Geriatric-patient flow-rate modelling

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We consider the application of a three-compartment mathematical model using difference equations in discrete time to model the flow of patients through departments of geriatric medicine. It has been shown empirically that the time spent in hospital since admission may be described by a two-term mixed-exponential distribution. Previous work has explained this empirical finding in terms of a two-compartment model of geriatric in-patient behaviour where the two compartments are acute/rehabilitative and long-stay care. Our model extends this approach by considering patients released from geriatric departments and their subsequent length of stay in the community. We have therefore, two states of patient behaviour while in hospital: (i) patients can be admitted to the acute-rehabilitative state, from which they may die or be released back into the community; (ii) patients can be admitted to the long-stay state, from which they eventually die. The community state currently has only one state, from which patients may be readmitted back into the geriatric department or they may die. The model may then be used to estimate the average numbers and lengths of stay for short-term and long-term patients and the average number and length of stay in the community for released patients, allowing for a significant improvement in the forecasting of future bed requirements to aid the planning of geriatric departments.

Keywords: geriatric medicine; hospital-bed management; mixed exponential; first-order linear difference equations.

1. Background

In recent years advances in the care of elderly people has meant a steady increase in life expectancy, and, therefore, in turn, an increasing proportion of elderly people in the population. At the same time, the proportion of young people participating in the workforce is declining. The problem of caring for the frail or sick elderly therefore requires urgent attention and careful planning, particularly in an environment of economic recession and government cutbacks.

The style of management employed in geriatric departments may vary between hospitals, mainly due to the resources available. However, whatever the style of management, the underlying principle remains the same: multidisciplinary assessment and rehabilitation, before admission to long-stay care.

Current guidelines for the assessment of hospital departments are based on a crude
turnover, or mean length of stay, for a patient. However, the existence of long-stay patients—who, although only a small proportion of the total number of patients passing through a department, may remain in hospital for months, or even years—suggests that the mean length of stay for a patient is no longer an appropriate measure of performance.

We have, therefore, developed a mathematical model that may be used to produce a performance indicator to analyse the pattern of bed occupancy in geriatric departments and the subsequent time they spend outside the department after release.

### 2. The empirical model

The model for in-patients has two compartments, representing short-stay and long-stay patients; it was developed as a result of the empirical observation (Millard, 1988) that the pattern of bed occupancy is best fitted by a two-term mixed-exponential distribution of the form

\[ T_c(x) = Ae^{-Bx} + Ce^{-Dx}, \]  

where \( x \) is the patient’s length of stay in hospital, up to the present, and \( T_c(x) \) is the number of patients who have been in the department for at least \( x \) days.

The model for released patients, which has one compartment representing an ex-patient back in the community, was developed from an empirical observation that the pattern of time spent back in the community by ex-patients is best fitted by a single-term exponential distribution of the form

\[ C_c(y) = E e^{-Fy}, \]  

where \( y \) is the ex-patients’ length of stay in the community, up to the present, and \( C_c(y) \) is the number of ex-patients who have been in the community for at least \( y \) days.

Our model explains these findings in terms of the flow rates of acute, long-stay, and released patients. The midnight bed states collected daily in all National Health Service hospitals provide data on \( x, T_c(x), y, \) and \( C_c(y). \) The model may then be used to predict future requirements for geriatric departments.

### 3. The theoretical model

The theoretical model is based on looking at the flow of patients around the system. The system is represented by a compartment model with three compartments. The flow of patients through the theoretical system is illustrated in Fig. 1. All the patients are initially admitted into acute care. From there they are discharged back into the community at a rate \( r_1(s) \) (where \( s \) is the time that a patient has spent in hospital), they die at a rate \( r_2(s) \), or they are transferred to long-stay care at a rate \( u(s) \). Long-stay patients stay in long-stay care until they die at a rate \( d(s) \). Ex-patients in the community may be readmitted from the community back into the department, at a rate \( u_1(s) \), where \( s \) is now the amount of time that the patient has spent in the community, or they may die at a rate \( u_2(s) \). This model is a generalization of a model by Harrison & Millard (1991) which describes the hospital part of the system.
By setting up equations for the model where $A(s, t)$ is the number of patients that have been in acute care for $s$ days at time $t$, $L(s, t)$ is the number of patients who were admitted to hospital $s$ days ago and are in long-stay care at time $t$, and $C(s, t)$ is the number of ex-patients that have been in the community for $s$ days at time $t$, then we can obtain the equations

$$
\begin{align*}
A(s + 1, t + 1) &= A(s, t) - r_1(s)A(s, t) - v(s)A(s, t) - r_2(s)A(s, t), \\
L(s + 1, t + 1) &= L(s, t) + v(s)A(s, t) - d(s)L(s, t), \\
C(s + 1, t + 1) &= C(s, t) - u_1(s)C(s, t) - u_2(s)C(s, t).
\end{align*}
$$

In order to solve these equations we make the following assumptions, which are similar to assumptions previously made by Harrison & Millard (1991).

**Assumption 1** The discharge rate per occupied bed and the conversion rates from acute care to long-stay care are independent of the length of stay in hospital ($s$); that is, $r_1(s) = r_1$, $v(s) = v$, etc., so

$$
\begin{align*}
A(s + 1, t + 1) &= A(s, t) - r_1 A(s, t) - vA(s, t) - r_2 A(s, t), \\
L(s + 1, t + 1) &= L(s, t) + vA(s, t) - dL(s, t), \\
C(s + 1, t + 1) &= C(s, t) - u_1 C(s, t) - u_2 C(s, t).
\end{align*}
$$

Assumption 1 may not be particularly valid, as it would seem to be natural that the length of time a patient spends in hospital will have an effect on their chances of leaving. However, the assumption is made for three reasons: first, the results of such a model would be very difficult to analyse; secondly, data to fit such a model is not presently available; and, thirdly, the empirical data that does exist supports such an assumption.

**Assumption 2** The total number of acute-care and long-stay patients are both kept at their maximum levels; that is,

$$A(s, t + 1) = A(s, t) = A(s), \quad L(s, t + 1) = L(s, t) = L(s), \quad C(s, t + 1) = C(s, t) = C(s),$$
so
\[
A(s + 1) = A(s) - r_1A(s) - vA(s) - r_2A(s),
\]
\[
L(s + 1) = L(s) + vA(s) + dL(s),
\]
\[
C(s + 1) = C(s) - u_1C(s) - u_2C(s).
\]
(5)

Assumption 2 is equivalent to saying that the department is in equilibrium, and therefore that the number of patients in each compartment at time \( t + 1 \) is the same as the number of patients at time \( t \).

By solving these equations, we now obtain
\[
A(s) = (1 - v - r_1 - r_2)^s A_0, \\
L(s) = kA_0(1 - d)^s - kA_0(1 - v - r_1 - r_2)^s \quad \text{where } k = \frac{v}{(v + r_1 + r_2 - d)}, \\
C(s) = C_0(1 - u_1 - u_2)^s,
\]
(6)

where \( A_0 \) is the number of patients entering acute care per day and \( C_0 \) is the number of patients released back into the community per day. From Fig. 1 and equation (6) we can see that no patients are admitted directly into long-stay care; therefore, \( L(0) = 0 \).

Although there may be a few empty beds in a department, in general, hospital administrators will try to run the department as close to full capacity as possible. This also implies that the number of patients in acute care, in long-stay care, and in the community have reached their equilibrium levels. Reaching such an equilibrium again may take a number of years after the introduction of a change in policy.

In order to compare the empirical and theoretical models we need to calculate the theoretical number of patients that have been in a compartment for more than \( x \) days. This is achieved by simply summing the equations gained from Assumption 2 from \( x \) to infinity. This leads to the following equations.

The number of patients who have been in acute care for \( x \) days or more is
\[
A_c(x) = \sum_{s=x}^{\infty} A(s) = \frac{A_0(1 - v - r_1 - r_2)^x}{v + r_1 + r_2}.
\]
(7)

The number of patients who have been in long-stay care in hospital for \( x \) days or more is
\[
L_c(x) = \sum_{s=x}^{\infty} L(s) = A_0k \left( \frac{(1 - d)^x}{d} - \frac{1 - v - r_1 - r_2)^x}{v + r_1 + r_2} \right).
\]
(8)

The total number of patients in hospital for \( x \) days or more is
\[
T_c(x) = A_c(x) + L_c(x) = A_0 \left( 1 - k \right) \frac{(1 - v - r_1 - r_2)^x}{v + r_1 + r_2} - \frac{k(1 - d)^x}{d},
\]
(9)

and the total number of beds \( T \) may be represented by
\[
T = A_c(0) + L_c(0) = A + L,
\]
(10)
where $A$ is the number of acute beds and $L$ is the number of long-stay beds. Similarly, the number of ex-patients who have been in the community for $x$ days or more is

$$C_e(x) = \sum_{s=x}^{\infty} C(s) = \frac{C_0(1 - u_1 - u_2)^x}{u_1 + u_2}. \quad (11)$$

We are also now able to calculate $C_0$, the number of patients being released from acute care per day:

$$C_0 = r_1 A_e(0) = \frac{r_1 A_0}{v + r_1 + r_2}. \quad (12)$$

4. Using the model

By using data based on admission rather than discharge dates we may gain an accurate picture of long-stay patients, who never appear in discharge statistics, rather than concentrating on short-stay patients. By focusing our attention on the simple principles that, at any time, the in-patients are first allocated to acute care before they are shifted to long-stay care, the model helps us to distinguish patterns that may otherwise be obscured by the complexities of the system.

In order to calculate the values of the theoretical model we need to equate our empirical and theoretical models and then solve for our theoretical parameters.

From the equation for hospital in-patients we obtain

$$A = \frac{(1 - k) A_0}{v + r_1 + r_2}, \quad e^{-B} = 1 - v - r_1 - r_2, \quad \left\{ \begin{array}{l}
A = \frac{A_0 k}{d}, \\
C = \frac{A_0 e^{-D}}{d}, \\
C = \frac{A_0}{d} \quad e^{-D} = 1 - d.
\end{array} \right. \quad (13)$$

Similarly from the equation for ex-patients in the community we obtain

$$E = \frac{C_0}{u_1 + u_2}, \quad e^{-F} = 1 - u_1 - u_2. \quad (14)$$

5. Application of the model

5.1 The data

The model was fitted to data collected by Professor Millard from 1969 to 1984 consisting of 6994 geriatric patients admitted between 1969 and 1984 to four sites in the health authority containing St George's Hospital, London (McClean & Millard, 1993a, b). We will fit the model to data for a hypothetical midnight bed return; that is, a census approach is used. A census approach enables relatively easy and available collection of data by the hospital administrators. The alternative, which is to fit the model to a cohort of patients, would of course be possible using our data, but this is less practicable for hospital managers looking for a quick and cost effective way of monitoring their service.
The choice of the census date for fitting our model was influenced mainly by the assumptions made in obtaining the model. The data is required to be in a steady state and therefore the number of beds needs to be fairly constant and all departments need to have settled down to an equilibrium state.

The first of December 1976 was chosen because in January 1977 Professor Millard checked the death records for all patients in the data. Therefore the data around this time is statistically complete. A Wednesday was chosen to avoid the phenomenon that patients tend to be admitted towards the beginning of the week and discharged at the end of the week. There may also be seasonal trends present within the data, however, and therefore we avoided dates soon after traditional holidays such as Christmas and Easter.

Visual analysis of the fitted curves for in-patients and ex-patients in the community (Figs. 2 and 3) shows an accurate fit for the observed data. The curves were fitted using nonlinear least squares.

5.2 Estimating the parameters

By using least squares we were able to fit the curves of the census data, on the basis of the mixed-exponential distribution for in-patients and the single-exponential model for ex-patients in the community. From our values for $A$, $B$, $C$, $D$, $E$, and $F$, and knowing the proportion of patients on their first admission to hospital, we calculated the values given in Fig. 4 for our theoretical parameters.
5.2 Calculation of the average time spent in a compartment

\[
\begin{align*}
\text{average time spent in acute care} & = \frac{1}{v + r_1 + r_2} = 37 \text{ days}, \\
\text{average time spent in long-stay care} & = \frac{1}{d} = 1000 \text{ days}, \\
\text{average time spent in a geriatric department} & = \frac{k}{d} + \frac{1-k}{v + r_1 + r_2} = 76 \text{ days}, \\
\text{average time spent in the community} & = \frac{1}{u_1 + u_2} = 714 \text{ days}. \\
\end{align*}
\]

The considerable difference between average time spent in acute care (37 days) and the average time spent in long-stay care (1000 days) illustrates the obvious drawback of using a simple average of the time that all the patients spend in hospital as a performance measure.

6. The effect of changes to the system

6.1 Introduction

Initially we need to calculate the current admission rates and the number of patients in acute and long-stay care. Although all patients enter acute care in the first instance,
as the department settles down to an equilibrium the admission rate decreases to

$$A_0 = \frac{d(r_1 + r_2 + v)}{v + d} T = 2.25 \text{ people per day},$$

(16)

and the number of patients in acute and long-stay care, respectively, may be calculated as

$$A = \frac{dT}{v + d} = 80 \text{ patients}, \quad L = \frac{vT}{v + d} = 80 \text{ patients}.$$  \hspace{1cm} (17)

The number of patients in acute care $A$ is therefore equal to the total number of beds $T$ (for the sample data there are 160 beds in the geriatric department) multiplied by the long-term death rate $d$ and divided by the sum of the transfer rate $v$ and the long-term death rate $d$; and the number of patients in long-stay care $L$ is equal to the total number of beds $T$ multiplied by the transfer rate $v$ and divided by the sum of the transfer rate $v$ and the long-term death rate $d$. Therefore, to achieve a permanent increase in admission rate either the total number of beds $T$ must be increased or the flow rates per bed $r_1$ or $v$ must be changed. (Changes in $r_2$ or $d$ will also affect the admission rate but changing these in order to increase the admission rate is ethically out of the question. However, changes may occur naturally.) Alternatively, we may increase the proportion of patients who are on their first admission since this implies that the department can serve a larger number of people.

6.2 Changes in the number of beds

If $\Delta T$ beds are added to the department, initially these beds will fill up with acute patients. However, over time, an equilibrium will be reached and a proportion of the
new beds will fill with long-stay patients. This results in new admission rate of
\[ A_0^* = \frac{d(v + r_1 + r_2)(T + \Delta T)}{v + d}, \]  
(18)
in a change in the admission rate of
\[ \Delta A_0 = \frac{d(r_1 + r_2 + \Delta T)}{v + d}, \]  
(19)
and in a relative change in the admission rate of
\[ \frac{\Delta A_0}{A_0} = \frac{\Delta T}{T}. \]  
(20)

Therefore, a 10% increase or decrease in the number of beds will eventually lead to
a 10% increase or decrease, respectively, in the number of admissions. This result is
independent of any of the factors controlling the department, that is, the death rates,
the transfer rates, and the release rates. Although an increase in the number of beds
has a significant effect on the admission rate for the majority of departments, the
expense involved in setting up new beds makes such an alternative impossible since
there would also need to be an increase in staff and in all the other resources required,
and in the long term these factors have the most significant costs.

6.3 Changes in the acute-care release rate

A change in the release rate \( r_1 \) will lead to a new admission rate of
\[ A_0^* = \frac{d(v + r_1 + \Delta r_1 + r_2)T}{v + d}, \]  
(21)
in a change in the admission rate of
\[ \Delta A_0 = A_0 \Delta r_1, \]  
(22)
and in a relative change in the admission rate of
\[ \frac{\Delta A_0}{A_0} = \frac{\Delta r_1}{r_1 + r_2 + v}. \]  
(23)

Therefore, the effect of a change in the acute release rate on the admission rate
depends on the death rate from acute care and on the proportion of patients
transferred to long-stay care.

By considering our sample data, and by applying a 10% increase to the acute
release rate, we can obtain
\[ \Delta A_0 = 0.12, \quad \frac{\Delta A_0}{A_0} = 0.053. \]  
(24)

Therefore, for our sample data, a 10% increase in the release rate from acute care
will lead to a 5.3% increase in admissions.
6.4 Changes in the conversion rate to long-stay care

A change in the conversion rate to long-stay care \( v \) by \( \Delta v \) will eventually lead to a new admission rate of

\[
A_0^* = \frac{d(v + \Delta v + r_1 + r_2)T}{v + d + \Delta v}, \tag{25}
\]

to a change in the admission rate of

\[
\Delta A_0 = \frac{\Delta v(d - r_1 - r_2)dT}{(v + d)(v + \Delta v + d)}, \tag{26}
\]

and to a relative change in the admission rate of

\[
\frac{\Delta A_0}{A_0} = \frac{\Delta v(d - r_1 - r_2)}{(v + \Delta v + d)(r_1 + r_2 + v)}. \tag{27}
\]

By considering our sample data, and by decreasing the transfer rate to long-stay care by 10%, we can obtain

\[
\Delta A_0 = 0.015, \quad \frac{\Delta A_0}{A_0} = 0.049. \tag{28}
\]

Therefore, while a 10% decrease in the transfer rate to long-stay care initially has a negligible effect on the admission rate, an increase in the admission rate of 4.9% will be observed once the department settles back to an equilibrium state (simulation has shown that this may take in the region of five years). Although this may be lower than the increase for a 10% increase in the release rate from acute care, a decrease in the transfer rate may be easier to obtain and less costly because the number of patients involved in order to obtain the decrease is far less.

6.5 Change in the time ex-patients spend in the community

If we are able to decrease the proportion of patients readmitted to the hospital by \( \Delta u_1 \), then, given that the number of admissions is

\[
A_0^* = A_N + \frac{(u_1 - \Delta u_1)C_0}{u_1 - \Delta u_1 + u_2}, \tag{29}
\]

and given that the number of new patients entering the department is

\[
A_N = \rho A_0, \tag{30}
\]

where \( \rho \) is the proportion of patients on their first admission, then, by solving for \( \rho \) we can obtain an equation for the change in the proportion of first-time admissions achieved by a decrease in the readmissions of \( \Delta u_1 \):

\[
\rho = 1 - \frac{r_1(u_1 - \Delta u_1)}{(u_1 - \Delta u_1 + u_2)(r_1 + r_2 + v)}. \tag{31}
\]

By applying these equations to our sample data and by considering a 10% increase...
in the time that patients stay in the community we can obtain a value for $p$ of 0.615, whereas our initial values for $p$ was 0.6. Therefore, a 10% increase in the time that ex-patients spend in the community leads to a 2.6% increase in the number of new patients that can be treated and to an effective increase in the size of the hospital's catchment area.

7. Conclusions

Justification for the allocation of hospital resources is becoming increasingly important for the decision makers in geriatric departments, especially in an environment of government cutbacks and economic recession. The model presented here represents:

(i) an accurate and easily interpretable model with intuitive appeal,
(ii) an analysis using information that is readily available from any geriatric department,
(iii) the ability given expert knowledge of potential feasible changes to the department and their associated costs to predict the long-term effects of these changes on the geriatric department.

REFERENCES


