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RESERVOIR MECHANISM

In an Aquifer of Arbitrary Boundary Shape

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The physical problem dealt with in this article is the unsteady ground-water flow in an aquifer of limited horizontal extent and arbitrary boundary shape.
The unsteady ground-water level is described by a differential equation, the solution of which includes the solution of an eigen-value problem by a numerical method.
Boundary conditions are provided by nature in the form of watertight rocks, rivers, lakes or any kind of constant water level in hydraulic contact with the aquifer.
The run-off is found to behave as a sum of flows through infinitely many linear reservoirs, corresponding to the eigen-functions.
The resulting equations for ground-water level and run-off discharge are sums of convolution integrals of the infiltration, an easy process to handle in a digital computer.
Finally, the mathematical model derived is used to analyse the inflow into a water reservoir in Iceland from a nearby lava field, and the run-off is compared with flow data.

MATHEMATICAL FORMULATION

We assume that the ground-water flow is practically horizontal and that the aquifer is isotropic and homogeneous. This kind of flow is governed by the differential equation:

$$\nabla^2 h \equiv \frac{S}{T}\frac{\delta h}{\delta t'} - \frac{R'(x',y',t)}{T}$$

where

$h(x,y,t)$	:	ground-water level, m
t'	:	time, sec
S	:	storage coefficient
T	:	transmissivity, m^2/sec
$R'(x',y',t')$	:	infiltration, m/sec
A	:	aquifer area, m^2

We now introduce the dimensionless variables:

$$x = x'/\sqrt{A}$$
$$y = y'/\sqrt{A}$$
$$t = t' \cdot T/AS$$
$$R = R' \cdot A/T$$

We have introduced:

a length scale	:	$\sqrt{A}$
a time scale	:	$A \cdot S/T$
an infiltration scale	:	T/A

Our equation of flow then reads:

$$\nabla^2 h = \frac{\delta h}{\delta t} - R \qquad (2.1)$$

Stationary boundary conditions are frequently encountered in nature, see Fig. 1.

$$h = h_0\,(x,y) \text{ on } C$$

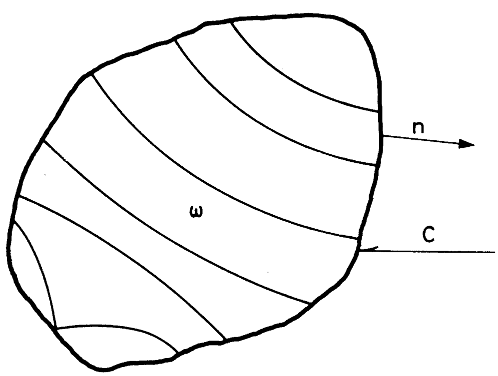

Fig. 1.
Contours of the stationary ground-water level h_0.

i.e. the boundary is stationary, rivers or lakes with fairly stationary surface levels make up the boundary curve, or

$$\frac{\delta h}{\delta n} = 0$$

that is impermeable rock at the boundary. n is the normal to the boundary curve.

$\delta h/\delta n$ given, but not zero, is also mathematically sufficient (given inflow or outflow) but is not represented in the following solution.

The ground-water level, h, is now divided up into a stationary part and a transient one:

$$h(x,y,t) = h_0(x,y) + h_1(x,y,t) \tag{2.2}$$

yielding two equations to be solved:

$$\nabla^2 h_0 = 0 \tag{2.3}$$

$$h_0(x,y) = h_0 \ \vee \ \delta h_0/\delta n = 0 \text{ on the boundary C.}$$

$$\nabla^2 h_1 = \frac{\delta h_1}{\delta t} - R \tag{2.4}$$

$$h_1(x,y,t) = 0 \ \vee \ \delta h_1/\delta n = 0 \text{ on the boundary.}$$

METHOD OF SOLUTION

Convolution integral

The problem is to solve eq. 2.4 with the homogeneous boundary conditions which still remain when the stationary h_0 ground-water level has been separated. Now, we assume that the infiltration varies with time only and not from one point to another within the aquifer

$$R(x,y,t) = R(t) \tag{3.1}$$

It is easy to show that the following convolution integral is then a solution to eq. 2.4.

$$h_1(x,y,t) = \int_0^{\infty} R(t-\tau)\, \Gamma(x,y,\tau) d\tau \tag{3.2}$$

By using the theory of Green's function it is easy to show that the function $\Gamma(x,y,\tau)$ has to satisfy the differential equation

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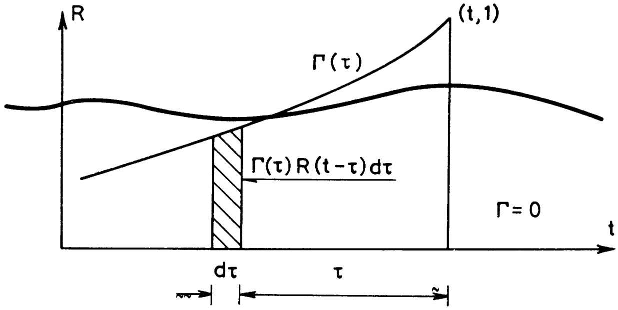


Fig. 2.
Convolution integral.

$$\nabla^2 \Gamma(x,y,\tau) - \frac{\delta \Gamma(x,y,\tau)}{\delta \tau} \equiv 0 \tag{3.3}$$

Furthermore, Γ must have a finite discontinuity in the point $\tau \equiv 0$, and it must be zero for negative τ. The necessary conditions for complete determination of Γ are as follows

$$\Gamma \equiv 0 \text{ for } \tau < 0 \tag{3.4a}$$
$$\Gamma = 1 \text{ for } \tau = 0 \tag{3.4b}$$
$$\Gamma \rightarrow 0 \text{ for } \tau \rightarrow \infty \tag{3.4c}$$
$$\Gamma = 0 \text{ on C} \tag{3.4d}$$

Later we shall see that Γ shows exponential decay everywhere in ω (the aquifer area bounded by C), and it can be expanded in series that converge reasonably quickly. In Fig. 2 the physical meaning of the convolution integral is visualized, the function $\Gamma(\tau)$ actually expressing the influence of the infiltration at the time t–τ on the ground-water level at the time t.

Separation of variables

We find the Γ-function by the method of separation of the variables:

$$\Gamma(x,y,\tau) \equiv \phi(x,y) \cdot f(\tau)$$

When this is inserted into eq. (3.3), we derive:

$$\frac{\phi(x,y)}{\nabla^2 \phi(x,y)} \equiv \frac{f(\tau)}{f'(\tau)} \equiv -\frac{1}{\alpha} \tag{3.5}$$

and subsequently the two equations:

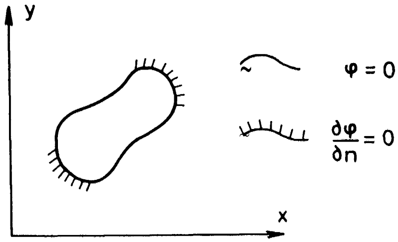


Fig. 3.

$$\nabla^2\phi + \alpha\phi = 0 \tag{3.6}$$

$$f'(\tau) + \alpha \cdot f(\tau) = 0 \tag{3.7}$$

The solution to eq. (3.7) is:

$$f(\tau) \equiv \text{constant} \cdot e^{-\alpha\tau} \tag{3.8}$$

Eq. (3.6) is the Helmholtz equation which, with the boundary conditions shown in Fig. 3, represents an eigenvalue problem with infinitely many eigenvalues, α_n.

It can be shown that α is real and $\geqq 0$. So, from now on we replace α by λ^2, a positive number.

The eigen-functions are known to be orthogonal, i.e.:

$$\int_\omega \phi_i \phi_j d\omega \equiv \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

The orthogonality allows us to write the Γ functions as an infinite series:

$$\Gamma(x,y,t) = \sum_{n=1}^{\infty} A_n \phi_n(x,y) e^{-\lambda_n^2 t} \tag{3.9}$$

Using eq. (3.4b) we find the constants A_n:

$$A_n \equiv \frac{\int_\omega \phi_n d\omega}{\int_\omega \phi_n^2 d\omega} \tag{3.10}$$

Insertion of eq. (3.9) into eq. (3.2) leads to the final equation for the transient ground-water level:

$$h_1(x,y,t) = \sum_{n=1}^{\infty} A_n \phi_n(x,y) \int_0^{\infty} R(t-\tau) e^{-\lambda_n^2 \tau} d\tau \tag{3.11}$$

The calculations can now easily be performed on a digital computer when the eigen-values and the corresponding eigen-functions have been determined.

The precision of calculated ground-water level when k eigen-functions are used can be found by

$$\lim_{k\to\infty} \sum_{n=1}^{k} A_n \phi_n(x,y) = 1 \tag{3.12}$$

Well inside the aquifer the precision is very good, but in the immediate neighbourhood of C it is not.

Run-off

The flow through the boundary is given by:

$$q(t) \equiv -\int_C \frac{\delta h}{\delta n} ds \equiv -\int_\omega \nabla^2 h d\omega$$

Eqs. (2.2) and (2.3) give:

$$q(t) \equiv -\int_\omega \nabla^2 h_1 d\omega$$

When we now insert the ground-water level given by eq. (3.11) and then replace $\nabla^2\phi_n$ by using eq. (3.6), we obtain the run-off:

$$q(t) = \sum_{n=1}^{\infty} A_n\lambda_n^2 \int_\omega \phi_n d\omega \int_0^\infty R(t-\tau)e^{-\lambda_n^2\tau} d\tau$$

A set of flow constants, C_n, is now defined:

$$C_n \equiv A_n \int_\omega \phi_n d\omega$$

We see that if the eigen-functions are normalized, i.e.

$$\int_\omega \phi_n^2 d\omega \equiv 1$$

is valid, then

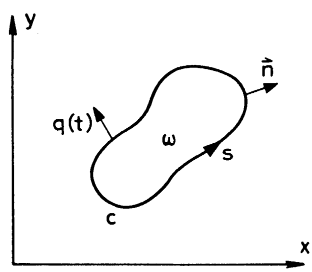


Fig. 4.

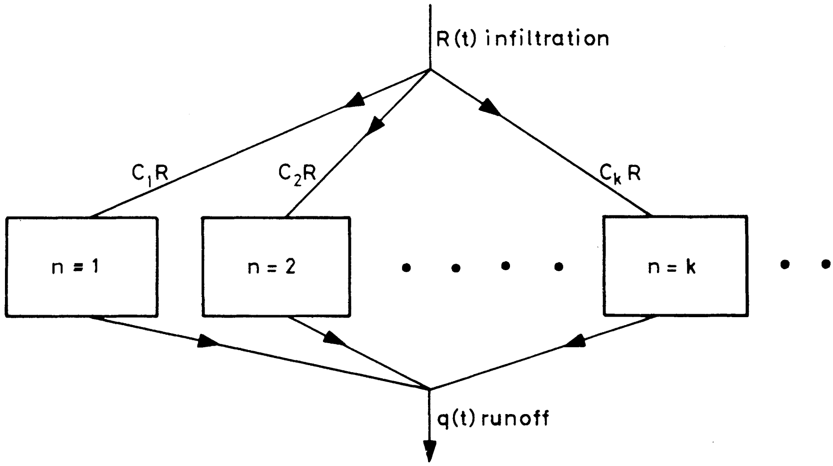


Fig. 5.

$$C_n \equiv A_n^2 \tag{3.13}$$

where A_n is given by eq. (3.10).

The run-off discharge now reads:

$$q(t) = \sum_{n=1}^{\infty} C_n \lambda_n^2 \int_0^{\infty} R(t-\tau) e^{-\lambda_n^2 \tau} d\tau \tag{3.14}$$

We now define:

$$q_n(t) \equiv C_n \lambda_n^2 \int_0^{\infty} R(t-\tau) e^{-\lambda_n^2 \tau} d\tau \tag{3.15}$$

Differentiating eq. (3.15) with respect to t, multiplying the result by $1/\lambda_n^2$ and adding it to eq. (3.15) yields the differential equation:

$$q_n(t) + \frac{1}{\lambda_n^2} \frac{dq_n}{dt} \equiv C_n R(t)$$

$$q(t) \equiv \sum_{n=1}^{\infty} q_n(t)$$

This is the differential equation for a system of infinitely many parallel, linear reservoirs, as illustrated in Fig. 5. The time constant of the n'th reservoir is $1/\lambda_n^2$.

If we call the amount of water stored in the n'th reservoir S_n, we have:

$$\frac{dS_n}{dt} = C_n \cdot R(t) - q_n(t)$$

$$q_n(t) = \frac{1}{\lambda_n^2} S_n(t)$$

These are the well known equations of a linear reservoir. The linear reservoir is the most widely used feature in hydrological modelling.

This linear reservoir mechanism is the basic concept of our run-off model.

It is easily verified that

$$\sum_{n=1}^{\infty} C_n = 1 \qquad (3.16)$$

which is an obvious necessity. Eq. (3.16) can be used to find the accuracy of the run-off when a limited number of eigen-functions are used.

Stationary ground-water level

The stationary ground-water level h_0 is given by eq. (2.3). This is Laplace's equation and it is solved by standard numerical methods, when the boundary conditions are given.

Physically, h_0 is the level towards which the ground-water would tend asymptotically, if all infiltration stopped.

Some physical features of the model will now be discussed briefly.

Averaged over time eq. (2.2) yields:

$$\overline{h}(x,y) = h_0(x,y) + \overline{h}_1(x,y) \qquad (3.17)$$

pertaining to a constant average infiltration. As $\nabla^2\overline{h} = \nabla^2\overline{h}_1(x,y)$, we find from eq. (2.4) that

$$\nabla^2\overline{h}(x,y) = -\overline{R} \qquad (3.18)$$

$$\overline{h}(x,y) = h_0 \vee \frac{\overline{\delta h}}{\delta n} = 0 \text{ on the boundary.}$$

This is Poisson's equation and it is solved by a numerical process similar to that of eq. (2.3).

Now recall that our dimensionless parameters are related to the physical parameters by:

$$x = x'/\sqrt{A}$$
$$y = y'/\sqrt{A}$$
$$R = R' \cdot A/T$$
$$t = t' \cdot T/AS$$

If the area under consideration has a known average ground-water level, $\overline{h}(x,y)$, in a series of observation wells, eq. (3.18) can be used to estimate the transmissivity, T.

The storage coefficient, S, can be estimated by comparing the amplitudes of the ground-water level fluctuations, $h_1(x,y,t)$, calculated by means of eq. (3.11) with measured variations in the observation wells.

In order to check actual calculations, the average values can be inserted into eq. (3.17). Further it is apparent that, averaged over time, eq. (3.11) yields:

$$\overline{h}_1 = R \cdot \sum_{n=1}^{k} \frac{A_n \phi_n}{\lambda_n^2}$$

when k eigen-functions are used.

The run-off can be checked by:

$$\overline{q} = \overline{R} \cdot \sum_{n=1}^{k} C_n$$

Eigen-value problem

As stated previously, we have an eigen-value problem when Helmholtz' equation, eq. (3.6), is to be solved with the boundary conditions:

$$\phi = 0 \vee \delta\phi/\delta n = 0$$

Analytical solutions of the problem exist for simple boundary curves, e.g. a circle and a rectangle. Where the boundary is arbitrary we must make use of numerical methods. In this particular case a method developed by Lic. techn. Knud Pontoppidan, "The iterative finite-element method", has been used. The method is a combination of finite-difference and finite-element methods. See ref. 2.

The eigen-values are given by the Rayleigh quotient

$$\lambda^2 \equiv \frac{\int_\omega (\nabla\phi)^2 d\omega}{\int_\omega \phi^2 d\omega}$$

which can be proved to be stationary for small variations in ϕ when ϕ is a solution.

It has been mentioned previously that the eigen-functions are orthogonal. If $\lambda_i^2 = \lambda_j^2$ for $i \neq j$ (degenerated eigen-values), ϕ_i and ϕ_j are chosen so that the orthogonality condition is satisfied. It may be demonstrated that degenerated eigen-values exist for boundary curves with a high degree of symmetry, and in such cases analytical approximations or direct analytical solutions usually exist.

The eigen-values increase with increasing order:

$$\lambda_1^2 \leqq \lambda_2^2 \leqq \lambda_3^2 \leqq \dots .$$

and thus the time constants for the flow reservoirs, $1/\lambda_n^2$, are decreasing with larger n.

Calculated flow constants corresponding to some areas of simple geometry are shown in Fig. 6 and the boundary condition $h_0 = 0$ on C is used.

For situation a) all $C_{2k} = 0$ and all $C_{2k-1} \neq 0$. Furthermore, it can be shown that for situation c) the flow constants may be arranged in a two-dimensional symmetrical matrix where the constants, C_{ij}, are related to the constants of situation a) by

$$C_{ij} \equiv C_i \cdot C_j$$

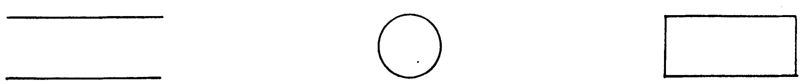
The Tables indicate that the flow constants for the circle and the rectangle are closely related in such a way that $\sum_k C_k \sim \sum_{k'} C_{ij}$, where $k' \equiv 2k-1$.

As the first reservoir governs the main proportion of the flow, the most important flow constants are C_{1j} (and C_{i1}).

Antimetric eigen-functions yield flow constants of value zero, and thus only one-fourth of the flow constants for the rectangle are non-zero.

Fig. 7 shows the four first eigen-functions for the area dealt with later in this article. The boundary conditions are $\delta\phi/\delta n \equiv 0$ along the x-axis, $\phi \equiv 0$ elsewhere. Thus the problem is solved for an area symmetric about the x-axis.

a) Parallel drains — b) Circle — c) Rectangular



a) Parallel drains		b) Circle			c) Rectangular	
i	C_i	k	C_k	$\sum_k C_k$	k′	$\sum_{i=1,j=1}^{i=k',j=k'} C_{ij}$
1	0.81057	1	0.6912	0.6912	1	0.6570
3	0.09006	2	0.1312	0.8224	3	0.8111
5	0.03242	3	0.0534	0.8758	5	0.8706
7	0.01654	4	0.0293	0.9051	7	0.9017

Fig. 6.

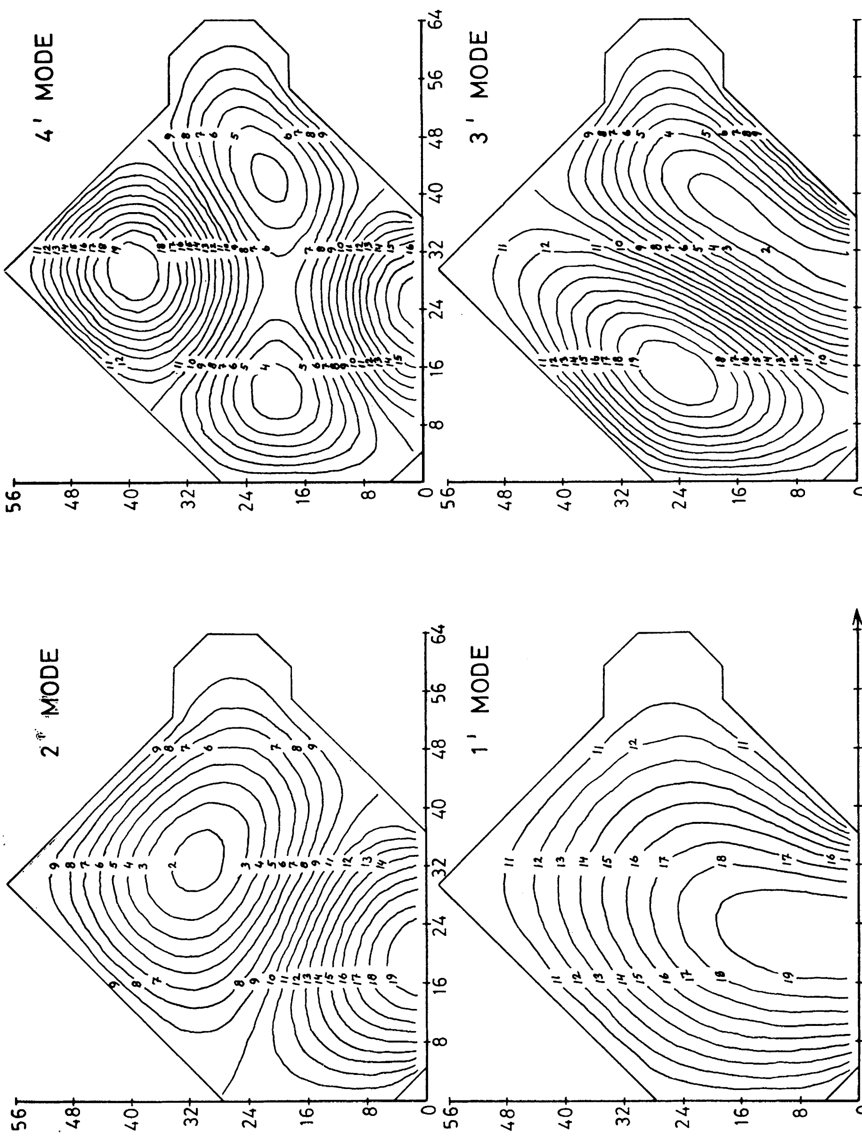


Fig. 7.
Eigen-functions.

Table 1.

No.	Eigenf.	C_{ij}
1	Φ_{11}	0.647
2	Φ_{31}	0.057
3	Φ_{21}	0.001
4	Φ_{51}	0.000
5	Φ_{31}	0.034
6	Φ_{31}	0.067

The resulting flow constants for the six first eigen-functions are shown in Table 1.

We will now compare this result with the analytical solutions previously mentioned. The first reservoir yields 64.7 % flow, whereas the rectangle yields 65.7 % and the circle 69.12 %. By summing up to the order three: $\sum_{i=1}^{3} C_{i1} =$ 80.6 %. By replacing the missing C_{33} by the corresponding value for the rectangle we get $\sum_{i=1}^{3} C_{i1} = 81.4$ %, whereas the corresponding sums for a rectangle and a circle are 81.1 % and 82.2 %. This similarity leads to an important conclusion.

As in most cases it is not practical to calculate a large number of eigen-functions for areas with arbitrary boundary curves, a possible way of increasing the accuracy of the model is to approximate the missing constants and eigen-values for eigen-functions of higher order by using the corresponding values for a rectangle or a circle or another aquifer shape as close to the actual one as possible but for which the eigen-functions can be found analytically.

HARMONIC ANALYSIS

When comparing hydrological series, a very efficient method is to use the Fourier transforms of the series instead of the series themselves.

The series we are dealing with here are the infiltration function, R(t), the run-off, q(t), and the transient ground-water level, $h_1(t)$, and we write them as the sum of their mean values and Fourier transforms:

Input:

$$R(t) = \overline{R} + \int_{-\infty}^{\infty} R(f)e^{2\pi ift} df \tag{4.1}$$

Output:

$$h_1(t) = \overline{h_1} + \int_{-\infty}^{\infty} H(f)e^{2\pi ift} df \tag{4.2}$$

and:

$$q(t) \equiv \overline{q} + \int_{-\infty}^{\infty} Q(f)e^{2\pi ift} df \tag{4.3}$$

By applying standard mathematical procedures it can now be shown that the Fourier transforms of the output, H(f) and Q(f), are related to the Fourier transform of the input, R(f), as follows:

$$H(f) = \sum_{n=1}^{\infty} \frac{A_n \phi_n}{\lambda_n^2 + i2\pi f} R(f) \tag{4.4}$$

$$Q(f) \equiv \sum_{n=1}^{\infty} \frac{c_n \lambda_n^2}{\lambda_n^2 + i2\pi f} R(f) \tag{4.5}$$

Bringing our flow model, Fig. 5, to mind we can define the Fourier transforms of the effects of a particular reservoir on ground-water level variation and run-off variation as follows

$$Q_n(f) = \frac{R_n(f)}{1 + \frac{i2\pi f}{\lambda_n^2}} \tag{4.6}$$

$$H_n(f) \equiv \frac{\phi_n}{\lambda_n^2 A_n} \cdot \frac{R_n(f)}{1 + \frac{i2\pi f}{\lambda_n^2}} = \frac{\phi_n}{\lambda_n^2 A_n} : Q_n(f) \tag{4.7}$$

where $R_n(f) = C_n \cdot R(f)$ and we have used $C_n \equiv A_n^2$.

Now we find the variance spectrum of the run-off from a particular reservoir

$$SQ_n(f) = Q_n(f) \cdot Q_n(f)^*$$

$$SQ_n(f) = \frac{1}{1 + \left(\frac{2\pi f}{\lambda_n^2}\right)^2} \cdot SR_n(f) \tag{4.8}$$

From this we see that the effect of the reservoir mechanism is reduction of variances on all frequencies. Now, the variance spectrum of hydrological series

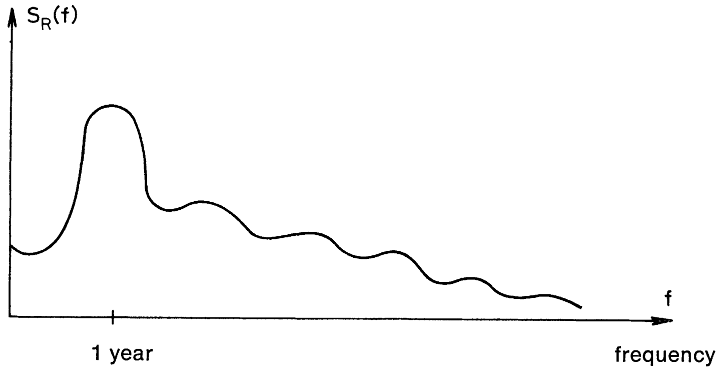


Fig. 8.

usually has the appearance sketched in Fig. 8 – that is, the main variance is on frequency one year $^{-1}$ and higher, i.e. the periods of the actual variations are one year or shorter. All these variances are very strongly damped when

$$\frac{2\pi f}{\lambda_n^2} \gg 1$$

Now, λ_1 is always the smallest eigen-value, so, in practice, the run-off will be almost constant when the physical time constant of the first reservoir is greater than 1 year. The physical time constant is defined as

$$K_n \equiv \frac{A \cdot S}{\lambda_n^2 T}$$

An Icelandic lindà (meaning a ground-water-fed river) usually has a very constant water discharge, because of the high time constants of the ground-water reservoirs of the extensive lava fields in Iceland, and it is a well known fact everywhere in the world that ground-water-fed rivers have a more constant flow rate than do other rivers. This is the direct consequence of the damping effect of the reservoir mechanism.

It is seen that by use of the Fourier transforms the convolution integral can be omitted in practical calculations. Calculations of ground-water levels and run-offs from known infiltration rates, or vice versa, can be done by means of eqs. 4.6 and 4.7, and in this connection use of the flow model in Fig. 5 makes the calculation much easier because the run-off of each reservoir can be found by dividing its infiltration transform by a function varying linearly with the frequency according to eq. 4.6, and the ground-water level effect is found by multiplying this by a constant.

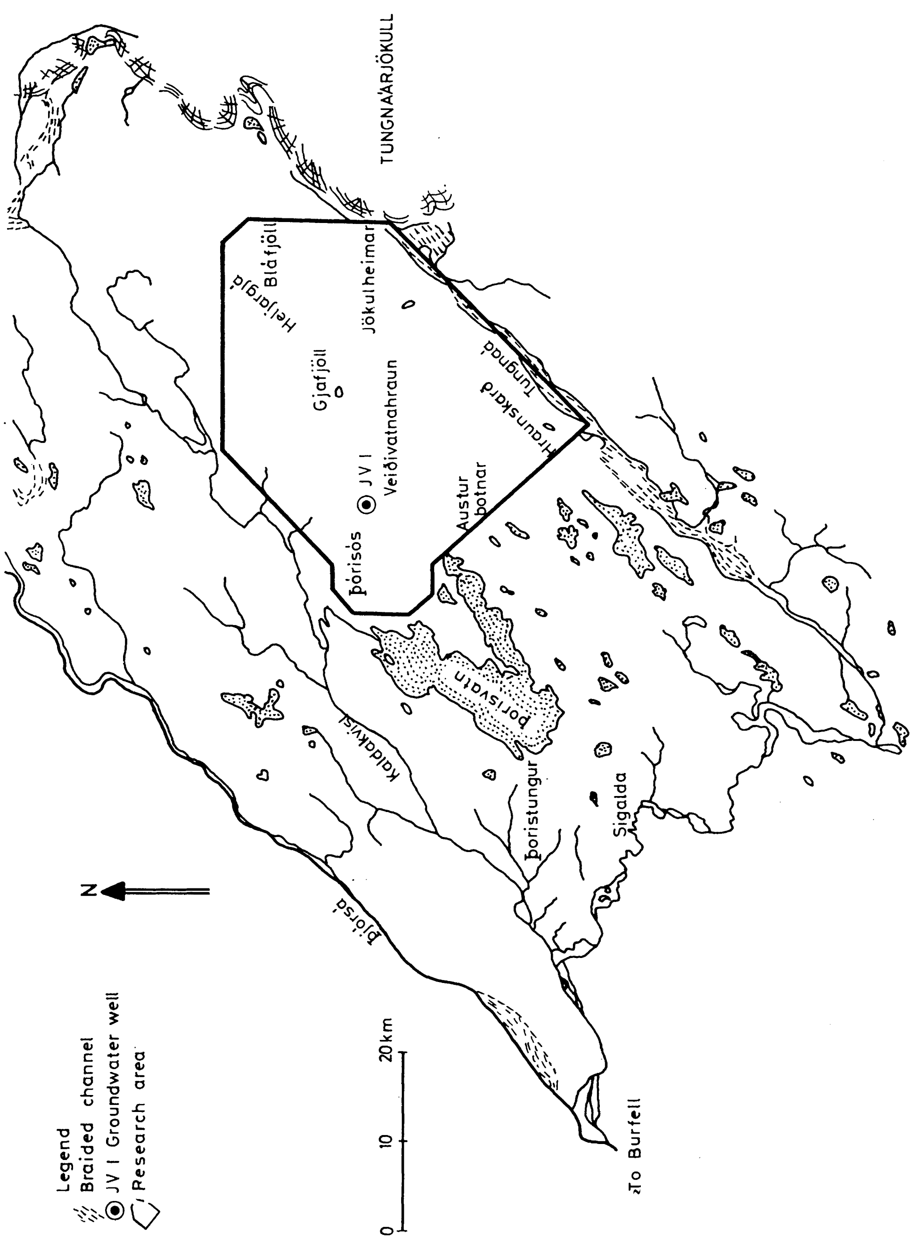


Fig. 9.
Model area.

USE OF THE FLOW MODEL

Numerical integration of the equations of ground-water flow is an easy process, at least in principle, so why use the flow model? First, there are many practical obstacles; including the fact that most of the programs are big and need both large machine installations and long execution times. Secondly, no qualitative information about the aquifer in question is obtained in a treatment that is entirely numerical, and the same big program is to be run each time information is needed.

For the flow model with the parallel linear reservoirs we have to solve the eigen-value problem, and for this purpose we need a large machine, but when this is done and the eigen-functions, the eigen-values and the A_n coefficients are found, we have once and for all gathered all necessary information on our aquifer, and we can build relatively simple series for the variation of ground-water levels in any particular point or the run-off through any particular section of the boundary. In our further research with this aquifer we can work with the series and build new ones without thinking more of the eigen-value problem which has been solved once and for all.

CASE STUDY

The theories have been used to calculate the ground-water yield from an aquifer in Iceland. The area in question is shown in Fig. 9 and the model area itself is within the frame shown.

The model area is at elevation 600–700 m.o.s.l. in the central highlands. The underground is composed mainly of postglacial and interglacial lava flows and is very pervious. There are no rivers or lakes within the model area. The only hydrological data available is the discharge of the river Thorisos coming from lake Thorisvatn. Meteorological observations are available from the station Haell 75 km southwest of the area.

The discharge of the Thorisos river was calculated. The hydrological model involved the features listed below:

1. Generation of meteorological series for the model area from those of Haell by linear regression.

2. Building of a snow storage.

3. Calculation of infiltration due to precipitation and melting of snow.

4. Calculation of the eigen-functions in Table 1. The four first are shown in Fig. 7.

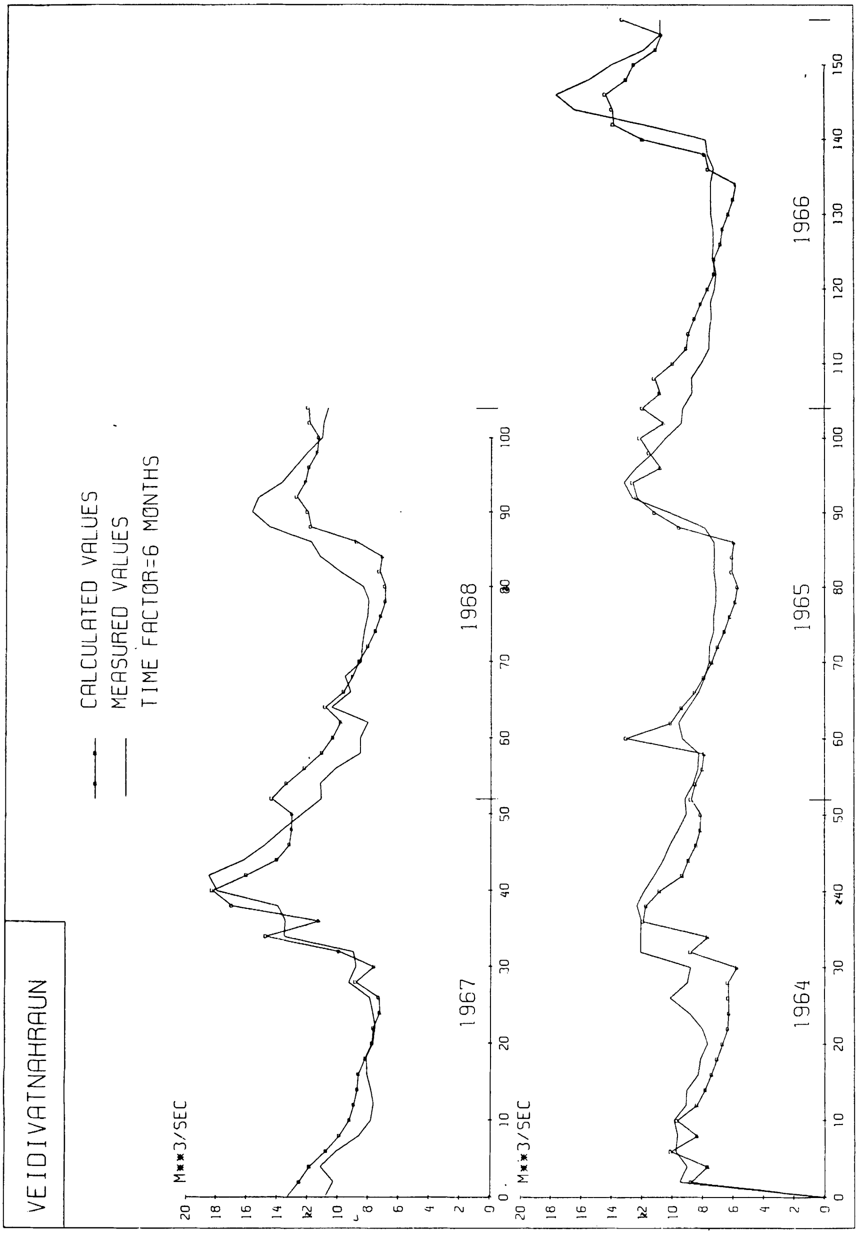


Fig. 10.
Run-off at Thorisos.

5. Calculation of run-off to lake Thorisvatn.
6. Calculation of the discharge in Thorisos.

The theory described above was used in steps 4 and 5 of the procedure. A complete description of the hydrological model and the entire procedure is given in ref. 3. This includes a linear relation between temperatures and precipitation series at Haell and Veidivatnahraun and the building up and melting of the snow storage. These features are interesting, but a further description of them is omitted as the purpose of this article is to give a description of reservoir mechanism.

Here, we shall only mention that the calculations were carried out on an IBM 360/75 using weekly mean values. It turned out that it was unncessary to use more eigen-functions than those given in Table 1 because their time constants were smaller than 1 week, so the output of the missing reservoirs could be put equal to the input. Furthermore, a time factor of 6 months was used to cover the reservoir effect of Thorisvatn.

The results are shown in Fig. 10. They comprise the discharge in the 5 years from 1964–68, and the values measured are also shown. The agreement is good, and the greatest part of the deviations may be shown to be due to inaccuracies of the linear regression of the meteorological data.

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