Wave Equations in Conformal Space
—Wave Equation for Nucleon—

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In 1936 Dirac\(^1\) investigated the wave equations in conformal space and concluded that there might be no simple way to generalize the Dirac equation so as to be invariant under the conformal group. Dirac considered the problem on the hyperquadric which lies in the five-dimensional projective space and corresponds to the assemblage of the points in the Minkowski space. If we consider the same problem in the whole five-dimensional projective space which corresponds to the assemblage of the spheres in the Minkowski space, any kind of wave equation can be described in a conformally invariant way: thereby the mass is introduced through a procedure which is similar to what was used by S. Watanabe in his investigation of wave equations in the de Sitter space.\(^2\)

We use six homogeneous coordinates \(X^\alpha\) which are related with the co-ordinates \(x^\ell\) in the Minkowski space and the radius \(r\) of the hypersphere around the point \(x^\ell\) as follows:

\[
\tau X^\ell = x^\ell, \quad \tau X^0 = 1, \quad \tau X^a = (x^a x^a - r^2)/2.
\]

The transformations which leave

\[
\sum_{\nu} X^\nu X^\nu = (X^\ell)^2 + (X^a)^2 - (X^0)^2 - 2X^0 X^a/r^2
\]

invariant form a conformal group. We require that any wave function or component of wave function satisfies the equation

\[
\frac{\partial^2 \psi}{\partial X^\mu \partial X_\mu} = 0. \quad (A)
\]

We require further that \(\psi(X)\) is homogeneous in \(X^\alpha\). Then eq. (A) becomes in the Minkowski space as follows:

\[
[\partial^2 / \partial x^\ell \partial x^\ell - \partial^2 / \partial r^2 + (2n + 3) \tau \partial / \partial \tau] \varphi = 0, \]

where \(\varphi\) is related with \(\psi\) by

\[
\varphi = \tau^n \psi,
\]

if \(\psi\) is of degree \(n\). Thus the mass is introduced through the equation
110

Letters to the Editor

\[ [d^2/dt^2 - (2n + 3)/r \, d/dr - m^2] \psi = 0. \]

For the spinor wave function it is natural to start from the equation which is obtained through a linearization of eq. (A):

\[ \beta_\mu \partial \psi / \partial X_\mu = 0, \tag{B} \]

where \( \beta_\mu \) are matrices satisfying the relations

\[ \beta_\rho \beta_\nu + \beta_\nu \beta_\rho = 2g_{\rho \nu}. \]

To represent the matrices \( \beta \)'s we need eight matrices, i.e., one more set of matrices like \( \rho \) and \( \sigma \) is necessitated. If we transcribe eq. (B) into the expression in the Minkowski space and investigate the correspondence to the usual Dirac equation, the following results can be found.

1) Eight independent states which correspond to the eight components of the wave function may be divided into two groups and these are distinguished through an interaction with the electromagnetic field; the one does interact with this field while the other doesn't. The interaction term can be introduced in a conformal and gauge invariant way. Therefore it may be admitted to regard eq. (B) as the equation for nucleon.

2) Isotopic spin operators are expressed with the aid of matrices \( \beta_\mu \).

3) The total inversion \( X^\rho \rightarrow -X^\rho \), which has nothing to do with the coordinates in the Minkowski space, can be considered the rotation around the \( r_1 \) axis through 180°.

4) The inversion \( X^0 \rightarrow -X^0, X^0 \rightarrow X^0, X^0 \rightarrow X^0, X^0 \rightarrow X^0 \) is a product of the usual spatial inversion and the rotation around the \( r_3 \) axis through 180°.

5) The linearized expression of \( r^i \), i.e., \( (X_\nu X^\nu)^{-1} \beta_\mu X^\mu \) plays the role of \( r_1 \).

6) When the interaction with the pseudoscalar boson is considered, only the direct couplings are permissible. (Because of the fact stated in (3), the charge symmetry is guaranteed though the charge independence is not; the reason is that the rotation in the isotopic spin space is not fully contained in the transformations in our six-dimensional space).

We have got these results through a provisional treatment of the newly introduced variable \( r \). The adequate treatment of this variable might be able to throw some light on the problem of the unstable heavy particles, mass spectrum, the value of the coupling constant etc.

The detailed account of this work will appear in the *Science Reports of the Saitama University, A*, around June 1955.