On Static Solutions of Einstein's Generalized Theory of Gravitation, II

Mineo IKEDA

Research Institute for Theoretical Physics, Hiroshima University
Takehara-machi, Hiroshima-ken

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Previously we considered a new expression of the electromagnetic field strength in the generalized theory of gravitation. In the present paper we study, on this basis, static solutions of the field equations for the magnetic case. At first a magnetostatic field is studied by introducing a scalar potential. Next, examining the feature of the field equations in Einstein's new theory, we show that a singularity of the field cannot be identified with the source, contrary to the classical field theories. By using this result it is further shown that a single magnetic pole does not exist. This conclusion is not contradictory to the observation, and consequently favorable to Einstein's new theory.

§ 1. Introduction

In a previous paper ¹ we have discussed static solutions of the generalized theory of gravitation ² for the electric case. The present paper is chiefly concerned with the magnetic case.

In Einstein's theory, there are several antisymmetric tensors of the second order which depend on the fundamental tensor $g_{\beta\mu}$ alone and may be interpreted as the strength of the electromagnetic field. Therefore, in I, we have investigated the mathematical expressions of the electromagnetic quantity, and have clarified that, in order for the treatment of the electromagnetic field to be kept as analogous as possible to the Maxwell theory, it is preferable to identify $F_{\lambda\mu}$ given by

$$F_{\lambda\mu} = - (\rho f_{\lambda\mu} + \epsilon_{\lambda\mu\nu\sigma} f^{\alpha\beta} g/2) / \sqrt{-g}, \quad (\alpha, \beta, \lambda, \mu = 1, \ldots, 4), \quad (1.1)$$

with the strength of the field, where

$$b_{\lambda\mu} = g_{\lambda\mu} = (g_{\lambda\mu} + g_{\mu\lambda})/2, \quad f_{\lambda\mu} = g_{\mu\lambda} = (g_{\lambda\mu} - g_{\mu\lambda})/2,$$

$$g = \det g_{\lambda\mu}, \quad g = \det b_{\lambda\mu}, \quad \rho = \epsilon^{\alpha\beta\gamma\delta} f_{\alpha\beta} f_{\gamma\delta}/2, \quad (1.2)$$

and both $\epsilon_{\lambda\mu\nu\sigma}$ and $\epsilon^{\lambda\mu\nu\sigma}$ are the Levi-Civita relative tensors. The indices of $f_{\lambda\mu}$ are raised and lowered by means of $b^{\lambda\mu}$ and $b_{\lambda\mu}$, the former being the conjugate of the latter. Then one of the field equations, (2.5) in I, is equivalent to

$$F_{\lambda\mu,\nu} + F_{\mu,\lambda\nu} + F_{\nu,\lambda\mu} = 0, \quad (1.3)$$

* Hereafter referred to as I.
where the comma denotes the ordinary differentiation with respect to the \( x \)'s. By means of this equation it is possible to introduce a four-vector potential in the same way as in the Maxwell theory.

When Wyman examined the static spherically symmetric solutions, he pointed out a certain difficulty about the boundary value problem in the electric case.\(^3\) He expected that this problem would be solved by obtaining a general solution including both the 'electric' and 'magnetic' parts. However, it was shown by Bonnor that such a solution does not throw any further light on this problem.\(^4\) Afterwards a method of avoiding the difficulty was put forward by the present author:\(^1\) He showed that if one uses the electrostatic potential, i.e., the fourth component of the vector potential above stated, then one can choose a suitable boundary condition, which is free from the difficulty. And the essential feature of I was the introduction of a 'scalar' potential for the electrostatic field.

In the present paper we shall deal with the magnetic case in a similar manner as in I. For that purpose we shall introduce for the magnetostatic field a scalar potential, whose existence is not a necessary consequence of the field equations. In the Maxwell theory a similar situation appears also, but for the discussion of the magnetostatic problems a scalar potential can often be introduced in many important cases, by means of which the treatment becomes very easy. Correspondingly we shall assume the existence of a potential \( \psi \) at the beginning of this work. \( \psi \) thus introduced can be expressed in terms of \( g_{\lambda\mu} \) for the field with or without spatial symmetry (§ 2). In particular, if \( h_{\lambda\mu} \) and \( \psi \) are assumed to be static s.s. (spherically symmetric),\(^*\) it follows that the form of \( f_{\lambda\mu} \) reduces to the general static s.s. tensor with vanishing electric part. This form of \( f_{\lambda\mu} \) has been used by other authors in studying the spherically symmetric solutions.\(^9\)-\(^7\) It is thus shown that for the static spherically symmetric field of the magnetic case hitherto studied a scalar potential can be introduced without any assumption (§ 3).

We shall further investigate the features of singularity of the field in the new Einstein theory, and shall show that, in contrast to the classical field theories, it is not appropriate to identify a singularity with the source of the field. If a single magnetic pole were existent as discussed by some authors, its field would be represented by a static s.s. solution without singularity. On the other hand, as will be shown, the general static s.s. solution for the magnetic case necessarily has singularity, that is, there is no coordinate system for which the solution is regular everywhere. Accordingly, we may conclude that our result indicates the non-existence of a single magnetic pole, which is consistent with the observation (§ 4). We shall lastly give some supplementary remarks to the previous and present papers, explaining our standpoint that \( F_{\lambda\mu} \) defined by (1·1) is identified with the strength of the electromagnetic field (§ 5).

\section*{§ 2. The magnetostatic field in general}

In the present work the strength of the electromagnetic field is represented by \( F_{\lambda\mu} \) in

\(^*\) A spherically symmetric (abbreviated as s.s.) tensor is, by definition, one which is form-invariant under the group of spatial rotations.
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(1·1). And this section, unlike the others, is concerned with the magnetostatic field for the general case, i.e., with or without spatial symmetry.

For the following purpose it is convenient to divide the electromagnetic field into the electric and magnetic parts, and to write the equations in a three-dimensional form. Further we shall use a coordinate system such as

\[ h_{\mu\nu} = 0, \quad (2·1) \]

which is analogous to the time-orthogonal system in general relativity. The index 4 is referred to the time coordinate, and the Latin indices are to the space coordinates and take the values 1, 2, 3. Then the equations to be treated, e.g., (2·2), (2·3), etc., are tensor equations for \( x' \), although not invariant for the general four-dimensional transformations of coordinates, since the index 4 appears in a special mode. This device is especially adequate in discussing the magnetostatic field, because the electric field may appear or the staticness of the field may be lost by a transformation of \( x' \).

Since the magnetostatic field is in question, it is reasonable to assume

\[ F_{\mu\nu} = 0 \quad (2·2) \]

by analogy with the Maxwell theory. Then we get from (1·1)*

\[ f_{\mu\nu} = 0, \quad (2·3) \]

a condition to be satisfied by \( g_{\lambda\mu} \) in the present case.

Later on we shall introduce a scalar potential for the magnetostatic field. For that purpose we must obtain from \( F_{\lambda\mu} \) the covariant components of the 'magnetic vector', which will be equated to the negative gradient of the potential. In forming such components there appears, unlike the case of general relativity, some arbitrariness due to the non-symmetric character of \( g_{\lambda\mu} \). So we shall follow here a procedure keeping a sufficient analogy with the Maxwell theory.

Now let \( \sigma \) be a space-like hypersurface defined by \( x^4 = \text{const.} \),** and consider in \( \sigma \) an axial vector

\[ H^4 = \epsilon^{4\mu\nu} \xi_{\mu\nu} / 2 \], \quad where \quad \xi_{\lambda\mu} = F_{\lambda\mu} / \sqrt{-g}. \quad (2·4) \]

This vector is equivalent to the spatial antisymmetric tensor of the second order with components \( F_{\epsilon\rho} \) and corresponds to the magnetic field strength in the Maxwell theory. (2·4) can also be written

\[ \xi_{ij} = \epsilon_{ij\nu} H^\nu, \]

which is used when the Maxwell equations are formulated in the Minkowski space-time referred to curvilinear coordinates. 8)

Since the magnetic vector \( H^4 \) lies in the three-dimensional space \( \sigma \), it is natural to

* See Appendix.
** \( \sigma \) may be regarded as a three-dimensional Riemannian space with the fundamental tensor \( h_{ij} \) (or equivalently \( g_{ij} \) by means of (2·3)).
define its covariant components by\(^*\)
\[
H_i = h_{ij} H^j,  \quad (2.5)
\]
where \(h_{ij}\) are the spatial components of \(h_{\mu\nu}\), i.e., the components without index 4. Then from the assumption (2·1) of time-orthogonality, we have
\[
H^i = \delta^i_4 H_4,  \quad (2.6)
\]
where \(\delta^i_4\) are the spatial components of \(\delta^{\mu}_4\), and form the conjugate of \(h_{ij}\) in \(\sigma\). It is to be noted that the right-hand member of (2·5) is equivalent to \(g_{\mu} H^\mu\), while the right-hand member of (2·6) is not necessarily coincident with \(g^{\mu} H_\mu\), as can easily be seen from (2·3).\(^*\)

In the Maxwell theory, a scalar potential is not necessarily useful for discussing a magnetostatic field. However, it is well known that, in important cases, it is possible to introduce a scalar potential, with which the problems are treated very easily. Moreover, in the Einstein theory, it was shown in I that a scalar potential is useful for the electrostatic case in discussing the boundary value problem. Thus we shall similarly assume that the magnetic force \(H_i\) can be derived from a scalar potential, i.e.,
\[
H_i = -\phi^\mu,  \quad (2.7)
\]
where \(\phi\) is a scalar function in \(\sigma\). It is obviously equivalent to the vanishing of the curl of \(H_i\). This means that our program corresponds to the classical case in which the curl of the field strength is zero (e.g., the magnetic dipole). We should notice that there is no such classical analogue for the spherically symmetric case. By substituting (2·5), (2·4) and (1·1), we have from (2·7)
\[
\phi^\mu = -h_{\alpha} f^{\star}_4 g/\sigma,  \quad (2.8)
\]
which expresses the derivatives of \(\phi\) in terms of \(g_{\mu\nu}\).

§ 3. The spherically symmetric solutions of the magnetic case

In this section we shall study the static spherically symmetric field of the magnetic case. In this case we may assume as in I
\[
h_{\mu\nu} = \begin{pmatrix}
-A(r) & 0 & 0 & 0 \\
0 & -B(r) & 0 & 0 \\
0 & 0 & -B(r) \sin^2 \theta & 0 \\
0 & 0 & 0 & C(r)
\end{pmatrix},  \quad (3.1)
\]

\(^*\) \(H^4\) can be regarded as a vector \(H^4\) in the four-dimensional space-time considered, where the fourth component is taken to be zero. By lowering the indices by means of \(h_{\mu\nu}\), we may obtain a covariant vector, which does not coincide with (2·5) in general. However, under the assumption (2·1), its fourth component is also zero and the spatial ones reduce to (2·5).

\(^{**}\) \(g^{\mu}_4\) are the spatial components of \(g^{\mu\nu}\) defined by \(g^{\mu}_4 g_{\mu\nu} = \delta^{\mu}_\nu\), and are not necessarily symmetric in general.
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\[ \psi = \psi(r), \quad (3.2) \]

using the 'polar coordinates' \( x^2 = (r, \theta, \varphi, ct) \). Then we obtain from (2.8)

\[ f_{34} = f_{44} = 0, \]
\[ \psi' = Af_{34}/\{AC - (f_{34})^2\}, \quad (3.3) \]

where the prime denotes the differentiation with respect to \( r \). Solving (3.3) for \( f_{34} \) we know that \( f_{34} \) is a function of \( r \) alone. Thus \( f_{3\mu} \) representing the magnetostatic field with spherical symmetry reduces to the form

\[ f_{34} = -f_{44} = f(r), \quad \text{other } f_{3\mu} = 0. \quad (3.4) \]

This form constitutes, together with (3.12) in I, the general form of the static antisymmetric s.s. tensor of the second order, and it was adopted by other writers as the starting point of their studies on the spherically symmetric solutions of the magnetic case.\(^9\)\(^-\)\(^7\) Thus for the case under consideration, supposing that the force of the magnetic field be derived from a scalar potential, we can obtain the general antisymmetric s.s. tensor without the electric part. This implies that for the spherically symmetric field of the magnetic case studied by many authors, a scalar potential can be introduced without any assumption, and hence (2.7) must be considered as the definition of \( \psi \).

We have thus seen that when the field is static spherically symmetric, the form of \( b_{\lambda \mu} \) and \( \psi \) can be obtained by integrating the field equations under the conditions (3.1) and (3.4), and then substituting the results in (3.3). For this case Papapetrou obtained a solution of the field equations (i.e., \( g_{\lambda \mu} \)), which is given by\(^9\)

\[ A = (1 - 2m/r)^{-1}, \quad B = r^2, \quad C = (1 - 2m/r)(1 + e_m^2/r^4), \quad (3.5) \]
\[ f = e_m^2/r^2, \quad (3.6) \]

\( m \) and \( e_m \) being arbitrary constants of integration. The most general solution under the assumption (3.1) and (3.4) is obtained from this by carrying out an arbitrary transformation of the radial coordinate. By substituting (3.5) and (3.6) in (3.3) and integrating, we have

\[ \psi = (e_m/2m)\log(1 - 2m/r) + \text{const.}, \quad (3.7) \]

which is the form of the potential for the present case.

It follows from (3.5) and (3.7) that

\[ b_{\lambda \mu} \rightarrow \text{Galilean values}, \quad \psi \rightarrow \text{const.} \quad (\neq \infty) \quad \text{as } r \rightarrow \infty, \quad (3.8) \]

which expresses the vanishing of the field at infinity and can be considered as a natural boundary condition. In the same way as in I we can show that this condition is, in a sense, independent of the coordinate system, that is, invariant under \( S \) (the class of transformations of \( x^\prime \) leaving spatial infinity invariant) and \( T \) (the class of linear transformations of \( x^\prime \)).

The above solution has been interpreted as representing a single magnetic pole by other authors.\(^9\)\(^,\)\(^8\) This interpretation was obtained from the traditional standpoint that singularities
of the field represent its sources. However, such interpretation is not possible in the generalized theory of gravitation, and the meaning of the solution must be considered physically according to the characteristic feature of the theory. This will be discussed in detail in the following section.

§ 4. Singularity and source of the field

Before we study the character of singularity of the field in Einstein's new theory, we shall briefly consider how a singularity has been interpreted in the classical field theories.

As is well known, in the Maxwell theory, the electrostatic potential satisfies Laplace's equation in a region where there are no charges. Now let a solution of this equation have a singular region. Then, in such a region, there should be sources which will produce the 'exterior' field corresponding to the solution, and the potential does not satisfy Laplace's equation but must satisfy Poisson's equation. As is easily seen, this interpretation is based on the existence of the charge density in the field equation.

In general relativity, a similar situation is seen for Einstein's gravitational field equation. The source of the field is represented by a singularity of the exterior solution, and in some region containing the singularity there should exist a suitable solution of the field equation with the non-vanishing energy-momentum tensor. Many investigations on this subject were made by Lichnerowicz and others. From their results, one can clearly see how a singularity of the exterior field is connected with the existence of the energy-momentum tensor in the field equation.

These show that, in the classical field theories, a singularity of the field is generally interpreted as the source, or the region with singularity is considered as being filled with the source of the field. This interpretation is possible by means of the common feature of the classical field theories that a phenomenological description of the source enters into the field equation.

On the other hand, in Einstein's new theory, the field equations are dependent on the field quantity (i.e., non-symmetric $g_{\mu\nu}$ alone, and have no quantity representing the source of the field (such as the charge density in Poisson's equation, or the energy-momentum tensor in Einstein's equation). Therefore, if the field equations have a solution which is not free from singularity, there exists a region in which the field equations are not satisfied by the solution. The new theory has no basis on which to interpret this region physically, and consequently it is a regular solution that may be required from the physical point of view. We must thus seek after those solutions which are regular everywhere.

This is characteristic of the new theory and can not be ignored in studying rigorous solutions of the field equations. Nevertheless, when many writers discussed the s.s. solutions, they kept to the traditional view that the sources are represented by singularities of the field. So we shall discuss Papapetrou's solution on the basis of the above consideration.

It is easily seen from (3·5) and (3·6) that Papapetrou's solution has a singular point $r=0$ and a singular surface $r=2m$ in the coordinate system in which $g_{\mu\nu}$ is given by (3·1) and (3·4) with $B=r^2$. These singularities cannot be removed by any (probably...
singular) transformation such as \( r' = r(r) \). For, even if \( C \) is made regular everywhere by a suitable choice of \( r'(r) \), \( B \) will simultaneously become singular at some value of \( r' \), since \( B \) is regular at \( r = 2m \) but \( C \) is not.\(^*\) We thus know that there is no choice of radial coordinate by which the solution becomes free from singularity.

A similar circumstance holds for a more general change of coordinates, namely, for a transformation leaving the boundary condition (3·8) invariant (i.e., one belonging to \( S \) or \( T \) in §3). This is trivial for \( T \). Also for \( S \) one may see this at a glance from the scalar character of \( \psi \) in (3·7), since a finite point is transformed into such a point by any transformation of \( S \). Now we are going to prove this in another way without using \( \psi \).

In the following argument \((x^\lambda)\) denotes the coordinate system in which \( g_{\lambda\mu} \) is given by (3·1) and (3·4) with \( B = r^2 \), and \((x^\lambda')\) a system obtained from \((x^\lambda)\) by a transformation of \( S \). Assume that there is a coordinate system \((x^\lambda')\) in which the solution is regular everywhere, and let a two-dimensional closed surface \( F \) be defined by \( r = \text{const.} \) in \((x^\lambda)\) in a space-like hypersurface \( \sigma \) given in §2. Then its interior domain \( V \) is finite in \((x^\lambda)_*\), and we have Stokes' theorem\(^1\)

\[
3 \int_V F_{\mu\lambda}' k^{\alpha\beta} d'v = \int_F F_{\mu\lambda}' k^{\alpha\beta} d's,
\]

(4·1)

where \( k^{\alpha\beta} d'v \) and \( k^{\alpha\beta} d's \) are a volume element of \( V \) and a surface element of \( F \) in \((x^\lambda')\) respectively, and antisymmetric with respect to all indices. This equation is invariant under a transformation of \( S \), because it is possible to replace the ordinary differentiation in it by the corresponding covariant one with respect to \( h_{ij} \).\(^*\) Since (1·3) in \((x^\lambda')\) is satisfied throughout \( V \) by the assumption of regularity, we have, by use of the antisymmetry of \( k^{\alpha\beta} \),

\[
\int_V F_{\mu\lambda}' k^{\alpha\beta} d's = 0.
\]

(4·2)

On the other hand, it holds that

\[
\int_F F_{\mu\lambda}' k^{\alpha\beta} d's = \int_F k^{\alpha\beta} ds,
\]

(4·3)

because of the invariant character of the integral. If we choose a surface element of \( F \) so that its bounding curves are given by \( \theta = \text{const.}, \varphi = \text{const.} \), we get

\[2 k^{\alpha\beta} ds = d\theta d\varphi, \quad \text{other} \quad k^{ij} ds = 0.\]

Then from (3·5) and (3·6), the right-hand member of (4·3) becomes \( e_m \) apart from a constant factor, and consequently \( e_m = 0 \) by comparing with (4·2). This means that the solution degenerates into the one for the 'pure gravitational' field, hence the field equations coincide with those of general relativity for empty space.\(^*\) In this case it is

\( * \) \( B \) and \( C \) transform as scalars by \( r' = r(r) \).

\( ** \) See footnote at page 267.

\( *** \) For the symmetric \( g_{\mu\nu}, \Gamma_{\mu\nu} \) is not necessarily symmetric. But in the s.s. case \( \Gamma_{\mu\nu} \) follows from \( f_{\lambda\mu} = 0 \). See the form of \( \Gamma_{\lambda\mu} \) in 7).
known that an asymptotically flat solution cannot be free from singularity, unless it reduces to the flat solution.\textsuperscript{10}

We have seen that, in the magnetic case, there exists no regular static spherically symmetric solution satisfying the boundary condition (3·8). Moreover if a single magnetic pole were existent, its field would be represented by a static s.s. solution. Thus we may say that a single magnetic pole cannot exist in the Einstein new theory, as in the Maxwell theory. This theoretical conclusion is consistent with the observation.

On the other hand, in the electric case, there is a regular solution which satisfies the boundary condition in (1). It is given by (5·1), (5·2) with (5·3), (5·9) in I, where \(C\) satisfies the following conditions: It takes positive values other than 1, and has derivatives of the required orders, the first of which vanishes nowhere. Further it reduces asymptotically to (5·12) in I.*

Since the above conclusions on the existence of regular solutions are not contradictory to the observation, we may state that Einstein's new theory is satisfactory at least concerning the subject now considered.

It is to be noted that our argument in this section is based on (1·3) and (3·8), and therefore that the expression (1·1) of the electromagnetic field strength and the introduction of a scalar potential play an important role in our discussion. A similar result may follow from the ordinary standpoint that the conjugate tensor of \(f_{\lambda\mu}\) is identified with the field strength, where, however, ambiguous considerations will be needed.

\section{5. Supplementary remarks}

We have studied on the basis that \(F_{\lambda\mu}\) in (1·1) represents the strength of the electromagnetic field. Comparing with the ordinary view that \(f_{\lambda\mu}\) is more basic, our treatment would seem to be complicated formally. But we can introduce naturally a scalar potential for the electrostatic and magnetostatic fields. Moreover, as a consequence of this introduction, the non-covariance of the boundary condition is removed, and it is possible to show the non-existence of a single magnetic pole. These are the salient features of our method. In this last section we shall clarify our standpoint adding further complementary remarks.

Many writers of various orders have studied the weak field with the help of the approximation method of various orders. In the first approximation it holds that

\[
F_{\lambda\mu} \sim \epsilon_{\lambda\mu\nu\rho} \gamma^{\alpha\rho} \gamma^{\beta\epsilon} f_{\beta\epsilon} \sim \epsilon_{\lambda\mu\nu\rho} f^{\alpha\rho},
\]

where \(\gamma^\nu = \gamma^\rho = \gamma^\beta = - \gamma^\mu = -1\) and other \(\gamma^{\lambda\mu} = 0\). Hence all the results hitherto obtained in the linear approximation are valid also for the new expression of the field strength.

In Einstein's theory the charge-current density is given by

\[\text{\tiny * Since } \sin\theta \text{ appears in the denominator of } \Gamma_{\lambda\mu\nu}, \text{ the domain } \sin\theta=0 \text{ may seem to be singular in a sense. This domain is not excluded, however, in discussing the field equations which depend on functions of } r \text{ alone, because we can understand it as the limiting cases } \theta \to 0, \pi. \text{ Thus we have only to consider singularities which may arise from the form of } A, B, C \text{ and } h.\]
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\[
\mathbf{j}^\lambda = \varepsilon^{\lambda\mu\nu} f_{\mu\nu}/3!.
\]  \(\text{(5.1)}\)

If one considers the dual relative tensor of \(f_{\lambda\mu}\), i.e., \(\varepsilon^{\lambda\mu\nu} f_{\mu\nu}/2!\), as the field strength, the divergence of this relative tensor coincides with the charge-current density, just as in the Maxwell theory. On the other hand, the divergence of \(F_{\lambda\mu}\) is not equal to the charge-current given by \(\text{(5.1)}\). Therefore, when \(F_{\lambda\mu}\) is interpreted as the strength, the analogy with Maxwell's theory comes to a stop in this respect. This analogy is, however, valid for the linear approximation, that is, the divergence of \(F_{\lambda\mu} = \gamma^{\lambda\alpha} \gamma^{\mu\beta} F_{\alpha\beta}\) is equal to \(\mathbf{j}^\lambda = j^\lambda / \sqrt{-g}\) in this approximation.

In the present work \(h_{\mu\nu}\), \(\phi\) and \(\psi\) were assumed to be spherically symmetric, namely, form-invariant under the group of spatial rotations. The same assumption of spherical symmetry was put on \(g_{\lambda\mu}\) in the studies of the other authors. Consequently it may seem at first sight that we did not deal with the two fields in a unified mode. In other words, in the usual method the condition is imposed only on one tensor, while in our new method on one tensor and two scalars. However, when remarking that \(g_{\lambda\mu}\) is reducible and decomposed into two irreducible tensors \(h_{\lambda\mu}\) and \(f_{\lambda\mu}\), we see that the condition is actually put on two tensors in the former treatment. We may thus say that it is merely a matter of appearance that the new treatment of both fields seems to be less unified. In fact, in § 3 of I and § 3 of the present paper, the spherical symmetry of \(f_{\lambda\mu}\) was derived from that of \(h_{\lambda\mu}\), \(\phi\) and \(\psi\), which shows the equivalency of both methods.

The general theory, developed in §§ 2, 3 of I and § 2 of the present paper, seems of little use for the spherically symmetric case, except the problems depending on the boundary conditions. However, it will provide a powerful means, when one tries to obtain a solution with other spatial symmetry. For instance, when the static axially symmetric solutions are in question, it is hardly clear how the form of \(g_{\lambda\mu}\) is to be assumed. In this case, it seems reasonable to assume \(g_{\lambda\mu}\) to be a general axially symmetric tensor, but this condition is too weak, and, hence, is of no use for the integration of the field equations. However, if we adopt our theory the procedure may be as follows: We take as \(h_{\lambda\mu}\) the metric tensor of Weyl by analogy with general relativity. As for the electromagnetic field, the potential \(\phi\) or \(\psi\) is assumed to be static axially symmetric, i.e., independent of \(\varphi\) and \(t\) in the 'cylindrical coordinates' \(x^\lambda = (\rho, z, \varphi, t)\). Then by calculations analogous to this work we can easily obtain the form of \(f_{\lambda\mu}\), which is found to be a special form of the axially symmetric tensor. The form of \(g_{\lambda\mu}\) thus obtained gives a more stringent condition than the above, and therefore, may simplify the integration of the field equations.

Conclusion

The present paper is concerned with static solutions of the generalized theory of gravitation for the magnetic case. For the magnetostatic field we introduced a scalar potential \(\psi\), which was useful for the problems treated in this work. A spherically symmetric solution was previously obtained by Papapetrou, and it has been thought of as corresponding to a single magnetic pole by some writers. Examining this solution, we saw that \(h_{\lambda\mu}\) and \(\psi\) satisfy a reasonable boundary condition, and that there is no coordinate system in which
the solution is free from singularity. On the other hand, we clarified that it is a regular solution that may be required from the physical point of view in the new theory. We may thus conclude that a single magnetic pole does not exist. This result is not contradictory to the observation and consequently favorable to Einstein’s theory.

In his concluding remarks, Einstein stated, "......we know no method to find singularity-free solutions for such a system of equations——not even any method by which to judge the very existence or non-existence of singularity-free solution." We may say that the discussion in § 4 gives an example of an approach to this subject for the particular case.

In general relativity the Schwarzschild solution brought some satisfactory results, which explained the famous three observed effects. In order that the spherically symmetric solution of the electric case discussed in I may similarly provide the conclusive evidence of the theory, it is necessary to find out a phenomenon such that the distribution of matter and charge is static spherically symmetric, and the mutual effect between them can not possibly be neglected. At the present stage, however, we know of no such phenomenon, so that we can say for the present that the validity of Einstein’s theory has not yet been determined.

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Appendix

Proof of (2·3)
By substituting (1·1), (2·2) reduces to
\[ pf_{14} + f_{1k}g = 0, \quad (i, j, k = \text{cycl.}(1, 2, 3)). \] (A·1)

Now let us choose the coordinates so that at an arbitrary point \( P \) of the space time
\[ h_{11} = h_{22} = h_{33} = -h_{44} = -1, \quad \text{other} \quad h_{\lambda\mu} = 0, \]
then (A·1) becomes at \( P \)
\[ \rho f_{14} - f_{1k} = 0, \quad (i, j, k = \text{cycl.}(1, 2, 3)). \] (A·2)

Multiplying \( f_{14} \) and summing for \( i \), we obtain by using (1·2)
\[ \rho^2 - (f_{14})^2 - (f_{24})^2 - (f_{34})^2 = 0. \] (A·3)
On the other hand, we have
\[ g = g + \rho^2 + g h^{15} h^{17} f_{17} f_{15}/2 \]
and consequently by using (A·2) and (A·3)
\[ g = -1 + (f_{14})^2 + (f_{24})^2 + (f_{34})^2. \] (A·4)
If \( \rho \neq 0 \), then we have \( g = 0 \) by means of (A·2) and (A·4). Therefore, \( \rho \) must vanish, hence \( f = 0 \) in consequence of (A·2).
References

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