The availability of fuzzy estimation methods providing fuzzy parameters, as well as imprecision due to indirect measurements or subjective derivations of parameters, suggest the development and application of methods to incorporate fuzzy parameters in modelling procedures. For groundwater flow simulations, this was done by Dou *et al*. (1995). For the unsaturated zone, Schulz & Huwe (1997) presented the incorporation of fuzzy soil-hydraulic properties into closed-form solutions of the Darcy–Buckingham equation to analyse the impact on 1-dimensional steady-state water flow and on the estimation of maximum evapotranspiration rates. This paper will provide an extension to that by considering 1-dimensional steady-state water flow in a layered soil profile and by comparing a closed form solution of the fuzzy flow problem with the incorporation of a finite difference solution into a nonlinear optimisation procedure.

FUZZY SET THEORY

Fuzzy sets

In general, fuzzy logic represents an extension of the classic binary logic, with the possibility of expressing sets without clear boundaries or partial memberships of elements to a given set. The central concept of fuzzy set theory is the membership function, which numerically represents the degree to which a given element belongs to the set. Formally a fuzzy set (subset) is defined as follows:

DEFINITION 1. *Let X be a universe set of elements. A is called a fuzzy (sub)set of X, if A is a set of ordered pairs:* $A=\{(x,\ \mu_A(x)),\ x\in X,\ \mu_A(x)\in[0,1]\}$, *where* $\mu_A(x)$ *is the grade of membership of x in A.*

The closer $\mu_A(x)$ is to 1, the more x belongs to the set A; the closer it is to 0, the less it belongs to A. In this way fuzzy sets allow flexible expression of uncertainties for set descriptions like 'the set of fast cars' or 'the set of possible values for saturated hydraulic conductivity'. In an alternative way, the membership grade (x) is also named as grade of credibility or grade of possibility for a given element and a fuzzy set describes our strength or degree of acceptance of the possible value of a specific variable or parameter.

Fuzzy numbers

A special case of fuzzy sets are the so-called fuzzy numbers, which are defined on the set of real numbers. They have to fulfil the following three conditions (Definitions 2–4):

DEFINITION 2. *A fuzzy subset* $A=\{(x,\ \mu_A(x))\ |\ x\in\Re\}$ *is called normal, if at least one x exists, such that* $\mu_A(x)=1$.

DEFINITION 3. *A fuzzy subset* $A = \{(x, \mu_A(x)) \mid x \in \Re\}$ *of a set of real numbers is called convex, if for any x, y, z* $\in A$ *with* $x < z < y$: $\mu_A(z)$ $\min[\mu_A(x), \mu_A(y)]$.

DEFINITION 4. *A fuzzy subset A is called a fuzzy number, if A is a normal, convex fuzzy subset of the set of real numbers.*

α-level cut, support

A 'horizontal representation' of fuzzy sets is given by its α-level cuts. Formally it is described as follows:

DEFINITION 5. *The* α *level cut* (A_α) *of a fuzzy subset A is the set of those elements, which have at least a membership value greater than or equal to* α:

$$A_\alpha = \{x \in X, \mu_A(x) \geq \alpha\}$$

The support of a fuzzy member (supp) *is the set of real numbers:*

$$\text{supp}(A) = \{x \mid \mu_A(x) > 0\}$$

Owing to the convexity assumption of fuzzy numbers, the level cuts describe sets of numbers in $\Re$ (intervals) with a given minimum likeliness (acceptance) α. With decreasing values of α, which is also called level of credibility or presumption, increasing widths of intervals of real numbers are obtained. This horizontal representation is very close to human thinking where uncertainty and imprecision is usually expressed by associating different 'intervals of confidence' (widths of intervals) with different 'levels of presumptions' (Kaufmann & Gupta 1991) and accepting as wide a range of parameter values as possible with a decreasing level of presumption.

Acquisition of fuzzy numbers

Figure 1 shows an example of a fuzzy number with an α-level cut and its support. This so-called triangular fuzzy number (TFN) is defined by specifying three numbers. Kaufmann & Gupta (1991) proposed the answering of three questions for the construction of a TFN out of an expert appraisal: 'What is the smallest value given to the

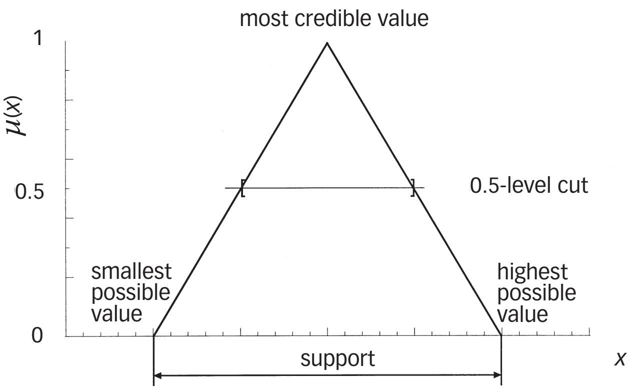

Figure 1 | Membership function of a fuzzy number, an a-level cut and its support.

uncertain parameter? What is the highest? Further, if we were authorised to give one and only one value, what value should we give?' The most credible value is given a membership value of 1; numbers that fall short of the smallest possible value and exceed the highest possible value get membership values of 0. Intermediate membership grades are obtained here just by linear interpolation.

In general, we are not restricted to TFN: any other type of membership function can be used as long as it fulfils Definitions 2–4. Some examples will be shown later. Further methods for the derivation of membership functions are given by, for example, Dubois & Prade (1980), Civanlar & Trussell (1986) and Turksen (1991).

Mathematical operations on fuzzy numbers

Mathematical point-to-point operations are extended to be defined on fuzzy sets by the use of the extension principle:

DEFINITION 6. *If* $X_1, \ldots, X_n$ *and Y are sets, and f is a mapping from the cartesian product* $X_1 \times \ldots \times X_n$ *to Y, then f can be extended to operate on the cartesian product* $A_1 \times \ldots \times A_n$ *of fuzzy subsets of* $X_1, \ldots, X_n$ *with membership functions* $\mu_{A1}(x), \ldots, \mu_{An}(x)$. *The image of* $A_1 \times \ldots \times A_n$ *in Y is the fuzzy subset B with the membership function:*

$$\mu_B(y) = \begin{cases} \sup\{\min(\mu_{A1}(x_1), \ldots, \mu_{An}(x_n)), y = \mathrm{f}(x_1, \ldots, (x_n)\} \\ 0, \text{ if no } (x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n, \\ \qquad \text{such that } \mathrm{f}(x_1, \ldots, x_n) = y \end{cases}$$

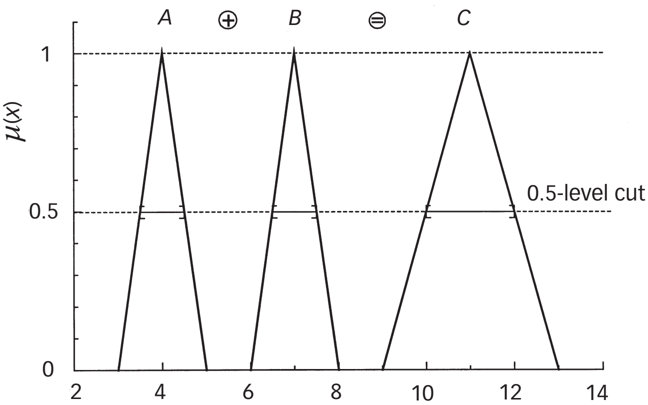

Figure 2 | Addition of two triangular fuzzy numbers.

The direct application of the extension principle is very often an involved and extensive procedure. By using α-level cuts (Definition 5), operations on fuzzy sets can be reduced to operations within the interval arithmetic (Dubois & Prade 1980; Kruse *et al*. 1993). As an illustration of this method, Figure 2 shows the simple addition of two fuzzy numbers *A* and *B*. The interval boundaries of the α-level cuts of the resulting fuzzy number *C* are obtained by the minimum and maximum possible values, when any numbers out of the corresponding α-level cuts of *A* and *B* are added. Here, the interval boundaries of the α-level cuts of *C* are just calculated by simply adding the left and right interval boundaries of the α-level cuts of *A* and *B*. In cases where more complex functions or operators have to operate on fuzzy numbers, this procedure will result in nonlinear numerical optimization problems. This will be shown in the next section, where a finite difference solution scheme to solve a steady-state water flow problem is extended to operate on fuzzy soil hydraulic properties and fuzzy boundary conditions.

WATER TRANSPORT

Steady-state water flow for a layered soil profile

Water flow in the unsaturated zone for a rigid, non-swelling porous media and in the absence of thermic and osmotic gradients can be described by the Darcy–Buckingham equation (Darcy 1856; Buckingham 1907) and is here formulated under the assumption of isotropy

$$\vec{q} = -K(\psi,\vec{x}) \cdot \nabla h, \tag{1}$$

with $\vec{q}$ = volumetric water flux density [L T^{-1}], $K(\psi,\vec{x})$ = hydraulic conductivity [L T^{-1}], ψ = soil water pressure [L], $\vec{x}$ = space vector [L], ∇ = gradient operator, z = vertical co-ordinate (pointing upward) [L] and $h = \psi + z$ = hydraulic potential [L].

To model water transport, a proper knowledge of the soil hydraulic properties, namely the hydraulic conductivity function $K(\psi)$ is required. This relation is generally expressed as a simple closed-form function, where a few parameters characterise the shape of the relation (Mualem 1986). One frequently used expresion in analytical solutions of Equation (1) is the exponential relation introduced by Gardner (1958):

$$K(\psi) = K_s \cdot \exp\,(c \cdot \psi), \tag{2}$$

with K_s = saturated hydraulic conductivity [L T^{-1}] and c = empirical coefficient related to soil texture with respect to pore size distribution [L^{-1}]. Assuming a 1-dimensional, homogeneous soil profile under steady infiltration conditions Equation (1) is solved by using Equation (2) as expression for $K(\psi)$. For the limit $\psi = \psi_{lb}$ at $z = 0$ (lower boundary condition such as groundwater level) and after some integration and rearrangement we obtain (Kutilek & Nielsen 1994):

$$\psi(z) = 1/c \cdot (\ln[\exp(-z \cdot c) \cdot q_0 + K_s \cdot \exp(-c \cdot (z - \psi_{lb})) - q_0] - \ln(K_s)), \tag{3}$$

with q_0 = water flux through upper boundary [L T^{-1}] and ψ_{lb} = soil water pressure at the lower boundary [L]. The recursive application of Equation (3) is necessary to describe steady-state water flow in a layered soil profile: by starting with the lowest horizon, the use of (3) allows calculation of the matrix potential at its upper boundary, which then in a next step is used as a lower boundary condition (ψ_{lb}) for the description of the soil water pressure heads in the next horizon (Kutilek & Nielsen 1994,

p. 134). This procedure can formally be described by an operator F:

$$\psi(z) = F(K_{s,1}, \ldots, K_{s,n}, c_1, \ldots, c_n, q_0, \psi_{lb}, z), \quad (4)$$

with $K_{s,i}$ = saturated hydraulic conductivity of horizon i [L T^{-1}], c_i = empirical coefficient related to soil texture of horizon i [L^{-1}] and F = operator for the multiple application of Equation (3) for a layered soil profile as described in the previous text.

Numerical solution

The method described by Equations (3) and (4) already indicates that even simple process descriptions with closed-form solutions may result in very complex calculation schemes. However, the consideration of, for example, depth-dependent sinks and sources, the extension of 2- or 3-dimensional flow problems or describing transient conditions even makes the application of numerical methods (e.g. finite difference or finite element) absolutely necessary. In the following, we solve the previously defined water-flow problem (Equations 3 and 4) by a finite difference method (see, for example, Hornung & Messing 1984). Applying Equation (1) under steady-state conditions and without sinks and sources results in:

$$\frac{\partial}{\partial z}\left(K(\psi,z)\cdot\frac{\partial h}{\partial z}\right) = 0, \quad (5)$$

with $\partial/\partial z$ = (partial) derivation to z. Discretising the profile into n equidistant compartments with nodes i and using a central differential quotient, Equation (5) can be approximated for every node by:

$$\begin{aligned} 0 &= \frac{\partial}{\partial z}\left(K(\psi,z)\cdot\frac{\partial h}{\partial z}(\psi + z)\right)\Bigg|_i \\ &\approx \frac{K_{i+0.5}}{\Delta z^2}\cdot(\psi_{i+1}-\psi_i) \\ &\quad -\frac{K_{i-0.5}}{\Delta z^2}\cdot(\psi_i-\psi_{i-1}) + \frac{1}{\Delta z}\cdot(K_{i+0.5}-K_{i-0.5}), \end{aligned} \quad (6)$$

with Δz = compartment length [L], ψ_i = soil water pressure of compartment i [L] and $K_{i+0.5}$ = hydraulic conductivity at the boundaries of compartments i and $i+1$ [LT^{-1}]. By using a known water flux at the upper boundary and a known soil water pressure at the lower boundary, Equation (6) results in a tridiagonal equation system which is solved by the Thomas algorithm (e.g. Hornung & Messing 1984). The nonlinearity due to the dependence of the hydraulic conductivity $K_{i\pm0.5}$ on the soil water pressure is solved by the iterative Picard procedure (Celia *et al.* 1990). By using Equation (3) to express hydraulic conductivity, this solution method can be also described as an operator:

$$(\psi_1, \ldots, \psi_n) = FD(K_{s,1}, \ldots, K_{s,n}, c_1, \ldots, c_n, q_0, \psi_{lb}), \quad (7)$$

with ψ_i, $K_{s,i}$, c_i as described before, but for every compartment i and FD = operator for the numerical solution of Equation (5) as outlined above.

Fuzzy water transport

If all input parameters in Equations (4) and (7) are precisely known, also the dependent variables are exactly defined with non-fuzzy (crisp) values. If we assume that the input parameters are imprecise and represented by fuzzy numbers, the resulting soil water pressures will also be fuzzy numbers characterised by their membership functions. Thus Equations (4) and (7) become 'fuzzy operators' and can be reformulated for every depth z or for every compartment i by the following:

$$\begin{aligned} \tilde{\psi}_i &= FD(\tilde{K}_{s,1}, \ldots, \tilde{K}_{s,n}, \tilde{c}_1, \ldots, \tilde{c}_n, \tilde{q}_0, \tilde{\psi}_{lb}) \\ \tilde{\psi}(z) &= F(\tilde{K}_{s,1}, \ldots, \tilde{K}_{s,n}, \tilde{c}_1, \ldots, \tilde{c}_n, \tilde{q}_0, \tilde{\psi}_{lb}, z), \end{aligned} \quad (8)$$

where parameters marked with a tilde represent fuzzy numbers. This extension of operators FD and F to operate on fuzzy sets is done by the use of the extension principle (see Definition 6). The solution of the fuzzy water transport problem (Equations 4 and 7) requires the determination of the membership functions of the output variable (soil water pressure with depth) for the given membership functions of the input parameters. As already illustrated in the previous section, mathematical operations on fuzzy sets are performed at different α-level cuts (see Definition 2 and Figures 1 and 2) and the determination of the

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resulting interval boundaries of the soil water pressures for a given α-level can be mathematically formulated as the following constrained nonlinear optimisation problem:

For any selected membership level $0 \le \alpha \le 1$, and for every depth z or compartment i, $i = 1, \ldots, n$ in Equations (4) and (7) calculate:

$$\min/\max \psi_i \quad \text{or} \quad \min/\max \psi(z),$$

$$\text{with: } \psi_i = FD(K^{\alpha}_{s,1}, \ldots, K^{\alpha}_{s,n}, \ldots, c^{\alpha}_1, \ldots, c^{\alpha}_n, q^{\alpha}_0, \psi^{\alpha}_{lb}),$$
$$\psi(z) = FD(K^{\alpha}_{s,1}, \ldots, K^{\alpha}_{s,n}, c^{\alpha}_1, \ldots, c^{\alpha}_n, q^{\alpha}_0, \psi^{\alpha}_{lb}, z) \quad (9)$$

under the constraints:

$${}_{ub}K^{\alpha}_{s,i} \ge K^{\alpha}_{s,i} \ge {}_{lb}K^{\alpha}_{s,i}, \quad {}_{ub}c^{\alpha}_i \ge c^{\alpha}_i \ge {}_{lb}c^{\alpha}_i \quad {}_{ub}q^{\alpha}_0 \ge q^{\alpha}_0 \ge {}_{lb}q^{\alpha}_0 \text{ and } {}_{ub}\psi^{\alpha}_{lb} \ge \psi^{\alpha}_{lb} \ge {}_{lb}\psi^{\alpha}_{lb},$$

where ${}_{ub}K^{\alpha}_{s,i}$, ${}_{ub}c^{\alpha}_i$, ${}_{ub}q^{\alpha}_0$ and ${}_{ub}\psi^{\alpha}_{lb}$ denote the right and ${}_{lb}K^{\alpha}_{s,i}$, ${}_{lb}c^{\alpha}_i$, ${}_{lb}q^{\alpha}_0$ and ${}_{lb}\psi^{\alpha}_{lb}$ the left interval boundaries of the α-level cuts of the fuzzy inputs in Equation (8).

The mathematical nonlinear constrained optimisation problem which is defined by Equation (8), is solved in this study with the NLPQL routine of Schittkowski (1986). By minimising and maximising the dependent variable (ψ), subject to the constraints given by the fuzzy input variables, the minimum and maximum values of the unknown soil water pressures for a given level of presumption α are obtained. The widths of the resulting intervals can be interpreted as a measure of the uncertainties due to the imprecision and vagueness in the soil hydraulic properties as well as boundary conditions. By repeating this procedure for several different α-level cuts, the complete membership functions of fuzzy output variables can be approximated.

SIMULATIONS

Profile description and boundary conditions

In this study, we describe steady-state water flow in a layered soil profile as it is illustrated in Figure 3. This profile consists of three different layers which are characterised in terms of soil texture as loams and clay. It is also known that there is a groundwater table at about 5 m depth and that there is a net infiltration rate of approximately 2 mm day^{-1}.

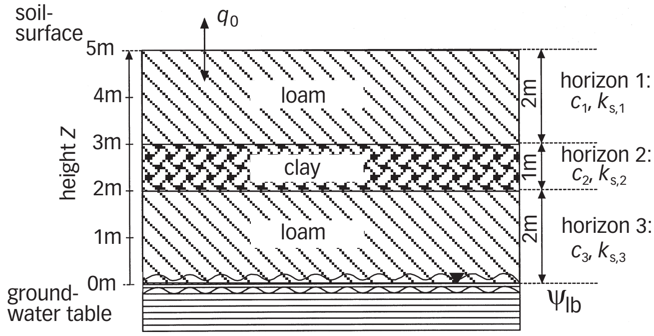

Figure 3 | Illustration of the layered soil profile.

However, although some descriptions of soil hydraulic properties (loam, clay) and the necessary boundary conditions are generally given, this is done in a fairly vague way. The values of saturated hydraulic conductivity for different soil texture classes may vary over orders of magnitude (Ünlü *et al*. 1990; Wierenga *et al*. 1991) and even if measured at certain locations, owing to spatial variations, effective representation of those characteristics will still be lacking (Roth 1995). The same problems exist for the description of the unsaturated hydraulic conductivity $K(\psi)$ via Equation (2) and the determination of factor c to describe the exponential decrease of K with decreasing soil water pressure. Some uncertainties also exist for the exact depth of the groundwater table (owing to variations over time or inexact meter readings, etc.) and the average net infiltration that is calculated as the difference between rainfall and evapotranspiration rates (e.g. imprecision of evapotranspiration estimates).

For instance, we have an expert statement/observation which says that 'the groundwater level is approximately at 5 m depth and will certainly not vary more than ± 10 cm over the time period of interest'. Although it would be very difficult to derive a specific probability distribution function within a stochastic/frequentistic uncertainty analysis, there exists quite an easy and obvious way to translate this statement into a

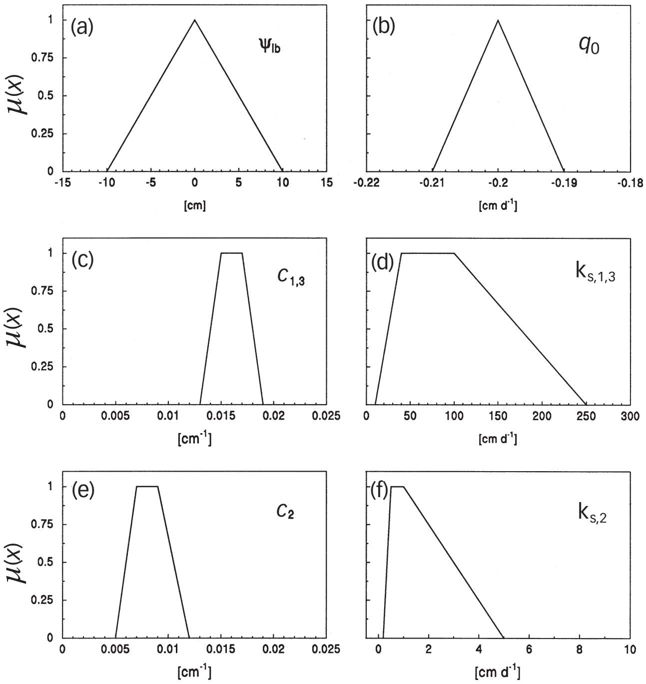

Figure 4 | Membership functions of (fuzzy) input parameters for the steady-state water flow problem.

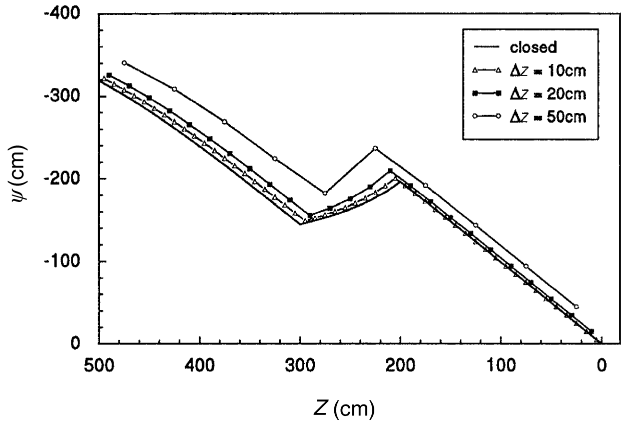

Figure 5 | Comparison of a closed-form solution of steady-state water flow in a layered soil profile with numerical solutions using different spatial discretisation lengths.

membership function for the now-fuzzy lower boundary condition (ψ_{lb}) of steady-state water flow. Using the concept proposed by Kaufman & Gupta (1991) (which has already been described above) results in the construction of a triangular fuzzy number which is shown in Figure 4a. We assigned a membership value of 1 to the most credible value for the soil water pressure in 5 m depth (0 cm) and membership values of 0 for all the soil pressures lying beyond the range of ± 10 cm given as upper and lower boundaries for possible values by the expert statement. The full membership function is then obtained just by linear interpolation of these three points. Thus the membership value for every single soil water pressure now reflects our 'belief' in or the 'likeliness', 'credibility' or 'possibility' for the occurrence of that value.

In a similar way, we have derived membership functions for the average net infiltration, the saturated hydraulic conductivities and the factor c describing unsaturated hydraulic conductivity for all the three soil layers, which are presented in Figure 4b–f. However, it has to be noticed that in contrast to the membership functions of ψ_{lb} and q_0, where we choose a single value for the most credible value, we selected intervals for the 1.0-level cuts (see Definition 5) of $K_{\mathrm{s},i}$ and c_i, thus representing even larger ranges of possible values for a high level of credibility and resulting in a so-called trapezoidal membership function.

Fuzzy calculations

In the following simulations we systematically investigate the impact of the uncertainties described above on the prediction of soil water pressures within the layered soil profile. Therefore to solve the fuzzy water flow problem, Equation (9) is applied at five different α-level cuts (0.0, 0.25, 0.5, 0.75 and 1.0). By embedding the finite difference scheme into the optimisation procedure, intervals of soil water pressures for every discretized depth are calculated for every α-level, whence the complete membership functions of the (fuzzy) soil water pressures are constructed.

In a first step, we had to select an appropriate spatial discretisation length Δz within the finite difference solution of the steady-state water flow problem. Figure 5 shows a comparison of the closed-form solution (Equation 3) using soil hydraulic properties and boundary conditions as given in Table 1 with numerical solutions (Equation 6) using different values for Δz. It can be seen

Table 1 | Input data for the comparison of closed form and numerical solution method for steady-state water flow in layered soil profile.

c_1/cm^{-1}	$K_{s,1}$/(cm d^{-1})	c_2/cm^{-1}	$K_{s,2}$/(cm d^{-1})	c_3/cm^{-1}	$K_{s,3}$/(cm d^{-1})	q_o/(cm d^{-1})	ψ_{lb}/cm
0.015	70	0.008	0.5	0.015	70	− 0.2	0.0

that we obtain very good results already for a discretisation length of $\Delta z = 10$ cm, which is therefore used in all further calculations.

In the first simulation, we only treat the boundary conditions as fuzzy parameters with membership functions as they are presented in Figure 4a; all other parameters (soil hydraulic properties) are fixed in values as given in Table 1. Figure 6a,b shows the resulting imprecision of the soil water pressure with depth represented by contour plots for the α-levels 0.0, 0.5 and 1.0. It can be seen for both calculations that these are only of minor dimensions and only led to very limited uncertainties (width of the α-level cuts) within all depths.

In the second simulation, we successively investigate the influence of imprecision in the soil hydraulic properties on the predicted soil water pressures, while the boundary conditions are assumed to be well known as given by Table 1. Figure 7a–d shows four examples where only the soil hydraulic properties of the loams (horizons 1 and 3) are assumed to be fuzzy and are characterised by membership functions that are shown in Figure 4c,d ($K_{s,1,3}$, $c_{1,3}$). In (a) and (c) only the soil hydraulic properties of horizon 1, in (b) and (d) only those of horizon 3 are treated as fuzzy inputs, but using different lower boundary conditions ($\psi_{1b} = -100$ cm in c, d). It can be seen from Figure 7a,c that assuming only the characteristics of horizon 1 to be imprecise leads to exactly defined soil water pressures within the lower two horizons ($z = 0$–300 cm), but resulting in more fuzzy predictions towards the upper boundary of the soil profile. For the upper boundary ($z = 500$ cm) the range of soil water pressures with, for example, membership values of greater than 0.0 spans from around −350 to 200 cm, thus being much wider than resulting uncertainties from assumed imprecision in the boundary conditions. A very similar result is obtained when using a different value for the lower boundary condition ($\psi_{1b} = -100$ cm), with only small shifts of the lower boundaries of the α-level cuts. In Figure 7b,d the same calculations are presented, but assuming that only the characteristics of the loam in horizon 3 are imprecise. Although under these conditions the predicted soil water pressures in the upper horizon have nearly not been affected, it has to be noted that the widths of the different α-level cuts are very much dependent on the lower boundary condition used.

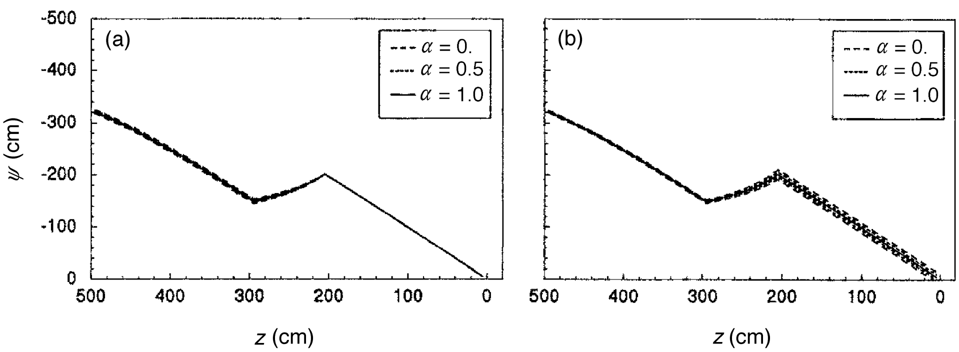

Figure 6 | Membership functions for soil water pressure with depth when (a) only the lower boundary condition ψ_{lb} and (b) only the upper boundary condition q_0 are treated as fuzzy inputs.

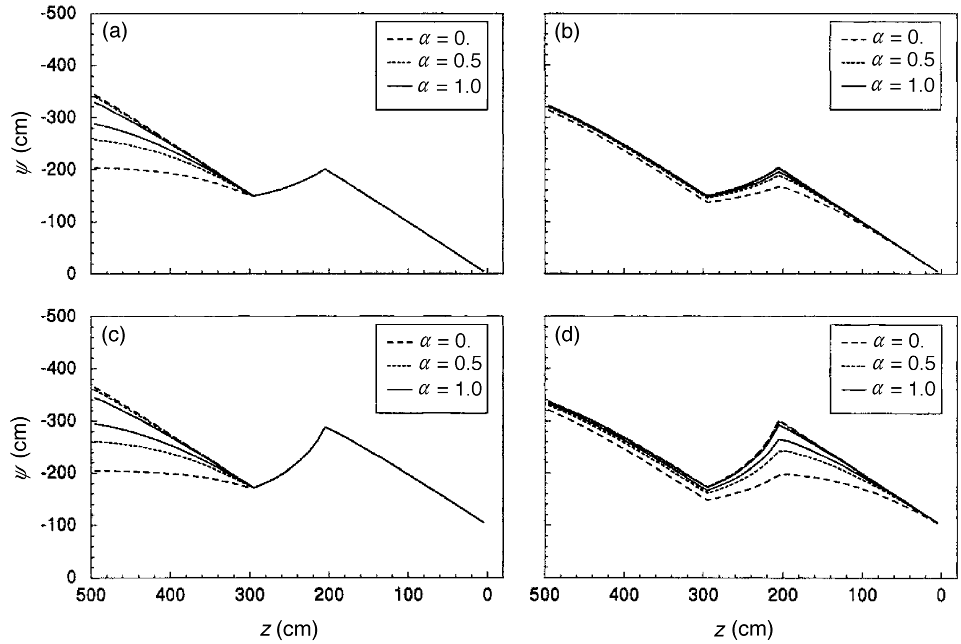

Figure 7 | Four solutions of the fuzzy water flow problem with imprecise parameters for the horizons 1 and 3 (loam). In (a) only those of horizon 1, in (b) only those in horizon 3, are given as fuzzy inputs (ψ_{lb}=0 cm). (c) and (d) show the corresponding results for a different lower boundary condition (ψ_{lb}=−100 cm).

Figure 8a shows the resulting membership functions when only the soil hydraulic properties of the clay (horizon 2) are assumed to be of a fuzzy nature, whereas for the calculations illustrated in Figure 8b all three horizons are considered to be fuzzy simultaneously. It can be seen that the imprecision of the soil hydraulic properties of horizon 2 ($z=200$–300 cm) still affects the soil water pressure at the upper boundary on a very high level (support -380 to -260 cm at $z=500$ cm, Figure 8a). This much greater influence compared with effects of horizon 3 (Figure 7b,d) is due to the fact that the uncertainty of saturated hydraulic conductivity of the clay horizon ranges very close to the absolute value of the upper boundary condition, and thus produces large variations in soil water pressure.

Finally, Figure 9 shows the resulting membership functions for the soil water pressures, when all parameters, boundary conditions as well as soil hydraulic

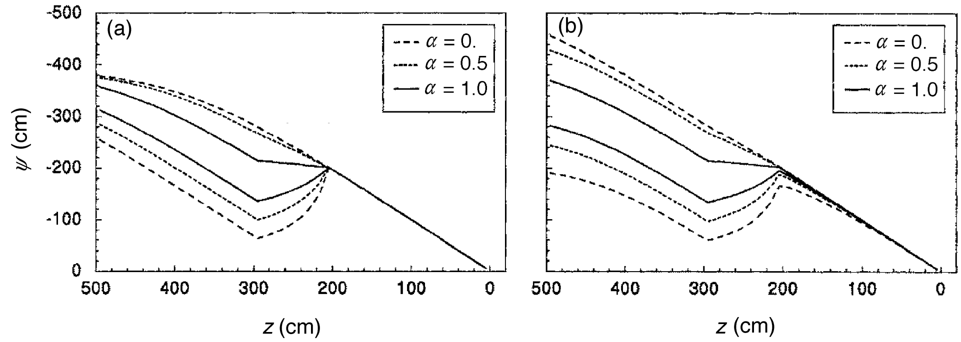

Figure 8 | Solution of the fuzzy water flow problem when (a) only soil hydraulic properties of horizon 2 are considered as fuzzy inputs, (b) when all three horizons are treated as fuzzy.

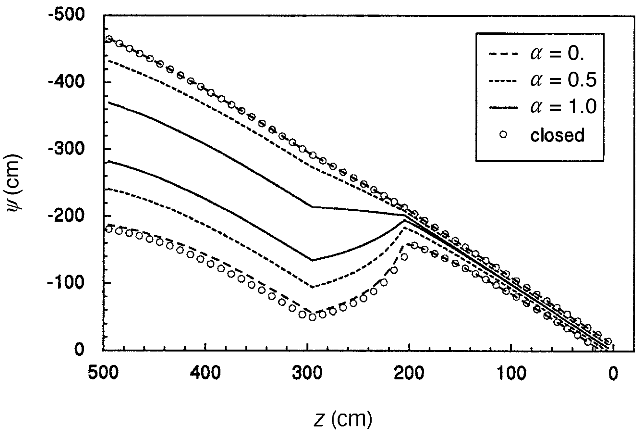

Figure 9 | The resulting membership functions for soil water pressure when soil hydraulic properties as well as boundary conditions are characterised with fuzzy inputs. A comparison with a closed form solution (see text) is given for the 0.0-level cut (○).

properties of all three horizons are treated to be fuzzy, expressed by membership functions illustrated in Figure 4. As was already analysed in the previous simulations, the fuzziness of soil water pressures is mainly dominated by the imprecision of the soil hydraulic properties. In addition to that, Figure 9 also allows a comparison of the applied method of embedding the finite difference solution method for steady-state water flow into the nonlinear optimisation code NLPQL as described by Equation (9), with a closed form solution to solve the fuzzy water flow problem.

Owing to continuity and the strict monotonicity of Equation (3) relative to all their input parameters, the minimum and maximum ψ_i values for every depth and for every α level can just be calculated by using the upper and lower boundary values of the corresponding α-level cuts of the input parameters:

$$\min \psi(z) = F({}_{\mathrm{ub}}K^{\alpha}_{\mathrm{s},1}, \ldots, {}_{\mathrm{ub}}K^{\alpha}_{\mathrm{s},n}, {}_{\mathrm{lb}}c^{\alpha}_{1}, \ldots, {}_{lb}c^{\alpha}_{n}, {}_{\mathrm{ub}}q^{\alpha}_{0}, {}_{lb}\psi^{\alpha}_{\mathrm{lb}})$$
$$\max \psi(z) = F({}_{\mathrm{lb}}K^{\alpha}_{\mathrm{s},1}, \ldots, {}_{lb}K^{\alpha}_{\mathrm{s},n}, {}_{ub}c^{\alpha}_{1}, \ldots, {}_{ub}c^{\alpha}_{n}, {}_{\mathrm{lb}}q^{\alpha}_{0}, {}_{\mathrm{ub}}\psi^{\alpha}_{\mathrm{lb}}), \quad (10)$$

with the same parameter description as for Equation (9). The comparison is shown for the 0.0-level cut and it can be seen that the differences are very similar to those found when the closed-form solution for the water flow problem is compared with the finite difference solutions with different discretisation lengths (Figure 5). This was repeated for several α-level cuts (no figure) and always showed a very good match of both methods.

DISCUSSION AND CONCLUSIONS

An approach based on fuzzy set theory has been presented to express imprecision of soil hydraulic properties and boundary conditions in a non-probabilistic sense. These uncertainties have been incorporated into steady-state water predictions. After focusing on the deduction of membership functions for the given information, which proved to be very simple and flexible, the methodology of extending the finite-difference solution scheme to operate on fuzzy input data has been applied within different simulations. By subsequently incorporating the imprecision of soil hydraulic properties and boundary conditions, the sensitivity of soil water pressure in different depths to the uncertainties of the input data has been investigated. It is obvious that imprecision increases with the number of imprecise input parameters; however, compared with the vagueness of the boundary conditions, it is concluded for this system that the imprecision of soil hydraulic properties is of major significance. It can also be seen that apart from the degree of uncertainty (number of uncertain parameters, width of α-level cuts) the effect on the predicted variables (here soil water pressures) is also very much dependent on the system context, which means the same 'kind of uncertainty' may lead to different consequences dependent on boundary conditions and their spatial distribution. The successful incorporation of the finite difference scheme into the numerical optimisation scheme to operate on fuzzy input data will allow us in the future to extend this method towards more complex numerical process descriptions, such as water and solute transport in the unsaturated zone under transient flow conditions, for two dimensions and/or with additional source and sink terms.

When using fuzzy set theory to express imprecision, the derivation process of a membership function for a specific parameter seems to be the most crucial step.

Different authors would certainly transfer the available information to different forms of membership function. However, by presenting the complete membership function (Figure 4), this subjective derivation process remains generally transparent and comprehensible. The membership function of the dependent variables (here soil water pressures) thus fully represents the resulting imprecision of the outcomes. To evaluate the obtained membership functions, a certain α-level has to be chosen (corresponding to an interval of real numbers) on which a decision, a further interpretation or proceeding (e.g. for evapotranspiration predictions) will be linked. This corresponds to choosing a 1σ, 2σ, or 3σ confidence interval in a stochastic framework. The α-level that will be used depends on the decision problem itself and the more financial and human safety aspects are affected, the lower this level will be selected. But as in statistics, this choice still remains a subjective one and no general application rule can be deduced.

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