The Determination of $Q$, Dynamic Viscosity and Transient Creep Curves from Wave Propagation Measurements*

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Summary

Perhaps the best theoretical approximation to the experimentally observed attenuation and spreading of seismic body waves is due to Futterman. Wuenschel has utilized the Pierre shale field data of McDonal et al. as well as his own two-dimensional model work using plexiglas sheet, to obtain remarkable confirmation of Futterman's approach. However, Futterman's assumption of a linear attenuation-frequency relation requires truncation at both ends of the frequency spectrum. The arbitrariness of this truncation leads to unnecessary parameters which make his approximation awkward to use. We discuss a power-law approximation to the attenuation characteristic which does not have this difficulty. Emphasis will be on computation of mechanical properties such as spatial $Q$, magnitude and loss angle of the modulus, dynamic viscosity and transient creep. These computations which are based upon the wave propagation measurements of Jordan will be seen to be in remarkable agreement with independent measurements by Lethersich, who utilized direct experimental procedures.

1. Introduction

One of the most important problems in seismology today is to understand the nature of the loss mechanism that contributes to the observed decay in amplitude and the associated spreading in time of the outgoing waveform as it passes through the Earth. Until very recently it had been widely thought that rock materials were so structurally complicated that it would be highly improbable that one could find a theoretical description containing a small number of parameters (say two or three) which could be easily evaluated from experimental data. The path toward such a successful theoretical description has been paved by many investigators. The work of Knopoff on the constant $Q$, and attenuation as a linear function of frequency, bears special mention. We should also mention the more recent work by Futterman (1962) and Lamb (1962), who separately realized that in order to maintain causality of the outgoing waveform the phase velocity characteristic, $v(f)$, must not be constant and must in fact be uniquely determined by the attenuation, $a(f)$. Then we must mention the even more recent two-dimensional model work of Wuenschel (1965), Toksöz et al. (1965), and Jordan (1966), who have given such strong experimental evidence of the correctness of the Futterman theory that one should be hesitant to look elsewhere for a better description. On the other hand, as is well known, there are several weak points in the structure of Futterman's theory that must be removed. His assumption of a linear frequency dependence for $a(f)$ cannot be extended to infinite frequency.

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as it would violate the Paley-Wiener principle of causality. On the other hand, it cannot be extended to zero frequency as it would then contribute to an unboundedness in the Kramers-Krönig or K-K relation applied to $\alpha(f)$. Since the precise value for this low-frequency cut-off is somewhat arbitrary, it introduces a vagueness in the parameters $\nu(f) = c$ and $Q(f) = Q_0$ associated with this cut-off frequency.

It is rather easy to modify Futterman's approach so that not only will the aforementioned difficulties be avoided, but at the same time a better 'fit' to the experimental data can be obtained. Our description will in a way be intermediate between that of Futterman, who assumed a truncated $\alpha(f)$, linearly frequency dependent, and Lamb, who assumed a square root frequency dependence for $\alpha(f)$. The sole objective of Futterman and Lamb was to determine the effect on the outgoing waveform of the dispersion that must be associated with the observed attenuation. Their technique was to use K-K relations to compute the dispersion (i.e. phase-lag) $\theta(f)$ from an assumed parametric form for $\alpha(f)$. The resulting complex propagation function $\gamma(f) = \alpha(f) + j\theta(f)$ was then used to Fourier invert $\exp \left[ -\gamma(f) R \right]$ to the time domain, thereby obtaining a spreading of the delta function excitation with increasing distance $R$. Their common conclusion was that for distances encountered in earthquake seismology the viscoelastic dispersion should be small and consequently difficult to measure. As we shall see later, our objective is much broader than that of these authors.

We shall begin by following the procedure of Lamb and Futterman except that we shall use Hilbert transforms rather than K-K relations to relate the assumed parametric approximations to $\alpha(f)$ and $\nu(f)$. We are going to assume a simple power law dependence for $\alpha(f)$ not just because the arbitrary exponent $s$ can include both the square root relation of Lamb and the almost linear relation of Futterman, but also because it appears to be the only frequency dependence whose Hilbert transform, as
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we shall see later, will account in a simple manner for the almost constant $Q$ behaviour of materials as observed in nature. As a result of the simplicity of our Hilbert pair, we shall find it easy to compute theoretical expressions for

1. the spatial $Q$,
2. the magnitude $G(f)$ and loss angle $\delta(f)$ of the complex dynamic modulus $E(f)$,

and

Fig. 2. Attenuation of vertically travelling compressional waves generated by a charge of one pound of dynamite at a depth of 260 ft (Record T). Pierre shale; after McDonal et al. (1958).

Fig. 3. Dispersion curve for Pierre shale (after Wuenschel 1965).
(3) the dynamic viscosity \( \eta(f) \) over an extrapolated frequency range of \( 10^{-4} - 10^4 \text{c/s} \) (a range of eight decades) where we shall find a rather striking agreement with the varied experimental work of Lethersich (1950).

Because the complex propagation function \( \gamma(f) = \alpha(f) + j\beta(f) \) formed from such a Hilbert pair can be re-expressed in operational form by \( p = j\omega \), we shall also find it quite simple to assume the applicability of the wave equation and Hooke's stress–strain relation to arrive at a transient creep curve which also meets remarkable confirmation in Lethersich's work.

2. Published experimental results

Since we are going to determine the parameters for our theoretical description from attenuation, \( \alpha(f) \), and phase velocity, \( v(f) = 2\pi f \beta(f) \), characteristics, let us briefly consider some published experimental curves for a few materials. In Fig. 1 we illustrate

![Graph](https://example.com/image)

**Fig. 4.** Wuenschel's experimental values for plate waves in plexiglas sheet (or compressional waves in pseudo-plexiglas) and his least square fit through these points. Different symbols denote that measurements were taken on different occasions.
measurements on a small cylinder of polythene (Hillier 1949). The attenuation characteristic $a(f)$ is essentially linear whereas the phase velocity characteristic continues to increase with increasing frequency but with diminished slope. This general pattern is also true of other materials. Unfortunately, although linear $a(f)$ type curves are abundant in the published literature, the associated $v(f)$ or $\theta(f)$ curves are not so plentiful. Pierre shale (see Figs. 2 and 3) has, according to McDonal et al. (1958) and Wuenschel (1965), curves which fit the same pattern. In this discussion we are going to make the assumption that these curves are typical of what one would
find for any solid at a very wide range of frequencies well below those where relaxation and scattering effects become important.

The $\alpha(f)$ and $\nu(f)$ curves for the high polymer, plexiglas, are of particular interest to us because it is this material that has been widely used recently by Wuenschel, Toksöz et al. and Jordan in their two-dimensional model work, and it is a material (perspex) that is very similar to plexiglas that Lethersich has used in his varied experimental work. These curves will give us the opportunity to obtain the parameters from two-dimensional wave propagation experiments and then to compare our theoretical predictions with Lethersich's experiments which are of a non wave-propagation type. In Fig. 4 we show Wuenschel's $\alpha(f)$ curve for plate waves in plexiglas sheet and in Fig. 5 Jordan's $\alpha(f)$ curve for shear waves in the same material. The curves are basically linear. In Figs. 6 and 7 we also see that the $\nu(f)$ curves for plate and shear waves have the same general pattern as was the case for polythene and Pierre shale.

3. The inability of spring–dashpot models to account for the experimental data

Before entering into the detailed discussion of these measurements in plexiglas, we shall digress briefly and consider how well this general pattern fits the classic two and three element spring–dashpot viscoelastic models that are usually attributed to Maxwell, Voigt and Zener. Excellent discussions of these models can be found in Kolsky (1960) and Meidav (1964). The Zener standard linear model has a spring shunting a Maxwell spring–dashpot series element whereas the second 3-parameter model has a dashpot instead of the spring as a shunting element. In Fig. 8 we show the expected $\alpha(f)$ and $\nu(f)$ curves for the series (Maxwell) and parallel (Voigt) models. This and the following figure were taken from a paper by Kolsky (1949). It is apparent that the Maxwell curve has a $\nu(f)$ characteristic which could fit the required pattern but the $\alpha(f)$ curve is far from linear over the required wide frequency range. On the other hand, the Voigt model does have a reasonable linear portion for $\alpha(f)$, but its associated $\nu(f)$ curve has a curvature in the wrong direction. In Fig. 9 we see that the
situation is somewhat better for the three-element Zener standard linear solid in that one might conceive of a frequency band that could be a fair approximation to our general pattern. This type of model has been extensively studied by Meidav (1964) and Horton (1959), who find that the model does not fit the data over a sufficiently wide frequency range. The inability of these classic models to fit the observed data is best illustrated in another figure taken from a later paper by Kolsky (1960). In Fig. 10 we see Kolsky's plot of the log tan of the loss angle \( \delta \) of the complex modulus as measured by Lethersich for the four polymer solids, (A) polyethylene, (B) perspex, (C) ebonite, and (D) polystyrene, over the frequency range from \( 10^{-4} - 10^{+3} \) c/s. Note that the curves are essentially independent of the frequency. Kolsky also plotted an overlay of four of the classic spring dashpot models, and we observe that even though the standard linear solid is the best fit, it is an acceptable fit for only two or three of the seven frequency decades. We have taken advantage of the fact that the temporal \( QT \) is essentially the negative of the tangent of the loss angle \( \delta \) to recalibrate the ordinate in terms of the \( QT \) so that it will be more in line with seismological thinking. If we further note that in any system where \( v(f) \) has such a gradual rise with frequency it can be shown that the phase and group velocities differ by no more than one or two per cent (this is particularly true of our theoretical description to follow), then it will not be necessary to distinguish between the temporal \( QT \) and the spatial \( QR \).

4. A three-parameter power-law approximation to the complex propagation function

For a plane wave, as is shown in the appendix, the spatial \( QR \) is simply the ratio of the phase lag function \( \theta(f) \) to twice the attenuation function \( \alpha(f) \)

\[
QR = \frac{\theta(f)}{2\alpha(f)}.
\]
Fig. 9. Velocity and attenuation in standard linear solid.

Fig. 10. log Q and log tan δ versus log frequency for four spring–dashpot models superposed on experimental values of Lethersich (1950). (After Kolsky 1960.)
In order for $Q_R$ to be essentially independent of frequency over at least seven frequency decades it is necessary that $\theta(f) = 2\pi f/\nu(f)$ should have a frequency dependence that is almost the same as that for $\alpha(f)$. It is, however, well known that precisely linear frequency dependence cannot, because of causality, be extended to infinite frequency. It is also well known that a power-law relation

$$\alpha(f) = k_0 f^s \quad (0 < s < 1) \tag{2}$$

can satisfy the Paley–Wiener condition (for example Papoulis 1962) no matter how close we take $s$ to be to 1 as long as we do not go to the limit. In our search for a Hilbert pair, $\alpha(f)$ and $\delta(f)$, which have essentially the same frequency dependence (in order to satisfy the constant $Q$ requirement) and which will satisfy the essentially linear frequency requirement, we could only find the power-law relation, which has the Hilbert transform

$$\delta(f) = \tan \left( \frac{s \pi}{2} \right) \text{sgn} f \alpha(f), \tag{3}$$

to meet the requirements. $\text{sgn} f$ is +1 for $f > 0$ and −1 for $f < 0$.

If we were to take $\theta(f)$ to be identical to $\delta(f)$, then our spatial $Q_R$ would be

$$Q_R = \theta(f)/2\alpha(f) = \frac{1}{2} \tan \left( \frac{s \pi}{2} \right), \tag{4}$$

**FIG. 11.** Wuenschel's attenuation data.
and thus precisely independent of frequency for all frequencies. Although this may be a desirable property in order to fit the Lethersich data, it fails to account for the fact that a propagating wave must, because of its finite travel time, have a pure delay (i.e. all-pass and thus not minimum-phase) behaviour inherent in the measured $v(f)$ data. Consequently, if we let $\tau R$ be the travel time for waves of infinite frequency ($\tau$ will then have the units of reciprocal velocity), then we can write

$$\theta(f) = 2\pi f \frac{v(f)}{u(f)} = \dot{\alpha}(f) + 2\pi \tau R. \quad (5)$$

The need for the third parameter was first suggested to the author by his colleague, Dr A. J. Seriff. This leads to a nonconstant $Q_R$ having the form of an extremely slowly increasing function of frequency

$$Q_R = \frac{1}{2} \tan \left( \frac{s \pi}{2} \right) + \frac{\pi \tau}{k_0} f^{1-s}. \quad (6)$$

Note also that at infinite frequency $v(f)$ has the finite value $1/\tau$.

We shall now take a look at the experimental results of Wuenschel and Jordan to see how well our assumptions fit their data. We consider Wuenschel's data for plate waves in plexiglas sheet first. Fig. 11 shows Wuenschel's original $\alpha(f)$ curve, and Fig. 12 contains a log-log plot of this same data where the slope yields $s = 0.9227$ (which is slightly less than 1) and the intercept yields $k_0 = 2.038 \times 10^{-5}$. Fig. 13

![Fig. 12. The data of Fig. 11 replotted on a log-log scale.](https://academic.oup.com/gji/article-abstract/13/1-3/197/919909)
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shows the power-law relation derived from the plot of Fig. 12 replotted on a linear scale. The excellent fit does not contain the small positive ordinate intercept that is characteristic of the Wuenschel and Jordan interpretation of the Futterman theory. In a similar manner we can obtain a power-law fit to Jordan's experimental data for shear waves. In Fig. 14 (dashed line) we show the results of doing just this. For shear waves, we see that $s = 0.866$, which is somewhat less than that we found for plate waves. The intercept yielded $k_o = 7.13 \times 10^{-5}$, which is over three times as large as the plate wave value.

Due to the presence of the all-pass delay term in $\theta(f)$ and $v(f)$, it is apparent that a log-log plot of velocity data will not be a straight line (it will, of course, be rather close to a straight line for values of $s$ very close to 1). However, the evaluation is simple if we use the values for $s$ and $k_o$ as obtained from $\alpha(f)$. For plate waves we find $\tau = 119.7 \mu s/ft$, which corresponds to a maximum phase velocity of 8354 ft/s at infinite frequency. For shear waves we obtain $\tau = 214.3 \mu s/ft$ or a maximum phase velocity of 4666 ft/s. With all three parameters now completely known, we can plot the theoretical phase velocity curves and compare them with the experimental values. Following the same procedure as for plate waves we have used Jordan's data to obtain the phase velocity characteristic for shear waves as shown in Fig. 15. Our theoretical curve (dashed) is somewhat superior to Jordan's interpretation from the Futterman theory at the low frequency end.
5. Comparison of predicted $Q$, complex modulus and dynamic viscosity with the experimental results

Another way of evaluating how well our theoretical description fits both $\alpha(f)$ and $\nu(f)$ data is to use the three parameters we have just determined to compute $Q_R(f)$, which we saw in equation (6) to be a slowly increasing function of the frequency. This is shown in Fig. 16, where aside from an understandable deviation of the $Q_R$ for plate waves at the lowest frequencies, it is in excellent agreement with the observed data. The $Q_R$ for plate waves appears to be about a constant value of 5 larger than that for shear waves for all frequencies. We should note here that the experimental and theoretical $Q_R(f)$ of Fig. 16 is not truly constant $Q$ and we should reconcile this behaviour with the essentially constant $Q$ (i.e. constant $\delta$) that Lethersich found for the seven frequency decades. In Fig. 17 we have taken the $Q_R(f)$ three-parameter relation as evaluated for the frequency range of the two-dimensional model experiment and extrapolated it down to Lethersich's lowest frequency of $10^{-4}$ c/s. The result is superposed on Fig. 10; we have, however, taken the liberty of removing the experimental points for three of the four polymers since we desire to compare our theory based upon a two-dimensional experiment using plexiglas sheet only with Lethersich's loading of a perspex rod. For the extremely low frequencies we seem to predict a $Q$ that is somewhat on the low side but the extremes of the variations in $Q(f)$ are about the same for our theory as for the Lethersich experiment.

We are now in a position to make use of the wave equation and Hooke's stress-strain relationship to determine parametric relations for the quantities measured by Lethersich. In order to do this we find it algebraically simpler to start by introducing the Laplace variable

$$p = j \omega = 2\pi f.$$  (7)
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One can then use equations (2) and (5) to express the complex propagation function $\gamma(f)$ in the operational form $\gamma(p)$:

$$\gamma(p) = Kp + \tau p$$  \hspace{1cm} (8)

where

$$K = \frac{k_0}{(2\pi)^s \cos \left(\frac{1}{2}s\pi\right)}.$$  \hspace{1cm} (9)

For the plane wave

$$u(R, t) = U(0, p) \exp \left[-\gamma(p) R + \tau t\right]$$  \hspace{1cm} (10)

---

**Fig. 15.** Phase velocity characteristic for shear waves (after Jordan 1966).

**Fig. 16.** $Q_R(f)$ for plexiglas sheet plotted over a range of experimental values according to the three-parameter ($\tau$, $k_0$ and $r$) theory. $\bigcirc$, $P$ waves; $+$, $S$ waves.
Fig. 17. Comparison between response of model solids and measured values of $Q_R$ and $\tan \delta$ in shear (Lethersich) and the predictions according to the three-parameter power law theory. \quad \cdots \cdots; P \text{ waves}; \quad \cdots \cdots, S \text{ waves.}

to satisfy the wave equation

$$\frac{\partial^2 u}{\partial R^2} = \frac{\rho}{E^*(p)} \frac{\partial^2 u}{\partial t^2}$$  \hspace{1cm} (11)

where $(E^*/\rho)^4$ is the complex velocity and $\rho$ is the density, it is easy to obtain the well-known relation

$$E^*(p) = \rho \left[ \frac{\rho}{\gamma(p)} \right]^2.$$  \hspace{1cm} (12)

Inserting $\gamma(p)$ from (8) into (12) we have

$$E^*(p) = \frac{\rho}{[K p^{\gamma-1} + \tau]^2}.$$  \hspace{1cm} (13)

In order to study the modulus magnitude $G(f)$, loss angle $\delta(f)$ and dynamic viscosity $\eta(f)$, we return the argument to the $f$-plane by equation (7) and re-express the resulting $E(f)$ from equation (13) in polar and rectangular forms:

$$E(f) = G(f) e^{-j(kf)} = E_1(f) + 2\pi f \eta(f).$$  \hspace{1cm} (14)

In this way we find other well-known relations:

$$G(f) = \frac{\rho (2\pi f)^2}{\alpha^2(f) + \theta^2(f)},$$  \hspace{1cm} (15)
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\[
\tan \delta(f) = \frac{2}{\theta(f) - \alpha(f)} = \frac{1}{Q_R(f) - \frac{1}{4Q_R(f)}}.
\]  

(16)

For $Q_R$ large compared with $1/2$, equation (16) has the approximate form

\[
\tan \delta(f) \approx \frac{1}{Q_R(f)}
\]

(17a)

or

\[
\log \tan \delta(f) \approx -\log Q_R(f).
\]

(17b)

It is this relation (17b) that was used earlier in the reinterpretation of the ordinates of Figs. 10 and 17.

With $G(f)$ so defined we can next compare our theoretical predictions of $G(f)$ with Lethersich's independent measurements of $G(f)$ for the seven frequency decades. We again use one of Kolsky's figures, where we see (Fig. 18) that our prediction, the superposed heavy solid line, is again in good agreement with Lethersich's experiment. The standard linear solid may appear to be almost as good, but the flattening at high and low frequencies is only weakly suggested by experiment. We should remember

![Figure 18](https://academic.oup.com/gji/article-abstract/13/1-3/197/919909/1)

**Fig. 18.** Comparison of modulus magnitude $G(f)$ as computed according to the three parameter power law theory using Jordan's shear wave data for plexiglas with Lethersich's (1950) experimental values for perspex and the predictions of three spring–dashpot models.
that all of Lethersich’s experiments are related to a shear stress. We have also com-
puted the predicted curve for dilatational stress, but the values are too large for plotting
on the scale of Kolsky’s figure. Therefore in Fig. 19 we have plotted the theoretical
curves for $G(f)$ using both plate and shear wave parameters for comparison with
Lethersich’s shear stress data. The magnitude modulus for plate waves is seen to
be about three times greater than that for shear waves which is in line with the ratio
one would expect from Poisson’s ratio for elastic wave propagation in plexiglas sheet.
The gradual increase of the predicted velocity ratio suggests a corresponding increase
of Poisson’s ratio with increasing frequency.

![Graph](https://example.com/graph.png)

**Fig. 19.** Shear modulus magnitude of Fig. 18 replotted with an extended vertical
scale to include the modulus for plate waves in plexiglas sheet (or compressional
waves in pseudo-plexiglas) as computed using the three-parameter power law
theory based on Wuenschel’s (1965) data. ———, $P$ waves; ———, $S$ waves.

Lethersich has also made independent measurements of the dynamic viscosity
$\eta(f)$ of the shear stressed perspex rod. We can easily derive $\eta(f)$ from equations
(14), (15) and (16), and we find

$$\eta(f) = \frac{2\rho \alpha(f) \theta(f)}{[\alpha^2(f) + \theta^2(f)]^2},$$

where $\alpha(f)$ and $\theta(f)$ are again given by equations (2) and (5).

For materials having a value of $Q_R$ much greater than unity, equation (1) indicates
that we may neglect $\alpha(f)$ in comparison with $\theta(f)$ so that $G(f)$ and $\eta(f)$ may be
expressed as approximations analogous to equation (17) for $\tan \delta(f)$:

$$G(f) \approx \rho \left[ \frac{2\pi f}{\theta(f)} \right]^2 = \rho v^2(f),$$

$$\eta(f) \approx \frac{4\pi f \alpha(f)}{[\theta(f)]^3} = \frac{G(f)}{2\pi f Q_R(f)},$$
where \( v(f) = 2\pi f/\theta(f) \) is the usual phase velocity which according to equation (3) and (5) can be expressed in the form

\[
\frac{1}{v(f)} = \frac{\tan \left( \frac{1}{2} \omega \theta k_0 \right)}{2\pi f^{1-s}} + \tau.
\]  

(21)

Returning to equation (20) and noting that in Figs. 17 and 18 we observe that both \( Q_R(f) \) and \( G(f) \) are extremely slowly increasing functions of the frequency it is apparent that \( \eta(f) \) should vary very nearly as the inverse first power of the frequency. In Fig. 20 we have plotted (dashed line) values of \( \eta(f) \) for plexiglas as computed from equation (18) together with Lethersich's experimental values. The high frequency experimental point had been added by Lethersich and attributed to Kolsky. It is seen that the agreement between our theoretical prediction and the experimental data over almost ten frequency decades is extremely close. It is even more remarkable when we remember that the parameters were originally determined from model data by Wuenschel and Jordan taken over only about one frequency decade at the high end of the curve.

6. Comparison of predicted creep curves with experimental results

With these relationships and comparisons with data established, we can now proceed to make good our earlier statement that because we have properly constructed the parametric form for \( \alpha(f) \) and thus \( \gamma(f) \), we can use the operational form \( E^*(p) \)
for the complex modulus (equation (13)) and Hooke's law to arrive at predicted creep curves which agree with experiment. Starting with Hooke's relation in operational form:

$$\sigma(p) = E^*(p) \varepsilon(p)$$  \hspace{1cm} (22)

where \( \sigma \) and \( \varepsilon \) as usual denote stresses and strains, we can arrive at the transient creep curve by taking \( \sigma(p) = \sigma_0/p \) to be a step-function excitation in the stress. \( \sigma_0 \) is a constant which we shall take to be equal to \( 7.1 \times 10^7 \text{dyn/cm}^2 \) in order to be in accord with Lethersich's experiment. \( E^*(p) \) has the form of equation (13) with the parameters \( s, k_0 \) and \( \tau \) determined from the Wuenschel and Jordan experiments. Returning the resulting \( \varepsilon(p) \) to the time domain by Laplace inversion we obtain

$$\varepsilon(t) = \frac{\sigma_0}{\rho} \left[ \frac{\tau^2}{\Gamma(2-s)} + \frac{K^2 t^{2(s-1)}}{\Gamma(3-2s)} \right] H(t).$$  \hspace{1cm} (23)

\( \Gamma \) is the gamma function and \( H(t) \) is the Heaviside step function. We see that the transient creep function is composed of three terms, two of which have a simple power-law behaviour with the time. In Fig. 21 we reproduce Lethersich's transient creep experimental data for shear loading for a duration from one millisecond to one second. Just beneath these points we show our theoretical prediction based upon the parameters from Jordan's shear wave propagation data. Considering that perspex and plexiglas may have different mechanical properties, we find the agreement rather remarkable. At the bottom of the same figure we give the theoretical prediction for the transient creep that we would expect to find for longitudinal stress loading. As we have stated, Lethersich did not make measurements on this type of loading.

In the same paper Lethersich does, however, give additional experimental transient creep data over a longer time duration from one millisecond to two years, using an entirely different experimental arrangement. His results together with our predictions are shown in Fig. 22. At about ten seconds, our transient creep curve begins a sharp upward turn at about the same rate that his data turns at the end of about \( 10^5 \text{s} \). We can think of many reasons why our results may differ here. The wave propagation experiment of Jordan takes only a few milliseconds to run during which the plexiglas

![Fig. 21. Strain against log time relations for times up to 1 s after loading.](https://academic.oup.com/gji/article-abstract/13/1-3/197/919909)
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The determination of the quality factor $Q$ is important in seismology, as it relates to the energy dissipation in the medium. The value of $Q$ can be calculated from the transient creep behaviour of materials. In the long-time limit, $Q$ is related to the strain against log time for long times as shown in Fig. 22.

$$\sigma_0 = 7.3 \times 10^7 \text{ dyn/cm}^2$$

$\sigma_0$ is the peak stress at time $t = 0$.

![Fig. 22. Strain against log time for long times.](https://example.com/figure22)

$\rightarrow P$ waves; $\rightarrow \rightarrow S$ waves.

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<tr>
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Temperature effects are also very important when the mechanical properties of polymers are to be considered. Differences in molecular weight of high polymers are known to affect the transient creep behaviour. Regardless of the cause of this difference, it is apparent that in both cases we have at long times a simple power-law behaviour in time where the exponent for plexiglas is about one-half of the well-known value of one-third in Andrade's creep law for metals. Further work to better our understanding of the long time behaviour is indicated here.

Although our primary goal was to obtain these transient creep curves from wave propagation data, we can go even further and evaluate the retardation spectrum for both the shear and longitudinal excitations. We shall not go into details here; one should refer to the book on Linear Viscoelasticity by Bland (1960). We find that the retardation spectrum is composed of two terms, one being a simple power-law with exponent $s$ and the other a similar term with exponent $2s - 1$. Since $s$ is slightly less than 1 for the solids of interest to us, both exponents have a value slightly less than 1. The retardation spectrum is therefore very broad without the significant peaks that are characteristic of the simpler spring–dashpot models. It appears then that an infinite number of these spring–dashpot elements will be necessary to describe the mechanical behaviour of the kind of solids that we see with the relatively long wavelengths used in seismological studies. It is interesting to note that on the basis of our assumption, below some characteristic frequency (which is different for different solids) all solids have a power-law frequency dependence for their propagation function $\gamma(f)$ and that except for the numerical values of the three parameters, one cannot at
these frequencies tell a high polymer from a metal alloy or a rock. In our discussion, we have also assumed that one can extend this power-law behaviour to infinite frequency and therefore use Hilbert transforms to relate attenuation and phase lag. In doing this we have purposely ignored the existence of relaxation or scattering effects, which we know to be present in solids at the higher frequencies. We assume that these effects are band-limited and that for frequencies above these bands, the attenuation and phase lag curves will return to the same power-law behaviour that we assumed for the low frequencies. The fact that the Hilbert transform is an integration over the infinite frequency range and that we have seen that we can use this transform applied to $\alpha(f)$ to yield a $\nu(f)$ that agrees well with the experimental $\nu(f)$ data indicates that this is not a bad assumption.

7. Suggested modifications which allow for deviations from the power-law construction

When we attempt to apply the three-parameter approximation to the rock Pierre shale, the parameter $s$ as determined from the log-log plot of $\alpha(f)$ data (see McDonald et al. 1958) turns out to be greater than unity. This implies a time advance rather than a time delay and thus violates the very reason for introducing $\tau$ into the theory. The reason for this behaviour, which is most likely due to the edge of a relaxation or scattering effect, is not of concern to us here. Of greater interest is the fact that a log-log plot of $\theta(f)$ data does not have a slope greater than one, suggesting that the perturbing effect is not too important. Under such a circumstance it may be desirable to return to a two-parameter (i.e. $s$ and $k_0$) approximation and attempt to determine both of the parameters from the $\theta(f)$ or $\nu(f)$ data. This is a valid approach so long as the above slope is less than unity because the $\alpha(f)$ that is related to such a $\theta(f)$ by the Hilbert transform will still obey the Paley–Wiener condition. However, the interpretation of causality is now that the signal cannot be picked up by the detector before the source is excited instead of until a time after that for the infinite frequency component to reach the detector after the source excitation. The resulting values of these two parameters will of course be different from their values in the three-parameter case. Our results show that the two-parameter approximation when applied to plate waves in plexiglas sheet is only slightly inferior to the three-parameter and that the wave propagation in plexiglas sheet at ultrasonic frequencies is very much like wave propagation in Pierre shale at seismic frequencies.

Even with the third parameter $\tau$ present the result is not completely acceptable in that the predicted strain curve has the form of a power law in the time $t$ which increases without limit for large $t$. Jeffreys (1958) has stated that such a behaviour cannot account for the fact that mountains or even buildings continue to stand. In accordance with the parametric nature of our approach, we can easily correct the simple power law to yield a strain curve that approaches a constant value at infinite $t$ by simply replacing $p^s$ in $\gamma(p)$ by

$$p^s \left[ 1 + \left( \frac{\omega_L}{p} \right)^{1-s} \right]^{-1}.$$ 

Here $\omega_L$ is a fourth real parameter which must be extremely small since such a modification is not needed in plexiglas for frequencies above $10^{-4}$ c/s.

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References


Appendix

Since the concept of the spatial $Q$, i.e. $Q_R$, is so central to our discussion of the absorption problem, we should be precise in our definition of $Q_R$. We reproduce here the usual derivation:

Consider a ‘snapshot’ of a plane wave of single frequency $\omega/2\pi$ at two points $R_1$ and $R_2$:

$$
\begin{align*}
    u(R_1, t) &= U(0, \omega) e^{-a(\omega)R_1} \cos [\theta(\omega)R_1 - \omega t], \\
    u(R_2, t) &= P(0, \omega) e^{-a(\omega)R_2} \cos [\theta(\omega)R_2 - \omega t].
\end{align*}
$$

(A1)
For a positive peak to decay by a factor $e^{-\Delta}$, where

$$\Delta \equiv \pi/Q_R,$$  \hspace{1cm} (A2)

in going from $R_1$ to its next adjacent peak at $R_2$ we must have

$$\frac{u(R_1, t)}{u(R_2, t)} = e^{-n(\omega R_1 - R_2)} \cos \left[ \theta(\omega) R_1 - \omega t \right] \cos \left[ \theta(\omega) R_2 - \omega t \right] = e^{-\Delta}.$$

That is:

$$\alpha(\omega)(R_2 - R_1) = \Delta = \pi/Q_R,$$ \hspace{1cm} (A3a)

$$\theta(\omega) R_1 - \omega t = 2n\pi,$$ \hspace{1cm} (A3b)

$$\theta(\omega) R_2 - \omega t = 2(n + 1)\pi,$$ \hspace{1cm} (A3c)

where $n$ is an integer. If we eliminate $n$ and $R_2 - R_1$ from these three equations, then we arrive at the expression

$$Q_R(\omega) = \frac{\theta(\omega)}{2\alpha(\omega)}.$$ \hspace{1cm} (A4)