What can be learned from rotational motions excited by earthquakes?

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Accepted 1996 December 16. Received 1996 November 21; in original form 1996 August 15

**SUMMARY**

One answer to the question posed in the title is that we will have more accurate data for arrival times of \(SH\) waves, because the rotational component around the vertical axis is sensitive to \(SH\) waves although not to \(P-SV\) waves. Importantly, there is another answer related to seismic sources, which will be discussed in this paper.

Generally, not only dislocations commonly used in earthquake models but also other kind of defects could contribute to producing seismic waves. In particular, rotational strains at earthquake sources directly generate rotational components in seismic waves. Employing the geometrical theory of defects, we obtain a general expression for the rotational motion of seismic waves as a function of the parameters of source defects.

Using this expression, together with one for translational motion, we can estimate the rotational strain tensor and the spatial variation of slip velocity in the source area of earthquakes. These quantities will be large at the edges of a fault plane due to spatially rapid changes of slip on the fault and/or a formation of tensile fractures.

**Key words:** earthquake-source mechanism, seismic modelling, seismology, theory.

### 1 INTRODUCTION

There are many reports about rotations of tombstones and of stone lanterns during large earthquakes (e.g. Yamaguchi & Odaka 1974). Only translational ground motions, however, have been observed in instrumental measurements of seismic waves, and quantitative measurements of rotational ground motions have not been made until quite recently. Bouchon & Aki (1982) simulated rotational ground motions near earthquake faults buried in layered media for strike-slip and dip-slip fault models, and obtained a maximum rotational velocity of \(1.5 \times 10^{-3} \text{ rad s}^{-1}\) produced by a buried 30 km long strike-slip fault with slip of 1 m. Their simulation shows that the rotational motions are small compared with the amplitude of the translational motions. The difficulty experienced in measuring the rotational motions excited by earthquakes is caused by the lack of technology for measuring such small rotational motions.

Recently, Nigbor (1994) succeeded in measuring rotational and translational motions with a new angular measurement sensor (Morris 1971) at a surface station during a non-proliferation experiment at the Department of Energy Nevada Test Site using a very large (1 kiloton) chemical explosion. The sensor will allow us to measure the rotational ground motions of seismic waves in the near future.

What will rotational motions excited by earthquakes tell us? We will have accurate data for arrival times of \(SH\) waves, because the rotational component around the vertical axis is sensitive to \(SH\) waves although not to \(P-SV\) waves. A vertically heterogeneous, isotropic, elastic medium is the first-order approximation of the Earth's interior, so that we can expect to have a clear \(SH\)-wave onset in records of the rotational component around the vertical axis. When we try to separate \(SH\) waves from \(P-SV\) waves using translational motions, we need to rotate two horizontal components into radial and transverse components. To do this, we have to know the incident directions of seismic waves. Now, we will be able simply to detect the onsets of \(SH\) waves using the rotational component only.

The purpose of this paper is to elucidate another possibility, which is related to seismic sources. The familiar source model of earthquakes is a dislocation model concerned with a discontinuity of displacement across internal surfaces in a continuum, but not with a rotation across the surfaces. The rotation naturally generates rotational seismic waves. Defects in the continuum other than dislocations inclusive of tensile fractures are sources of such rotational motions. Teissreyre (1973) discussed, based on micropolar theory, the possibility of rotational motions in source processes. There are several theories that deal with elastic continua with internal defects (e.g. Kondo 1949a,b, 1957; Bilby, Bullough & Smith 1955; Kröner 1958; Amari 1962, 1968; Mura 1963, 1972; deWit 1973; Kossecka & deWit 1977a,b). Among them, the most general perspective is provided by the geometrical theory of defects.
(Kondo 1949a,b, 1957). Based upon this theory, we will derive a general expression for rotational motions of seismic waves. This, as well as a general expression for translational motion, completely specifies the seismic waves.

We start with a sketch of the geometrical theory of defects. After outlining the fundamental concepts in Section 2.1, we present a space–time formulation in Section 2.2 so that we can treat time-dependent problems. An expression for defects is generalized by using geometrical quantities introduced in these subsections. Two kinds of defects, dislocations and disclinations, play important roles. Interestingly, their motions are closely related to each other, being characterized by continuity equations. Section 2.3 is devoted to their derivation, using the space–time formulation of Section 2.2. The strain related to earthquakes will be less than 10^{-3}, the magnitude of strain of granite rocks just before brittle fracture measured in triaxial compression tests (e.g. Mogi 1970). So that we can employ a linear approximation to obtain our main result, simple formulae (69) and (70) for rotational and translational motions excited by earthquakes are used. Finally in Section 4, we present a simulation of rotational motion excited by an earthquake, and discuss the possibility of detection in real situations.

2 GEOMETRICAL THEORY OF DEFECTS

2.1 Fundamental concepts

Take an elastic material with internal defects, and cut it into many small pieces free of defects. Then, information on the defects will be reflected in the relative location of the neighbouring pieces after being relaxed to a strain-free state. Using this idea in the framework of non-Riemannian geometry, a general theory was developed by Kondo (1949a,b, 1953, 1955, 1957), and his successors (e.g. Amari 1962, 1968, 1981; Shiozawa 1980). First we give a sketch of their theory.

Let x(t = 1, 2, 3) be a coordinate system in the real Euclidean space $E_3$ in which a material body $M$ is immersed. The coordinate system can be Cartesian or spherical etc., although it need not be orthogonal. To each point of $M$ we attach the coordinate $x(t = 1, 2, 3)$ that the point occupies in $E_3$. We now have a manifold, called a material manifold hereafter. Let $e_i(P)$ (i = 1, 2, 3) be the natural bases (vectors directed along the $x'$ axis) at point $P$ of the manifold. The tangent space $T_P$ is a vector space spanned by the vectors $e_1(P), e_2(P), e_3(P)$.

The material body, in general, is not free from strains, because of external forces applied from the outside or plastic defects existing inside. Take out of the material an element small enough to be free from defects. The element can then be transformed elastically to the strain-free state by cutting it off from the surroundings and releasing it from the constraints of the surroundings. This procedure is named naturalization. The states before and after the naturalization are called the natural state and the natural state, respectively (see Fig. 1). We assume that the natural state is realized by an affine transformation of the torn small material element.

Suppose the vectors $e_1, e_2, e_3$ become $\hat{e}_1, \hat{e}_2, \hat{e}_3$, respectively, through naturalization. The squared length $dS^2$ of $dx'$ after naturalization is written as

$$dS'^2 = g_{ij}dx'dx',$$

with

$$g_{ij} = (\hat{e}_i, \hat{e}_j),$$

which defines a metric tensor in the tangent space $T_P$. Here $(,)$ indicates the inner product in $E_3$. The summation convention for repeated indices is followed throughout.

Take two points $P(x')$ and $P'(x' + dx')$ in $M$. We cut off two neighbouring material elements located around $P(x')$ and $P'(x' + dx')$ from the surroundings, and release them to the natural state. In this process, the two elements are kept connected. By this naturalization, the basis vectors $e_i$ at $P$ and $e'_i$ at $P'$ are transformed to $\hat{e}_i$ and $\hat{e}'_i$, respectively. In general, $e_i \in T_P$ and $e'_i \in T_{P'}$ are vectors belonging to different vector spaces, so they are not comparable. However, we can compare the naturalized vectors $\hat{e}_i$, and $\hat{e}'_i$ because they are in the same Euclidean space $E_3$. The difference $\hat{e}'(P) - \hat{e}(P)$ is expressed as $dx^\gamma \Gamma^\alpha_{\beta\gamma,\alpha}$ up to $O(dx^2)$ with a suitable quantity $\Gamma^\alpha_{\beta\gamma}$. This quantity $\Gamma^\alpha_{\beta\gamma}$ enables us to relate two tangent spaces $T_P$ and $T_{P'}$: the origin of $T_P$ is mapped to $dx$ in $T_P$,

$$dx = dx'\hat{e}_1,$$

and the basis vector $e_i(P')$ is mapped to $e_i(P) + dx^\gamma \Gamma^\alpha_{\beta\gamma}e_i(P)$ in $T_P$, $\text{Av}_i = dx^\gamma \Gamma^\gamma_{\beta\gamma}e_i$. (3)

Eqs (3) and (4) define an affine connection in the material manifold $M$, where $\Gamma^\gamma_{\beta\gamma}(P)$ are called parameters of affine connection. We have thus incorporated the effects of the defects into the rules of connection.

We assume that there is no strain gradient between neighbouring naturalized elements; that is, $T_P$ is mapped to $T_{P'}$ by rotation only. This connection is called a metric connection, characterized by the vanishing of the covariant derivative of the metric tensor $g_{ij}$:

$$V_kg_{ij} = \frac{1}{2}g_{ij}V_k - \Gamma^\alpha_{\beta\gamma}g_{m\gamma} - \Gamma^\alpha_{\beta\gamma}g_{m\gamma} = 0,$$

where $V_k$ and $\partial_k$ denote a covariant derivative and a partial derivative with respect to $x^k$, respectively.

Let us see what the connection implies. Take a closed circuit $L$ in $M$ which encircles an area $\Sigma$ passing through a point $P$, and develop tangent spaces along $L$ using the above connection, as is shown in Fig. 2. Let $P = P_0, P_1, P_2, ..., P = P$ be a sequence of neighbouring points on $L$. We can map the tangent space $T_{P_0}$ to the tangent space $T_{P_{n+1}}$. By repeating this procedure, the tangent space $T_{P_0} = T_{P_{n+1}}$ is mapped to $T_{P_n}$ and then to $T_{P_{n+1}}$, and so on, and finally we have a mapping from $T_{P_0}$ onto itself by going round a loop $L$. Let the origin of $T_{P_0}$ be mapped to a point $\Delta\theta \hat{e}_i \in T_{P_0}$ and let a vector $\delta^\beta\hat{e}_k$ be mapped to $(\delta^\beta + \Delta\theta^\beta)\hat{e}_k$. Then, we have

$$\Delta^\theta = \int_\Sigma S_{ij}^\gamma df'^\gamma,$$

$$\Delta^\theta = \int_\Sigma \frac{1}{2} \delta^\beta \int_{R_{ij}^\gamma df'^\gamma},$$

where the integration is taken over $\Sigma$, and $df'^\gamma$ represents small surface elements of $\Sigma$. The quantities $S_{ij}^\gamma$ and $R_{ij}^\gamma$ in the integrands are a torsion tensor and a curvature tensor, respectively, and are defined by

$$S_{ij}^\gamma = \Gamma_{ij\alpha}^\gamma,$$

$$R_{ij}^\gamma = 2\partial_k \Gamma_{ij\alpha}^\gamma + 2\Gamma_{ij\alpha}^\gamma \Gamma_{\alpha\beta\gamma}.$$

Here $[\gamma]$ denotes the alternation operation applied to indices in $[\gamma]$, except for those in $[\gamma]$. See Appendix A for explicit expressions.
The relations (6) and (7) allow us the following physical interpretations. Let us cut a loop of small material elements from the surroundings along \( L \). This forms a ring, which cannot in general be put in the natural state without cutting it further into a non-ring form. When we cut the ring at point \( P \) and release the strain, we have a sequence of naturalized elements, which does not form a ring in the natural state without cutting it further into a non-ring form. When we cut the ring at point \( P \) and release the strain, we have a sequence of naturalized elements, which does not form a ring in the natural state (see Fig. 2). The discrepancy of position \( \Delta x^i \) is given by eq. (6) and the change \( \Delta v^k \) of the vector \( v^k \) is given by eq. (7). The defect due to the torsion tensor is called the dislocation, which is commonly used in earthquake source models. The curvature tensor, on the other hand, gives another defect called disclination.

We further introduce two strain tensors, a strain tensor \( e_{ij} \) and a rotational strain tensor \( \gamma_{ijk} \), defined by

\[
e_{ij} = \frac{1}{2} (a_{ij} - g_{ij}),
\]

\[
\gamma_{ijk} = \Lambda_{ijk} - \Gamma_{ijk},
\]

where \( a_{ij} = (e_i, e_j) \) is a metric tensor in \( E_3 \), agreeing with \( g_{ij} \) in the case of Cartesian coordinates. In eq. (11),

\[
\Lambda_{ijk} = \frac{1}{2} \{ \partial_i a_{jk} + \partial_j a_{ik} - \partial_k a_{ij} \},
\]

and \( \Gamma_{ijk} \) is defined as \( \Gamma_{ijk} = g_{ik} \Gamma_{jk} \).

The strain tensor \( e_{ij} \) represents how the material element at \( P \) is deformed from the natural state: the difference between the length in the real state and the one in the natural state is given as \( (a_{ij} dx^i dx^j)^{1/2} - (g_{ij} dx^i dx^j)^{1/2} \approx e_{ij} dx^i dx^j \). On the other hand, the antisymmetric part of \( dx^m \Gamma_{ikl}^m \) shows the relative rotation between the two bases at \( P \) and \( P' \) in the natural state. \( \Lambda_{ijk} \) is the corresponding quantity when there are no defects, so that \( \gamma_{ijk} \) contains information on the intrinsic rotation between two neighbouring elements in the natural state.

The torsion tensor and the curvature tensor can be rewritten in terms of the strain tensor and the rotational strain tensors as

\[
S_{ijk} = -\gamma_{ijk} - \hat{\partial}_i e_{jk},
\]

\[
R_{ijk} = -2 \partial_i \gamma_{ijk} - 2 e^{klm} \gamma_{ilm} \gamma_{jkm}.
\]

The compatibility condition of \( e_{ij} \) and \( \gamma_{ijk} \) is characterized by the vanishing of the torsion and the curvature tensors: \( S_{ijk} = R_{ijk} = 0 \). The compatibility condition in the linear approximation means that there exist a vector \( u_i \) and a tensor \( \omega_{ij} \) such that

\[
e_{ij} = \hat{\partial}_i u_{jk},
\]

\[
\gamma_{ijk} = \hat{\partial}_i \omega_{jk} + \hat{\partial}_j \omega_{ik},
\]

where \((...)\) and \([...]\) denote mixing and alternation, respectively (see Appendix A).

### 2.2 Space–time formulation

We extend the formulation in the previous section to deal with moving defects. We add a time axis to the Euclidean space \( E_3 \) to make a 4-D Euclidean space \( E_4 \). Each point of a deformed material can be marked with a 4-D coordinate system \( x^a (\mu = 0, 1, 2, 3) \), where \( x^a = t \) and \( x^i (i = 1, 2, 3) \) denote the time and the space coordinates, respectively. We then have a 4-D material manifold \( M \). Hereafter, the Greek indices run over both the time and the space coordinates, while Roman indices run over the space coordinates only, unless otherwise stated.
The metric tensor defined by \( g_{\mu \nu} \), tensor and a curvature tensor as of the naturalization involving time as follows. Let Comparing eqs (20), (21), and (23), we have We can easily obtain through the naturalization as in the previous subsection. Since we deal with the non-relativistic case, we define the metric connection in the 4-D sense, \( \Gamma^\mu_{\nu \lambda} \) is defined above agree with those in the previous subsection because \( \Gamma^\mu_{\nu \lambda} = 0 \). We also note that the connection is a metric connection in the 4-D sense, \( \tilde{V}_k g_{\mu \nu} = 0 \), by virtue of eqs (5), (24), and (25).

From the definition of the strain tensor, \( e_{\mu \nu} \), and eq. (26), we have We define, in the same manner as the 3-D case, a torsion and a curvature tensor as

\[
S_{\kappa \nu \\ \lambda} = \Gamma^\kappa_{\nu \lambda},
\]

\[
R_{\mu \nu \kappa \lambda} = 2\Gamma^\kappa_{\mu \lambda} \Gamma^\mu_{\nu \kappa} + 2\Gamma^\mu_{\nu \mu} \Gamma^\mu_{\kappa \lambda},
\]

and a strain tensor and a rotational strain tensor as

\[
e_{\mu \nu} = \frac{1}{2} (a_{\mu \nu} - g_{\mu \nu}),
\]

\[
\gamma_{\mu \nu \kappa} = \Lambda_{\mu \nu \kappa} - \Gamma_{\mu \nu \kappa}.
\]

Here,

\[
\Lambda_{\mu \nu \kappa} = \frac{1}{2} [\partial_\mu a_{\kappa \nu} + \partial_\kappa a_{\mu \nu} - \partial_\nu a_{\mu \kappa}],
\]

and \( a_{\mu \nu} = (e_{\mu \nu}, e_{\nu \mu}) \) is a metric tensor; the components relating to the time axis are \( a_{00} = 1 \) and \( a_{i0} = a_{0i} = 0 \). A 4-D counterpart of eqs (13) and (14) now reads:

\[
S_{\lambda \kappa \nu \mu} = -\gamma_{\lambda \nu \kappa \mu} - \gamma_{\kappa \mu \lambda \nu}.
\]

\[
R_{\mu \nu \kappa \lambda} = -2\delta_{\nu \kappa} \gamma_{\mu \lambda} + 2g^{\mu \nu} \gamma_{\lambda \kappa \mu}.
\]

We note that the 3-D components of the torsion and curvature tensors defined above agree with those in the previous subsection because \( \Gamma^\mu_{\nu \lambda} = 0 \). Second, by Galilei transformation, the vector \( \mathbf{e}_0 + \mathbf{v}' \mathbf{e}_i \), indicating the motion of \( \mathbf{e}_i(P) \), is naturalized to a unit vector of the time axis, \( \mathbf{e}_0(P) \), by virtue of eq. (19), eqs (21), and (25).

\[
\partial_\mu v^\mu = -\gamma_{\mu \lambda} v^\lambda.
\]

From the definition of \( \gamma_{\mu \nu \kappa} \), \( \gamma_{0 \mu \nu} \) is represented as

\[
\gamma_{0 \mu \nu} = -\frac{1}{2} (s_{\kappa \lambda} \Gamma^\kappa_{\mu \lambda} - g_{\kappa \lambda} \Gamma^\mu_{\kappa \lambda}) \approx -\frac{1}{2} \partial_\nu v^\mu,
\]

which gives a gradient of the velocity.

2.3 Density and current tensors of defects

In this section we confine ourselves to linear problems where higher-order terms of geometrical quantities are negligible. We will derive systematically the results that were obtained based on a linear theory of elasticity by Mura (1963, 1972, 1982), de Wit (1973), and Kosseca & de Wit (1977a,b) and so on. First we introduce two density tensors, \( \tau_{\mu \lambda} \) for dislocation and \( \theta^{ij} \) for disclination, using the torsion tensor (32) and the curvature tensor (33) as follows:

\[
\tau^{ij} = \varepsilon^{ijk} e_k / \sqrt{g},
\]

\[
\theta^{ij} = \varepsilon^{ijkl} e_{kmn} R_{kmn}/4g.
\]

Here and in the following \( e_{ijk} \) is Eddington’s epsilon:

\[
e_{ijk} = \varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of } (1, 2, 3) \\ -1 & \text{if } ijk \text{ is an odd permutation of } (1, 2, 3), \\ 0 & \text{otherwise} \end{cases}
\]

and \( g = \det(g_{ij}) \). The meanings of \( \tau_{ij} \) and \( \theta^{ij} \) can be seen from relations (6) and (7). Since \( \varepsilon^{ij} / \sqrt{g} \) behaves as a tensor under the 3-D transformation, \( i \to i' = i, x \to x' = x'(x) \), so do \( \tau^{ij} \) and \( \theta^{ij} \). The inverse relations of eqs (37) and (38) are written as

\[
S_{ik} = \frac{1}{2} \sqrt{g} e_{ijk} \varepsilon^{ij},
\]

\[
R_{klnm} = g_{ik} h_{jmn} \theta^{ij}.
\]

Their current tensors \( I^m_k \) and \( J^m_{km} \) are defined as

\[
I^m_k = S_{m0} e^k_0 = -S_{0m} e^k_0,
\]

\[
J^m_{km} = \frac{1}{2} e^{opc} R_{kopq} / \sqrt{g}.
\]

Let us now derive continuity equations in the linear approximation, which will justify the above introduction. Hereafter we assume that the coordinate system \( x' \) is Cartesian.

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The continuity equations take the form
\[\dot{\varepsilon}_{ij} + \dot{\gamma}^{ik} \delta_{kj} + \varepsilon_{ij} J_p^p = 0,\]  
(44)
\[\dot{\gamma}^{ij} + \dot{\gamma}^{ik} \delta_{kj} = 0.\]  
(45)
At the same time, the dislocation tensor and inclination tensor are related to each other by
\[\dot{\varepsilon}_{ij} + \varepsilon_{ij} \rho^{ik} = 0,\]  
(46)
\[\dot{\gamma}^{ij} = 0.\]  
(47)
These equations follow from Bianchi's identities, which are fulfilled by the torsion and curvature tensors (Schouten 1954),
\[\dot{R}_{ijkl}^k = 2S_{[ik}^{,s} \Gamma_{l]jk}^s,\]  
(48)
\[R_{ijkl}^* = 2V_{[ik}^{,s} S_{l]jk}^s - 4S_{ik}^{,s} S_{ljk}^s,\]  
(49)
and an identity for the curvature tensor under the metric connection: \(R_{ijkl}^k = 0\) (see Appendix B). Here we note that the geometrical theory presents a more general framework than the dislocation theory: dislocation is a defect accompanied by the vanishing of the curvature tensor \(R_{ijkl}\) and is sometimes called a plasticity of distant parallelism.

We will rewrite the above expression using the strain tensor \(\gamma_{ij}\) and the rotational strain tensor \(\dot{\gamma}_{ij}\) for later convenience. In general, the deformation is not completely elastic but partly plastic, so we decompose the tensors into elastic and plastic components as
\[\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{pl},\]  
(50)
\[\gamma_{ij} = \gamma_{ij}^{el} + \gamma_{ij}^{pl},\]  
(51)
where the superscripts \(E\) and \(*\) denote the elastic part and the plastic part, respectively. When defects are present, a displacement field has a discrepancy and a change of vector on a defect surface, as shown in Fig. 2, which are given by eqs (6) and (7) in general. Therefore, it is found that the spatial derivative of displacement has a singularity on the defect surface, and this singular part is found to be the plastic parts of these strain tensors. When no defects are present, the plastic fields are compatible, i.e. the plastic parts of these strain tensors can be derived from a plastic displacement, \(u_t^*\), and a plastic tensor, \(\omega_{ij}^*\), using eqs (15) and (16) which satisfy the compatibility conditions, \(S_{ij} = R_{ijkl} = 0\). We sometimes conventionally introduce the total displacement \(u_t\) and \(\omega_{ij}\) satisfying eqs (15) and (16), even when \(u_t^*\) and \(\gamma_{ij}^{pl}\) do not vanish. It should be noted, however, that \(u_t\) and \(\omega_{ij}\) are no longer single-valued functions because of the incompatibility. This is easily understood in the dislocation model of the earthquake: \(u_t\) is discontinuous on the fault plane. The above convention will be permitted as long as multi-valuedness does no harm (see Section 3).

Because the elastic deformation satisfies the compatibility conditions, \(\varepsilon_{ij}^{pl}\) and \(\gamma_{ij}^{pl}\) do not contribute to the torsion tensor and the curvature tensor. Therefore, neglecting the higher-order terms, the torsion tensor and the curvature tensor are written as
\[S_{ij} = -\gamma_{ij}^{pl} - \varepsilon_{ij}^{pl},\]  
(52)
\[R_{ijkl} = -2\eta_{ij}^{pl} \Gamma_{kjl}^{pl}.\]  
(53)
Substituting eqs (52) and (53) into eqs (37) and (38), we have
\[\varepsilon_{ij} = -\varepsilon_{ij}^{pl} (\gamma_{ij}^{pl} + \varepsilon_{ij}^{pl}) + \varepsilon_{ij}^{pl},\]  
(54)
\[\gamma_{ij} = -\gamma_{ij}^{pl} \varepsilon_{ij}^{pl}.\]  
(55)
where
\[\kappa_i^j = \frac{1}{2} \varepsilon_{ijk}^* \varepsilon_{jk}^* \text{ or } \gamma_{ij}^* = \varepsilon_{ijk}^* \varepsilon_{kj}^* .\]  
(56)
The current densities \(I_{mn}\) and \(J_{m}^*\) are similarly given by
\[I_{mn} = \frac{1}{2} (\varepsilon_{mn} \kappa^i_j + \varepsilon_{j}^* \kappa^i_m - \varepsilon_{j}^* \kappa^i_m),\]  
(57)
\[J_{m}^* = -\varepsilon_{mn} \kappa^i_m + \varepsilon_{jm}^* \kappa^i_m .\]  
(58)

3 FORMULATION OF ROTATIONAL AND TRANSLATIONAL MOTIONS DUE TO EARTHQUAKES

In order to develop equations for seismic waves from buried plastic deformations, we shall describe such sources in terms of a transformational (or stress-free) strain introduced in the source volume. This problem is a generalization of Eshelby's 'transformation problem' to an anisotropic medium and an inhomogeneous stress-free strain (Eshelby 1957). We note here that the concept of the transformational strain in Eshelby (1957) is identical to the plastic strain of this paper. Similar approaches have been made by Mura (1972, 1982), deWit (1973), Kossecka & deWit (1977a,b), etc., who gave displacements and distortions caused by dislocations and/or disclinations in a linear elastic continuum. Kossecka & deWit (1977b) proposed a simple expression for the rotational velocity of a distribution of moving dislocations and disclinations. However, they assumed the existence of a 'plastic velocity' whose physical meaning was not clear.

Since we are interested in the ground motions excited by an earthquake, we assume that there exists a bounded region \(V_t\) outside of which defects vanish and the material deforms elastically. It should be noted that an earthquake is an internal process of the Earth, and the displacement field excited by an earthquake has a quiescent past. In this section, we will derive a simple formula for rotational motion caused by an earthquake.

To help understand the following formal derivation, let us briefly illustrate Eshelby's recipe of how to implement defects in the material. First, we separate the source material \(V_t\) by cutting along the surface \(S\) enclosing \(V_t\) and removing \(V_t\) from the surroundings. Second, we let the source material undergo plastic strain (stress-free strain), which causes deformation without changing the stress within \(V_t\). Third, we apply extra surface tractions or fictitious body forces that will restore the source volume to its original shape. Then, we put the source material back in the hole and weld the material across \(S\), third, we apply extra surface tractions or fictitious body forces that will restore the source volume to its original shape. Then, we put the source material back in the hole and weld the material across \(S\), and find the resulting displacement caused by the plastic strain. So, the moral of Eshelby's method is that a defect described by the plastic strain \(\varepsilon_{ij}^p\) can be simulated by the extra traction given by \(-C^{ijkl}\varepsilon_{ij}^p\), or the fictitious body force given by \(-C^{ijkl}\varepsilon_{ij}^p\), where \(C^{ijkl}\) is a tensor of elastic constants (e.g. Aki & Richards 1980).

The above procedure is simply done as follows. Keeping in mind that \(\varepsilon_{ij}^p\) is stress-free, we apply Hooke's law,
\[\sigma_{ij} = C^{ijkl} \varepsilon_{ij}^p,\]  
(59)
where \(\rho_{ij} \dot{\sigma}_{ij} = \partial_j \sigma_{ij} \) to the equation of motion without body forces:
\[\rho \ddot{\rho}_{ij} \dot{\sigma}_{ij} = \partial_j \sigma_{ij} .\]  
(60)
Here \( \rho \) is the density of the medium. The tensor \( C^{ijkl} \) is a symmetric tensor with respect to the \( i \) and \( j \) indices \((C^{ijkl} = C^{jikl})\) due to a symmetric characteristic of elastic stress, and with respect to the \( k \) and \( l \) indices \((C^{ijkl} = C^{ijlk})\) due to a symmetric characteristic of elastic strain. Therefore, we obtain the equation of motion as follows:

\[
\rho \ddot{\mathbf{u}}_m = \int_{-\infty}^{\infty} dt \int C^{ijkl} \partial_j \partial_k G_m(x, t - \tau; \xi, 0) \times \mathbf{e}_i(\xi, \tau) \; dV(\xi),
\]

(61)

Using the Green’s function \( G_m(x, t - \tau; \xi, 0) \), a seismic displacement (translational motion) generated by a plastic strain can be expressed as

\[
u_m(x, t) = - \int_{-\infty}^{\infty} dt \int C^{ijkl} \partial_j G_m(x, t - \tau; \xi, 0) \mathbf{e}_i(\xi, \tau) \; dV(\xi),
\]

(62)
after integration by parts. The Green’s function \( G_m(x, t - \tau; \xi, 0) \) satisfies an equation of motion,

\[
\rho \ddot{G}_m = \int_{-\infty}^{\infty} dt \int C^{ijkl} \partial_j \partial_k G_m(x, t - \tau; \xi, 0) \times \mathbf{e}_i(\xi, \tau) \; dV(\xi),
\]

(63)

and represents the \( n \)th component of the displacement resulting from a unit impulse source of the \( i \)th component applied at \( x = \xi \) and \( t = \tau \) (e.g. Aki & Richards 1980). \( \mathbf{e}_i \) and \( \delta(x - \xi) \) or \( \delta(t - \tau) \) denote the Kronecker delta and delta functions, respectively. Differentiating eq. (62), we have

\[
\partial_m \nu_m(x, t) = - \int_{-\infty}^{\infty} dt \int C^{ijkl} \partial_j \partial_k \partial_m G_m(x, t - \tau; \xi, 0)
\]

(64)

\[
\times \mathbf{e}_i(\xi, \tau) \; dV(\xi),
\]

where we have used integration by parts with respect to space and the boundary condition that the plastic strain \( \mathbf{e}_m^p \) vanishes outside \( V_e \). The partial differentiation with respect to \( \tau \) or an \( m \)th component of \( \xi \) is explicitly shown by an index with a prime, for example \( \partial_\tau' \) and \( \partial_m' \). From eq. (54), we have

\[
\partial_m \nu_m(x, t) = \int_{-\infty}^{\infty} dt \int C^{ijkl} \partial_j G_m(x, t - \tau; \xi, 0)
\]

(65)

\[
\times \mathbf{e}_i(\xi, \tau) \; dV(\xi),
\]

where \( \kappa^p = \kappa_i^p j \) from which it follows that

\[
\partial_m \nu_m(x, t) = \int_{-\infty}^{\infty} dt \int C^{ijkl} \partial_j G_m(x, t - \tau; \xi, 0)
\]

(66)

\[
\times \{ \mathbf{e}_m^p(\xi, \tau) - \kappa^p \mathbf{e}_i(\xi, \tau) \} \times \mathbf{e}_i(\xi, \tau) \; dV(\xi),
\]

where we have used integration by parts again and the symmetric property of \( C^{ijkl} \). Here, a reciprocal relation of Green’s function, \( G_m(x, t - \tau; \xi, 0) = G_m^*(x, t - \tau; \xi, 0) \), is also employed. The second term on the right-hand side of eq. (66) vanishes because we assume the plastic strain outside the source volume \( V_e \) to be zero. Partially differentiating eq. (66) with respect to time and applying the continuity equations, we finally obtain

\[
\partial_\tau \partial_m \nu_m(x, t) = - \int_{-\infty}^{\infty} dt \int C^{ijkl} \partial_j \partial_k G_m(x, t - \tau; \xi, 0)
\]

(67)

\[
\times \{ \mathbf{e}_m^p(\xi, \tau) - \kappa^p \mathbf{e}_i(\xi, \tau) \} \times \mathbf{e}_i(\xi, \tau) \; dV(\xi),
\]

The details are shown in Appendix C. The second term on the right-hand side of eq. (67) is zero because defects are limited in \( V_e \) and \( x \) is outside \( V_e \). The integrand of the third term can be simplified using eq. (57), and eq. (67) is rewritten as follows:

\[
\partial_\tau \partial_m \nu_m(x, t) = - \int_{-\infty}^{\infty} dt \int C^{ijkl} \partial_j \partial_k G_m(x, t - \tau; \xi, 0)
\]

(68)

\[
\times \{ \mathbf{e}_m^p(\xi, \tau) - \kappa^p \mathbf{e}_i(\xi, \tau) \} \times \mathbf{e}_i(\xi, \tau) \; dV(\xi),
\]

This equation is a generalized version of eq. (38.36) in Mura (1980). We have, then, a simple expression for the rotational velocity of a seismic wave as follows:

\[
\partial_\tau \partial_\theta \mathbf{u}(x, t) = \frac{1}{2} \{ \mathbf{V} \times \partial_\theta \mathbf{u} \}^q
\]

(69)

\[
= - \int_{-\infty}^{\infty} dt \int \mathbf{e}_m^p C^{ijkl} \partial_j \partial_k G_m(x, t - \tau; \xi, 0)
\]

\[
\times \{ \mathbf{e}_p^m(\xi, \tau) - \kappa^p \mathbf{e}_i(\xi, \tau) \} \times \mathbf{e}_i(\xi, \tau) \; dV(\xi),
\]

where we have used integration by parts again and the symmetric property of \( C^{ijkl} \). Here, a reciprocal relation of Green’s function, \( G_m^*(x, t - \tau; \xi, 0) = G_m^*(x, t - \tau; x, 0) \), is also employed. The second term on the right-hand side of eq. (68) vanishes because we assume the plastic strain outside the source volume \( V_e \) to be zero. Partially differentiating eq. (66) with respect to time and applying the continuity equations, we finally obtain

\[
\partial_\tau \partial_m \nu_m(x, t) = - \int_{-\infty}^{\infty} dt \int C^{ijkl} \partial_j \partial_k G_m(x, t - \tau; \xi, 0)
\]

(70)

\[
\times \{ \mathbf{e}_m^p(\xi, \tau) - \kappa^p \mathbf{e}_i(\xi, \tau) \} \times \mathbf{e}_i(\xi, \tau) \; dV(\xi),
\]

These two equations show that we can estimate the tensors \( \gamma_{\alpha \beta}^p, \hat{\epsilon}_{\alpha \beta}^p \) using the translational and rotational motions of seismic waves simultaneously.
4 POSSIBILITY OF ESTIMATING A ROTATIONAL STRAIN TENSOR DUE TO AN EARTHQUAKE

To obtain information about $\gamma_{0kl}$, $\delta_k v_l$ due to an earthquake based on observed seismic waves, it is necessary that $\gamma_{0kl}$, $\delta_k v_l$, appearing in eq. (69) should have a magnitude equal to or higher than $\epsilon v_{G*}$. Let us estimate their magnitudes in a simple case. In this section, we also assume that the coordinate system $x'$ is Cartesian.

From eqs (6), (7), (40), and (41), the discrepancy $\Delta u_k$ and the change $\Delta \sigma_{ij}$ of a vector $v_i$ due to defects are rewritten as

$$\Delta u_k = \int_S \rho_{pk} d^f \rho_k = \int_S \phi_{ij} d\Sigma_i,$$

$$ \Delta \sigma_{ij} = \frac{1}{2} \int_S \rho_{pk} d^f \rho_k = \frac{1}{2} \int_S \phi_{ij} d\Sigma_i = \epsilon_{ijk} \theta^{ij} d \Sigma_i,$$

where $d^f \rho_k = \rho_{pk} d^f \Sigma_i / \sqrt{g}$ and $d \Sigma_i$ is the $i$th component of the area vector of $\Sigma$.

Now, we consider a simple model of an earthquake such that a material below a surface $S$ has been plastically displaced with respect to a material above $S$ by a constant amount which represents a rigid motion; the upper plane (denoted by $S'$) slips by a constant Burgers vector $b^*$ relative to the lower plane (denoted by $S$), and $S'$ twists against $S$ by a constant Frank vector $a^*$ at point $x_0$ (see Fig. 3). Introducing $\phi_{ij}^p$ and $\beta_{ij}^p$ satisfying

$$ \phi_{ij}^p = -\epsilon_x^p (b_i^j b_j^p + \epsilon_{ijk} \phi_{k}^{*p}), $$

$$ \beta_{ij}^p = -\epsilon_x^p \phi_{ij}^{*p}, $$

and defining $\pi_{ij}^{*p} = \epsilon_x^p \phi_{ij}^{*p}$, we see that the integrand of the first term on the right-hand side of (67) reads

$$ I_{mn} = \frac{1}{2} (\pi_{xm}^{*p} + \beta_{mn} b_{mm}^* - \delta_m v_n), $$

where

$$ \phi_{ij} = -\delta_i (S) \Omega_j^{*s}, $$

$$ \beta_{ij} = -\delta_i (S) \{ b_i^* + \epsilon_{ijk} \Omega_j^{*p} (x^k - x^k) \}, $$

$$ \pi_{0ij} = \epsilon_{ijk} \phi_{ij}^{*p} = -\epsilon_{ijk} \delta_0 (S) \Omega_j^{*p} \delta (S). $$

Here $\delta_i (S)$ is defined by

$$ \delta_i (S) = \int_S (x_i - x_i) n_i dS (x_i) = \int_S \delta (x_i - x_i) dS_i $$

(see Appendix D).

During the last decade, waveform inversion of strong motion and teleseismic data excited by earthquakes have been used to determine the spatial and temporal variations of slip on the fault plane (e.g. Hartzell & Heaton 1983; Kikuchi & Fukao 1987; Takeo 1988), clarifying that the slip distribution on the fault plane of large earthquakes is generally very complex. The spatial resolution of the slip distribution obtained in these inversions is more than a few kilometres at best; the resolution depends on the minimum wavelength of the seismic waves employed in the inversion. Now, we assume that the Burgers vector is constant in the region $S$ whose length-scale is shorter than $10^3$ m, because we have no information about slip distributions in source areas smaller than several square kilometres. The order of $\Omega^*$ is not clear because the value related to an earthquake has not been obtained until now. One possible rotational deformation during an earthquake is a tensile fracture at an edge of the fault plane, as illustrated in Fig. 4. Let us assume that the order of $\Omega^* (O(\Omega^*))$ is $10^{-2}$ rad ($\approx 1'$) and the area ($S'$) concerning this phenomenon is an order of magnitude less than the area of the fault plane ($S$), i.e., $S' \approx S \times 0.1$. The spatial variation of slip velocity ($v_i$) also has a large uncertainty due to the lack of observations. It seems, however, reasonable to assume that the spatial scale of variation is not shorter than the scale of $S$. Taking appropriate values for other parameters, $O(b_i^*) \approx 1$ m, $O(v_i) \approx 1$ m s$^{-1}$, and $O(\Omega^*) \approx 1$ km$^{-1}$, we get $O(\pi_{0ij}) \approx 10^{-3}$, $O(\beta_{ij}^p) \approx 10^{-3}$, and $O(\delta_m v_n) \approx 10^{-3}$. This result shows that $\pi_{0ij}$, $\beta_{ij}^p$, and $\delta_m v_n$ have similar orders of magnitude, and it is possible to estimate $\pi_{0ij}$ and $\delta_m v_n$, i.e. $\gamma_{0kl}$ and $\delta_k v_l$, using the translational and rotational motions of seismic waves simultaneously.

Figure 3. Schematic figures of dislocation (upper) and disclination (lower). The dislocation line is defined as part of the boundary of a slip plane $S$. The disclination line is created by twisting surface $S''$ against surface $S'$ by rotation through an angle $\Omega^*$ at point $x_0$. $L$ is a Burgers circuit. When vector $\Omega^*$ is normal to $S'$, the disclination is of a twist type, and it is of a wedge type when $\Omega^*$ is on $S'$. Figure 4. Schematic image of a tensile fracture at an edge of a fault plane.
Takeo and H. M. Ito

Velocity (cm/sec)

\[ \omega_z \Delta = 1 \text{ km} \]

\[ \omega_r \Delta = 5 \text{ km} \]

\[ \omega_r \Delta = 10 \text{ km} \]

\[ \omega_r \Delta = 15 \text{ km} \]

\[ \omega_r \Delta = 25 \text{ km} \]

\[ \omega_r \Delta = 50 \text{ km} \]

\[ u_t \text{ (velocity: cm/sec)} \]

\[ 105 \]

\[ 28 \pm 1.5 \]

\[ 26 \]

\[ 4.0 \]

\[ 2.7 \]

\[ 0.68 \]

\[ 0.15 \]

\[ 18 \text{ sec} \]

Table 1. Velocity structures used in the simulation of rotational and translational ground motions.

<table>
<thead>
<tr>
<th>( V_p )</th>
<th>( V_s )</th>
<th>( \rho )</th>
<th>depth</th>
<th>( Q_p )</th>
<th>( Q_s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(km/s)</td>
<td>(km/s)</td>
<td>(kg/m³)</td>
<td>(km)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \times 10^3 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.80</td>
<td>1.30</td>
<td>2.30</td>
<td>0.00</td>
<td>200</td>
<td>100</td>
</tr>
<tr>
<td>5.60</td>
<td>2.90</td>
<td>2.50</td>
<td>2.70</td>
<td>400</td>
<td>200</td>
</tr>
<tr>
<td>6.00</td>
<td>3.40</td>
<td>2.60</td>
<td>6.10</td>
<td>500</td>
<td>230</td>
</tr>
<tr>
<td>6.80</td>
<td>4.00</td>
<td>3.00</td>
<td>19.00</td>
<td>600</td>
<td>270</td>
</tr>
</tbody>
</table>

\( 2 \times 10^{-3} \text{ rad s}^{-1} \) at a hypocentral distance of 5 km (\( \Delta = 5 \text{ km} \)). This simulation shows that this angular sensor will be able to record rotational motion excited by an earthquake of moment magnitude 6 (seismic moment: \( 10^{18} \text{ N m} \)) up to a hypocentral distance of 25 km.

5 CONCLUSION

We extend the dislocation model of earthquakes to cover phenomena generated by rotational motions in the source area. The extended model includes defects, dislocation and disclination, which are shown to be completely characterized by geometrical quantities, a torsion tensor and a curvature tensor. We derive a set of continuity equations that are dependent on densities of dislocation and disclination and their currents.

Employing the continuity equations, we derive a simple expression for the rotational velocity of seismic waves. Combining the rotational motions of seismic waves with the translational motions, we can estimate the tensors \( \gamma^e_{mn} \) and \( \delta_{mn} \), i.e. the rotational strain tensor and the spatial variation of slip velocity. These quantities will be large at the edges of a fault plane due to spatially rapid changes in the slip on the fault and/or a formation of tensile fractures. We estimate from a simulation that the angular sensor now available will detect the rotational motions from earthquakes with magnitude 6 or larger if the hypocentral distance is shorter than 25 km.

ACKNOWLEDGMENTS

This research was partially supported by a Grant-in-Aid for Scientific Research (B), the Ministry of Education, Science, Sports and Culture, Japan.

REFERENCES


APPENDIX A: ALTERNATION AND MIXING OPERATION

The alternation [] over \( p \) indices is obtained by adding \( p! \) signed isomers with permuted indices and by dividing the alternation by \( p! \), where the sign is positive if the permutation is even and negative if odd. If indices have to be singled out, the sign \([\cdot]\) is used for example,

\[
P_{(x,y)} = \frac{1}{6} \left( P_{x,y} + P_{y,x} + P_{y,x} - P_{x,y} - P_{y,x} - P_{x,y} \right).
\]

Eqs (8) and (9) are written explicitly as

\[
S_{ij}^k = \Gamma_{(ij)}^k = \frac{1}{2} \left( \Gamma_{ij}^k - \Gamma_{ji}^k \right),
\]

\[
R_{ij}^k = 2\partial_{ij}^k + \frac{1}{2} \left( \Gamma_{ijkl} + \Gamma_{jilk} \right) = \partial_{ij}^k - \partial_{ji}^k + \Gamma_{lm}^k \Gamma_{ij}^l - \Gamma_{lm}^k \Gamma_{ji}^l.
\]

The mixing \( \cdot \) over \( p \) indices is effected in the same way as the alternation except that the sign is always positive. For example,

\[
P_{(x,y)} = \frac{1}{6} \left( P_{x,y} + P_{y,x} + P_{y,x} + P_{x,y} + P_{y,x} + P_{x,y} \right),
\]

and eq. (15) is written as

\[
e_{ij} = \frac{1}{2} (\tilde{e}_{ij} + \tilde{e}_{ji}).
\]

APPENDIX B: CONTINUITY EQUATIONS

From the definition of the curvature tensor, it is obvious that \( R_{ijkl}^k = 0 \). The connection treated in this paper is metric \( \nabla_{ijkl} = 0 \), so we have \( R_{ijkl} = 0 \) (Schouten 1954). Eq. (49) becomes

\[
R_{ijkl}^k = 2\partial_{ij} S_{kl}^k,
\]

if we neglect terms of order higher than \( O(\Gamma^2_{ij}) \). Employing this equation, we obtain

\[
\partial_p x^p_m = \partial_p e^{kl} S_{ijm} = 6\partial_1 (S_{23})_m = \frac{1}{2} \delta_{lp} R_{klpm}.
\]

The second term of eq. (46) can be rewritten as follows:

\[
e_{ip} \partial_p = \frac{1}{4} e_{ip} \pi^{kl} \delta^{mp} R_{klp} = \frac{1}{2} \delta_{lp} R_{klpm}.
\]

if we use \( g = 1 \) for the Cartesian coordinate system and \( R_{ijkl} = 0 \). Then, we have the relation

\[
\partial_p x^p + e_{ip} \partial_p = 0.
\]

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Employing a formula for Eddington's epsilon of
\[
g_{ij} \epsilon^{mn} = g_{ij}^{\mu} g^{\nu m} g^{\rho n} \delta^\nu_\rho \delta^\mu_\nu \delta^\rho_j \delta^i_p,
\]
we see that \(\theta^{ij}\) defined by eq. (38) agrees with the Einstein tensor \(G^{ij}\).
\[
G^{ij} = R^{ij} - \frac{1}{2} R g^{ij},
\]
in a 3-D space. Here \(R^{ij} = g^{ij} g^{mn} R_{mn} - R \delta^{ij}\). The Einstein tensor \(G^{ij}\) generally satisfies
\[
\nabla_i G^i_j = -3 S_{ijk} R_{k\rho}^\rho g^{ij}.
\]
Hence, neglecting the higher-order terms, we have
\[
\delta \theta^{ij} \approx 0.
\]
(47) again
The formula (B6) is obtained from eq. (48) in three dimensions, i.e. the indices run over space coordinates 1, 2, 3, as follows. Multiplying by \(\delta^i_j\) and contracting along \(j\), the left-hand side term of eq. (48) becomes
\[
\delta^i_j V_{i\nu} R_{\nu\rho}^\rho = \frac{1}{3} (V_{i\nu} R_{\nu\rho}^\rho + V_{\nu} R_{\nu\rho}^\rho - V_{\rho} R_{\rho\nu}^\rho),
\]
where \(R_{\mu\nu} = R_{\nu\mu}\). Again, multiplying by \(g^{i\rho}\) and contracting along \(i\), the term on the left-hand side becomes
\[
-(2/3) V_{\nu} (R_{\nu\rho}^\rho - \frac{1}{2} \delta^\rho_\nu) R_{\rho\nu}^\rho),
\]
where \(R_{\rho\nu}^\rho = R_{\nu\rho}^\rho\). When we apply the same procedure to the right-hand side of eq. (48), it becomes \(\delta^\rho_\nu R_{\rho\nu}^\rho \delta^\rho_\nu R_{\rho\nu}^\rho\). Because \(G_{\rho\nu} = R_{\rho\nu}^\rho - \frac{1}{2} \delta^\rho_\nu R_{\rho\nu}^\rho\), we obtain eq. (B6) in three dimensions.

Let eq. (37) be partially differentiated with respect to time:
\[
\delta_0 \epsilon^{* \nu} = \epsilon^{* \nu} \delta_0 S_{i\nu} = \frac{1}{2} \epsilon^{* \nu} R_{i\nu}.
\]
Here, the relation \(R_{i\nu} = 2 S_{i\nu}^*\) deduced from eq. (B1) is used. From the definition of \(J_i^*\), we have the following expression:
\[
\epsilon_{i\nu} J_i^* = R_{i\nu}.
\]
(99) again
Then, eq. (B8) can be rewritten as
\[
\delta_0 \epsilon^{* \nu} + \epsilon^{* \nu} (2 \delta_0 I_{i\nu} + \epsilon_{i\nu} J_i^*) = \frac{1}{2} \epsilon^{* \nu} R_{i\nu}.
\]
Neglecting the higher-order terms in eq. (33) and using eq. (36), the following relation holds:
\[
R_{i\nu} = \frac{1}{2} (\delta_0 J_{i\nu} - \delta_0 J_{\nu i}) = 0,
\]
and, finally, we have
\[
\delta_0 \epsilon^{* \nu} + \epsilon^{* \nu} (2 \delta_0 I_{i\nu} + \epsilon_{i\nu} J_i^*) = 0.
\]
(44) again
In a linear approximation, we have from eq. (48) that
\[
\delta_0 R_{i\nu} = 0,
\]
and it follows that
\[
\delta_0 R_{i\nu} = - \delta_0 I_{i\nu} - \delta_0 R_{i\nu}.
\]
(131)
Let \(\epsilon^{* \nu}\) be partially differentiated with respect to time:
\[
\delta_0 \epsilon^{* \nu} + \epsilon^{* \nu} \delta_0 R_{i\nu} = - \epsilon^{* \nu} \delta_0 J_i^*.
\]
(143)
then we have
\[
\delta_0 \theta^{\nu \rho} + \epsilon^{* \nu} \delta_0 J_i^* = 0.
\]
(45) again

**APPENDIX C: DERIVATION OF EQ. (67)**
The partial differentiation of eq. (64) with respect to time yields
\[
\delta_0 \delta_{0 \mu} \mu_i = \int_{-\infty}^{\infty} dt \int \epsilon_{\mu \nu \rho} C^{\nu \rho} \delta_0 \mu_i (x, t, \xi, 0) \times \delta_0 \mu_i (t, \tau, \xi, 0) dV(\xi)
\]
\[
- \int_{-\infty}^{\infty} dt \int \epsilon_{\mu \nu \rho} C^{\nu \rho} \delta_0 \mu_i (x, t, \xi, 0) \times \delta_0 \mu_i (t, \tau, \xi, 0) dV(\xi) + \epsilon_{\mu \nu \rho} \epsilon_{\mu \nu \rho} dV(\xi).
\]
(151)
The first term \((I_1)\) on the right-hand side of the above equation can be rewritten using eq. (44) as
\[
I_1 = - \int_{-\infty}^{\infty} dt \int \epsilon_{\mu \nu \rho} C^{\nu \rho} \delta_0 \mu_i (x, t) \times \delta_0 \mu_i (t, \tau, \xi, 0) dV(\xi)
\]
\[
- \int_{-\infty}^{\infty} dt \int \epsilon_{\mu \nu \rho} C^{\nu \rho} \delta_0 \mu_i (x, t) \times \delta_0 \mu_i (t, \tau, \xi, 0) dV(\xi) + \int_{-\infty}^{\infty} dt \int C^{\nu \rho} \epsilon_{\mu \nu \rho} f_i \delta_0 \mu_i dV(\xi).
\]
(152)
where we also use the relations
\[
\epsilon_{\mu \nu \rho} \delta_0 \mu_i = \delta_0 \mu_i - \delta_0 \mu_i \delta_0 \epsilon_i^\rho,
\]
\[
C^{\nu \rho} \epsilon_{\mu \nu \rho} = \rho \delta_0 \mu_i (x, t) \times \delta_0 \mu_i (t, \tau, \xi, 0) \delta(\xi - x) \delta(t - \tau),
\]
and
\[
\epsilon_{\nu \rho} C^{\nu \rho} = 0\] (due to \(C^{\nu \rho} = C^{\rho \nu}\)).
(154)
From eq. (58), we have a relation
\[
\delta_0 \nu_0 = - J_i^* + \delta_0 \nu_i^*.
\]
(66)
Substituting this equation into the integrand of the second term \((I_2)\) on the right-hand side of eq. (C1), we have
\[
I_2 = - \int_{-\infty}^{\infty} dt \int C^{\nu \rho} \epsilon_{\mu \nu \rho} f_i \delta_0 \mu_i dV(\xi)
\]
\[
- \int_{-\infty}^{\infty} dt \int C^{\nu \rho} \epsilon_{\mu \nu \rho} \delta_0 \mu_i (x, t) \times \delta_0 \mu_i (t, \tau, \xi, 0) dV(\xi)
\]
\[
- \int_{-\infty}^{\infty} dt \int C^{\nu \rho} \epsilon_{\mu \nu \rho} f_i \delta_0 \mu_i dV(\xi).
\]
(17)
Finally, eq. (C1) is written as
\[
\delta_0 \delta_{0 \mu} \mu_i (x, \tau, \xi, 0) = - 2 \int_{-\infty}^{\infty} dt \int C^{\nu \rho} \epsilon_{\mu \nu \rho} f_i \delta_0 \mu_i (x, t, \xi, 0) \times \delta_0 \mu_i (t, \tau, \xi, 0) dV(\xi)
\]
\[
\times I_i (\xi, \tau - t) dV(\xi).
\]
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APPENDIX D: BURGERS VECTOR AND FRANK VECTOR

We will derive expressions for the dislocation tensor $\sigma^{ij}$ and the disclination tensor $\phi^{ij}$ in the case of a constant Burgers vector $b$ and a constant Frank vector $\Omega$. Since the discrepancy $\Delta u_j$ and the change $\Delta \phi_j$ of a vector $\phi_j$ are related to the Burgers vector $(b_j)$ and the Frank vector $(\Omega^j)$ by

\begin{align}
\Delta u_j &= b_j + [\Omega \times (x - x_0)]_j, \\
\Delta \phi_j &= (\Omega \times \phi)_j = \epsilon_{ijk} \Omega^k \phi^j,
\end{align}

by comparing these with eqs (71) and (72), we have formulae relating $\sigma^{ij}$ and $\phi^{ij}$ to $b_j$ and $\Omega^j$:

\begin{align}
\Delta \phi_j &= \epsilon_{ijk} \Omega^k \phi^j = \delta^i \epsilon_{ijk} \int_{\Sigma} \phi^{ij} d\Sigma, \\
\therefore \quad \Omega^j &= \int_{\Sigma} \phi^{ij} d\Sigma,
\end{align}

Employing eqs (73) and (74), we have

\begin{align}
\Omega^j &= - \oint_L \phi^i dL', \\
b_j &= - \oint_L [\beta^i - \epsilon_{ijk} \phi^k (x^j - x^j_0)] dL',
\end{align}

where we have used Stokes' theorem. It is straightforward to check that eqs (76) and (77) satisfy (D6) and (D7) if we note that

\begin{align}
\int_S \delta^i(L) dS &= \int_L \delta^i(L) dL = \left\{ \begin{array}{ll}
1 & \text{if } L \text{ crosses } S \text{ positively,} \\
0 & \text{if } L \text{ does not cross } S, \\
-1 & \text{if } L \text{ crosses } S \text{ negatively,}
\end{array} \right.
\end{align}

because the curve $L$ crosses the surface $S$ once as shown in Fig. 3. Here $\delta^i(L)$ is defined by

\begin{align}
\delta^i(L) &= \int_L \delta(x - x^i_0) dL(\xi) = \int_L \delta(x - x^i_0) dL.
\end{align}