Tidal Flow and Mass Transport in a Slowly Converging Estuary

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Summary

A variable eddy viscosity technique is used to investigate the structure of the semi-diurnal tidal flow in the Humber Estuary. The associated current system is determined theoretically along the river as far as Saltend and compared with observations made in the mid-channel. The tidal solution is then used to estimate the distribution of tidal induced mass transport above the bed of the river. The calculations indicate that there is a convergence of mid-channel mass transport off Grimsby. It is suggested that this mechanism is that primarily responsible for the development of the 'middle shoal' in the neighbourhood of Grimsby.

1. Introduction

Theoretical treatments of tidal flow in river estuaries have formed the subject of investigations by several authors (e.g. Hunt 1964, Johns 1966). A characteristic of both the works cited has been the use of the linearised shallow water equations. In the work of Hunt, several mathematical models were proposed for investigating both frictionless and frictional oscillations in tidal estuaries; the resultant theories yielded formulae for the tidal elevation and depth mean tidal currents. The present author used a more refined eddy viscosity technique to investigate the vertical structure of the current system at a fixed position in the estuary. However, the model did not take account of any of the topographical features of the estuary such as convergence or variable depth.

The contribution to the mass transport from the non-linear terms in the governing equations of tidal flow has been evaluated by Abbott (1960), Hunt (1961) and Hunt & Johns (1963). The first of these investigations was applied explicitly to the Thames estuary, and a region of deposition predicted in the approximate neighbourhood of the Thames mud reaches. The theoretical work of Hunt predicts a general landward transport of sediment in slowly converging estuaries. The investigations of Hunt and Johns were concerned with mass transport in large scale tidal current systems in which the Coriolis acceleration has a significant effect. The importance of these mechanisms as contributory factors to the movement of bed material has been fully stressed by these authors.

The purpose of the present work is to construct a theoretical model of a slowly converging estuary using the variable eddy viscosity technique described by Johns (1966). The resultant solutions are then used as the first order terms in a small amplitude expansion of the tidal solution near the estuary bed and an expression is obtained for the particle drift, or mass transport, above the river bed. The solution
is uniquely determined by specifying the amplitude and phase of the tidal elevation at the end points of a section of the estuary.

The theory is applied to the (dominant) semi-diurnal tides in the Humber estuary and computations are undertaken as far upstream as Saltend by dividing the river into sub-sections. The seaward section, which extends landward to Grimsby, is of slowly varying cross section. Upstream of Grimsby, the width of the river is assumed to decrease more rapidly as far as Saltend. The depth is constant throughout each section but may differ in different sections.

The resultant calculations determine (1) the amplitude and phase of the tidal elevation; (2) the amplitude and phase of the tidal current along the estuary; (3) the magnitude and direction of the mass transport at a fixed height above the estuary bed computed at intervals along the river. Using parameters representative of conditions in the Humber, it is found that the mass transport is landward downstream of Grimsby whilst it is seaward upstream of Grimsby. The transport passes through a zero in the neighbourhood of Grimsby and it is concluded that this indicates a point of accumulation corresponding approximately to the position of the 'middle shoal'.

2. Formulation of the tidal problem

All spatial conditions in the estuary are referred to rectangular axes Ox, Oz in which Ox is located in the undisturbed free surface and directed landward along the river from a point O at the seaward end of a section of the estuary. The axis Oz is measured vertically upwards. Denoting the (averaged) tidal current by u, the horizontal (Reynolds) stress by \( \tau_{xx} \), the uniform water density by \( \rho \) and the surface elevation by \( \zeta \), the shallow water equations governing the tidal flow may be written

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial \zeta}{\partial x} - \frac{1}{\rho} \frac{\partial \tau_{xx}}{\partial z},
\]

(2.1)

\[
\frac{1}{b(x)} \frac{\partial}{\partial x} \left( b(x) \int \limits_{-\infty}^{\infty} u \, dz \right) + \frac{\partial \zeta}{\partial t} = 0,
\]

(2.2)

where \( b(x) \) and \( h(x) \) denote respectively the width and depth of the estuary at the position \( x \).

Following Johns (1966), the horizontal shearing stress \( \tau_{xx} \) is related to the current system by

\[
\tau_{xx} = -\rho N_{xx} \frac{\partial u}{\partial z},
\]

(2.3)

where \( N_{xx} \) is a coefficient of eddy viscosity. The boundary conditions consistent with no slippage of the water at the estuary bed and zero applied surface stress are

\[
\begin{align*}
\quad u &= 0 \text{ at } z = -h \\
\tau_{xx} &= 0 \text{ at } z = \zeta
\end{align*}
\]

(2.4)

In order to simplify the problem, the equations are linearised for a first approximation and the depth \( h \) is taken as uniform in a given section of the estuary. The width \( b(x) \) is expressed in a form consistent with a convergence of the estuary by writing

\[
b(x) = b_0 e^{-2ax}.
\]

(2.5)
The equation (2.2) (in linearised form) then becomes

$$\left(\frac{\partial}{\partial x} - 2a\right) \int_{-h}^{0} u \, dz + \frac{\partial \zeta}{\partial t} = 0,$$  \hspace{1cm} (2.6)$$

and, on prescribing a co-oscillating tide of period $2\pi/\sigma$, a solution is written in the form

$$\zeta = A(x)e^{-i\sigma t} \hspace{1cm} (2.7)$$

$$u = -\frac{i\sigma}{\sigma} A'(x) U(z)e^{-i\sigma t},$$  \hspace{1cm} (2.8)$$

only real parts being of significance.

Upon substitution from (2.3), (2.7) and (2.8) into (2.6) and the linearisation of (2.1),

$$U = 1 + \frac{i}{\sigma} \frac{dU}{dz} \left( N_{xx} \frac{dU}{dz} \right),$$  \hspace{1cm} (2.9)$$

$$\frac{d^2 A}{dx^2} - 2a \frac{dA}{dx} + \frac{\sigma^2}{g} \int_{-h}^{0} U \, dz = 0.$$  \hspace{1cm} (2.10)$$

At this stage it is convenient to introduce the distribution of eddy viscosity used by Johns (1966):

$$N_{xx} = v \left\{ 1 + \alpha \left( 1 + \frac{z}{h} \right) \right\}^2.$$  \hspace{1cm} (2.11)$$

Such a distribution was found to yield a satisfactory vertical structure of the tidal flow. Defining a variable $\xi$ by

$$\xi = 1 + \alpha \left( 1 + \frac{z}{h} \right),$$  \hspace{1cm} (2.12)$$

equations (2.9) and (2.10) reduce to

$$U = 1 + \frac{i\sigma^2 v}{h^2 \sigma} \frac{d}{d\xi} \left( \xi^2 \frac{dU}{d\xi} \right),$$  \hspace{1cm} (2.13)$$

and

$$\frac{d^2 A}{dx^2} - 2a \frac{dA}{dx} + \alpha(\sigma^2/gh) \int_{1}^{\xi+\alpha} U \, d\xi = 0.$$  \hspace{1cm} (2.14)$$

3. The tidal solution

The solution of (2.13) is readily expressed in the form

$$U = 1 + C_1 \xi^{n_1} + C_2 \xi^{n_2},$$  \hspace{1cm} (3.1)$$

where

$$n_1, n_2 = -\frac{1}{2} \pm \left( \frac{1}{4} - \frac{ih^2 \sigma}{\alpha^2 v} \right)^{1/2}.$$  \hspace{1cm} (3.2)$$
The quantities $C_1$ and $C_2$ must be determined by application of the boundary conditions (2.4) which lead to

$$C_1 + C_2 + 1 = 0 \tag{3.3}$$

and

$$n_1 C_1 (1 + \alpha)^{n_1} + n_2 C_2 (1 + \alpha)^{n_2} = 0. \tag{3.4}$$

A solution of (2.14) might be obtained by prescribing either a standing or progressive type tidal wave in the estuary. This would lead to a single arbitrary constant in the solution for $A(x)$. However, for a real estuary, this procedure rarely gives a good approximation to the tidal flow and is unnecessary if the amplitude and phase of the tide be prescribed at two stations in the estuary. The solution of (2.14) is accordingly written in the form

$$A = A_1 e^{m_1 x} + A_2 e^{m_2 x}, \tag{3.5}$$

where

$$m_1, m_2 = a \pm \left( a^2 - \frac{\alpha (\sigma^2 / gh)}{1 + \alpha} \right)^{1/2}. \tag{3.6}$$

The quantities $m_1$ and $m_2$ will determine the attenuation of the tide as it propagates up the estuary—an effect of critical importance when assessing the positions of the zeros of mass transport. The structure of (3.6) indicates that the attenuation of the tidal wave will be influenced by all the parameters of the estuary including the distribution of eddy viscosity.

For the purpose of computing the values of $m_1$ and $m_2$, it should be observed from (2.4) and (2.13) that

$$\int_1^{1+\alpha} U d\xi = \alpha - \frac{i \alpha^2 v}{h^2 \sigma} \left( \frac{dU}{d\xi} \right)_{\xi=1}$$

$$= \alpha - \frac{i \alpha^2 v}{h^2 \sigma} (n_1 C_1 + n_2 C_2). \tag{3.7}$$

Upon taking the real part of (2.7) and (2.8), the tidal elevation and current may be written in the form

$$\zeta = |A(x)| \cos (\sigma t + \phi), \tag{3.8}$$

where

$$\cos \phi = \Re A(x)/|A(x)| \quad \text{and} \quad \sin \phi = -\Im A(x)/|A(x)|, \tag{3.9}$$

and

$$u = B(x, z) \cos (\sigma t + \psi), \tag{3.10}$$

where

$$B(x, z) = \frac{g}{\sigma} |A'(x) U(z)|, \tag{3.11}$$

$$\cos \psi = \Im A'(x) U(z)/|A'(x) U(z)|, \quad \sin \psi = \Re A'(x) U(z)/|A'(x) U(z)|. \tag{3.12}$$

Numerical evaluation of these equations will follow in Section 5.
4. The mass transport

The mass transport, or particle drift velocity, will be computed at a position above the estuary bed by a method analogous to the boundary layer techniques used by the before-mentioned authors.

It is postulated that there exists a shear layer of thickness $O(\delta)$ adjacent to the estuary bed beyond which the turbulent shear ceases to have a significant influence on the flow. Furthermore, as in the tidal solution, it is assumed that the pressure $p$ within the shear layer obeys the hydrostatic law so that

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = g,$$  \hspace{1cm} (4.1)

and

$$p = -g \rho z + \Pi(x, t),$$  \hspace{1cm} (4.2)

$\Pi(x, t)$ being a function of integration.

At the outer edge of the shear layer, the pressure is given by

$$p_{z+h-\delta} = -g \rho (-h+\delta) + \Pi(x, t),$$  \hspace{1cm} (4.3)

and so

$$-\frac{1}{\rho} \left( \frac{\partial p}{\partial x} \right)_{z+h-\delta} = -\frac{1}{\rho} \frac{\partial \Pi}{\partial x}. \hspace{1cm} (4.4)$$

The non-linear equation governing the tidal flow is

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{1}{\rho} \frac{\partial \tau_{xx}}{\partial z}, \hspace{1cm} (4.5)$$

and, upon neglect of the term involving $\tau_{xx}$ (when $z > -h+\delta$), it follows that

$$-\frac{1}{\rho} \left( \frac{\partial p}{\partial x} \right)_{z+h-\delta} = \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right)_{z+h-\delta}, \hspace{1cm} (4.6)$$

whereupon

$$-\frac{1}{\rho} \frac{\partial \Pi}{\partial x} = \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right)_{z+h-\delta}. \hspace{1cm} (4.7)$$

If the shear layer is relatively thin (in comparison with the total water depth), it will be assumed that the eddy viscosity may be taken as approximately constant. This may be an unsatisfactory approximation but is, at present, unavoidable without further extensive investigation. It might be added, however, that such an approximation will not influence the position of the zeros in the transport (these being determined by the full tidal solution).

With these simplifications, the equation in the shear layer (the boundary layer equation) may be written in the form

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial \Pi}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2}, \hspace{1cm} (4.8)$$

where $w$ is the (small) vertical component of current and $-(1/\rho)(\partial \Pi/\partial x)$ is to be computed through use of (4.7) and the tidal solution of Section 3. We therefore have

$$(u)_{z+h-\delta} = -\frac{ig}{\sigma} A'(x) U(-h+\delta) e^{-ist}, \hspace{1cm} (4.9)$$

and (4.8) is to be solved by successive approximation subject to an outer boundary condition given by (4.9).
The statement of the problem is now in the form considered by earlier authors and their results (e.g. Hunt & Johns 1963, equation (54)) may be applied to predict the mass transport just beyond the shear layer. This is readily found to be given by

$$\bar{U} = -\frac{g^2}{4\sigma^3} |U(-h+\delta)|^2 \text{Re}((3-5i)A'(x)A''(x)^*)$$

(4.10)

where the asterisk denotes the complex conjugate.

The major difficulty in applying this formula arises from the imprecise value of $\delta$. In the case of a laminar boundary layer, it is well known that $\delta \sim (v/\sigma)^{1/2}$, but in the present work it is more meaningful to identify $-h+\delta$ with that value of $z$ for which the flow is approximately free of shear. However, neither method is entirely satisfactory and the predictions concerning the magnitude (and height above the bed) of the transport should be regarded as indicating tendencies rather than absolute values. On the other hand, the location of the zeros of

$$\text{Re}((3-5i)A'(x)A''(x)^*)$$

which determine the zeros of the transport, is independent of the value of $\delta$. The predictions concerning the direction of transport and possible regions of shoaling are, therefore, of greater significance.

5. Numerical evaluation

The quantities $m_1$ and $m_2$ determine the important characteristics of the tide along the estuary. As is readily seen from (3.6) they will depend upon the rate of convergence of the estuary, the distribution of the eddy viscosity and the water depth. In a practical application, the first of these is of uncertain value owing to the largely unknown position of the effective lateral boundaries of the tidal flow. The eddy viscosity cannot be measured directly and is again of uncertain value. Finally, the depth varies across the river and so a mean value must be substituted for $h$. Numerical computation has indicated, moreover, that the tidal characteristics are greatly altered by varying these parameters. In the present work, therefore, it is necessary to prescribe a criterion by means of which it may be decided when suitable values have been used. In the present computations, the basic parameters have been chosen in such a way that the distribution of tidal current along the estuary (with regard to both amplitude and phase) is in essential agreement with observation.

In the first computations performed, the estuary was treated as a single section having a uniform depth throughout its length. Using mid-channel semi-diurnal tidal data supplied by the Hydraulics Research Station, the resultant tide curves were found to be in fair agreement with observation except in the neighbourhood of Grimsby. The associated mass transport was found to be in the landward direction.

The above-mentioned deficiency in the predictions near Grimsby was attributed to a change in the topographical characteristics (depth and convergence) of the estuary at that position. Accordingly, detailed computations were subsequently undertaken in which the river upstream to Saltend was treated as two sections meeting at Grimsby. Such a procedure allows the use of different parameters upstream and downstream of Grimsby by means of which it was hoped to improve theory with observation. This approach will certainly improve the predicted tidal elevation at Grimsby as the present model ensures the correct amplitudes and phases at the ends of a section.

We give here the results of three of the numerical computations for which the appropriate physical parameters are listed in Table 1. The semi-diurnal tidal data appropriate to the three reaches are given in Table 2. The theoretical surface current amplitudes, $B(x) = B(x,0)$, in series 1, 2, 3 are shown in Fig. 1 together with extrapolation from observational data on mid-channel currents. Also included are the
observed surface amplitudes for the semi-diurnal and fourth-diurnal constituents. Within the region of interest, the neglect of the latter of these is seen to be quite justifiable. Moreover, it is clear that series 1 and 3 yield the most satisfactory current amplitudes near Grimsby. The corresponding current phases, $\psi$, are given in Fig. 2 and it is readily seen that series 2 and 3 yield those results in best accord with observation.

### Table 1

**Parameters used in the mathematical models**

$x$ is measured from a reference point outside the estuary in the North Sea. The position of Grimsby corresponds to $x = 75000$ ft.

<table>
<thead>
<tr>
<th>Series number of calculation</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Seaward limit (ft)</td>
<td>0</td>
<td>47500</td>
<td>75000</td>
</tr>
<tr>
<td>Landward limit (ft)</td>
<td>75000</td>
<td>75000</td>
<td>137600</td>
</tr>
<tr>
<td>Mean depth (ft)</td>
<td>45</td>
<td>40</td>
<td>35</td>
</tr>
<tr>
<td>$\nu$ ($ft^2$ s$^{-1}$)</td>
<td>$2 \times 10^{-2}$</td>
<td>$4 \times 10^{-2}$</td>
<td>$2 \times 10^{-2}$</td>
</tr>
<tr>
<td>$N_{xy}$ at surface ($ft^2$ s$^{-1}$)</td>
<td>50</td>
<td>40</td>
<td>15</td>
</tr>
<tr>
<td>$a$ (ft$^{-1}$)</td>
<td>$5 \times 10^{-6}$</td>
<td>$8 \times 10^{-6}$</td>
<td>$8 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

### Table 2

**Field observations from the Humber estuary**

| $10^{-3}x$ (ft) | $|A(x)|$ (ft) | $\phi$ (deg) |
|-----------------|--------------|--------------|
| 0               | 10.0         | 0            |
| 47.5            | 10.04        | -8.1         |
| 75.0            | 10.12        | -13.0        |
| 137.6           | 10.98        | -24.1        |

**Fig. 1.** Amplitude of mid-channel surface tidal current, $B(x)$, and amplitudes of tidal elevations, $A(x)$, as functions of position along the estuary. ——— amplitude of current—theory; ——— amplitude of current—from observation; ——— amplitude of semi-diurnal tidal elevation—from observation; ——— amplitude of fourth-diurnal tidal elevation—from observation.
In Fig. 3, we give the velocity amplitude, $B(z)$, profile evaluated in the mid-channel off Grimsby. Also included is an extrapolation from observations made at five depths in the river. From this, it is clear that the vertical structure of the predicted current system is largely satisfactory and, moreover, that the bulk of the shear flow is within about 8ft of the bed. This aspect of the solution is advanced as justification of the methods used in Section 4 and we take $\delta \sim 5$–8 ft.

The foregoing results (and the comparison thereof with observation) were then used as a criterion in selecting the appropriate mass transport distribution in the estuary. The results of the calculation of this quantity are given in Fig. 4. From these, it is apparent that, near the bed, the mass transport upstream of Grimsby will be in the seaward direction whilst downstream of Grimsby both series 1 and 2 indicate a net landward drift of suspended sediment.

There is, therefore, strong supporting evidence for a mid-channel convergence of mass transport off Grimsby with an associated region of sediment accumulation.
corresponding approximately to the observed position of the 'middle shoal'. Moreover, the magnitude of the transport currents is sufficient to move the considerable loads of material which are necessary to lead to the rapid development of the 'middle shoal'.

In conclusion, attention is drawn to the fact that the rate of convergence of the estuary is of great significance in determining the transport pattern. This raises the interesting possibility of controlling the sedimentation in the estuary by introduction of appropriately converging barriers into the river.

Acknowledgments

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