Rayleigh-wave dispersion function for a transversely isotropic layered spherical earth using the Thomson–Haskell matrix method

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SUMMARY

We consider the two coupled differential equations of the two radial functions appearing in the displacement components of spheroidal oscillations for a transversely isotropic (TI) medium in spherical coordinates. Elements of the layer matrix have been explicitly written—perhaps for the first time—to extend the use of the Thomson–Haskell method to the derivation of the dispersion function of Rayleigh waves in a transversely isotropic spherical layered earth. Furthermore, an earth-flattening transformation (EFT) is found and effectively used for spheroidal oscillations. The exponential function solutions obtained for each layer give the dispersion function for TI spherical media the same form as that on a flat earth. This has been achieved by assuming that the five elastic parameters involved vary as \( r^p \) and that the density varies as \( r^q \), where \( p \) is an arbitrary constant and \( r \) is the radial distance. A numerical illustration with \( p = -2 \) shows that, in spite of the inhomogeneity assumed within layers, the results for spherical harmonic degree \( n \), versus time period \( T \), obtained here for the Primary Reference Earth Model (PREM), agree well with those obtained earlier by other authors using numerical integration or variational methods. The results for isotropic media derived here are also in agreement with previous results. The effect of transverse isotropy on phase velocity for the first two modes of Rayleigh waves in the period range 20 to 240 s is calculated and discussed for continental and oceanic models.

Key words: anisotropy, layered media, normal modes, Rayleigh waves, seismic-wave propagation.

1 INTRODUCTION

The study of seismic waves, and the interpretation of the information contained in them regarding the medium they travel through, is based on the solution of the equations of motion of waves, namely

\[ \rho \left( \frac{\partial^2 u_j}{\partial t^2} \right) = C_{jkll} \left( \frac{\partial^2 u_m}{\partial x_k \partial x_l} \right), \]

in the usual tensor notation. There have been two basic approaches to performing surface-wave dispersion computations for spherical earth models:

1. that in which exact structural specification is employed, with approximate methods for solving the differential equations using numerical integration or variational methods (Takeuchi & Saito 1972); and
2. that in which exact mathematical methods are applied by approximating the earth's structure by homogeneous layering (Haskell 1953; Kennett 1983).

Numerical schemes for solution in a laterally homogeneous, transversely isotropic, gravitating earth have been presented by Takeuchi & Saito (1972) and applied and improved by Woodhouse (1981) and Dziewonski & Anderson (1981). In numerical schemes, when we integrate the second-order differential equations step-by-step, we come up against a major source of inaccuracy. At any step, to obtain the value of \( R(r) \) for the next step we must integrate \( R \) and \( R \) numerically, and as such there is a loss of accuracy. Moreover, there is a choice of radial steps and methods of interpolation to find the material constants at the large number of intermediate points between the given set of hypothetical radial steps of an earth model (Lapwood & Usami 1981). This introduces a degree of arbitrariness into the numerical method. Nevertheless, many authors obtained excellent results with numerical integration. Friederich & Dalkolmo (1995), while discussing the methods applied so far to the study of wave propagation, pointing out the advantages and drawbacks of each of them, show that there is still a need to have more exact methods for improved earth models.
It is considered that a nearly complete solution of wave propagation in a spherical earth can be obtained by the reflectivity method. This was originally developed for layered half-spaces, but has been extended to spherical problems by Muller (1985) using approximate earth-flattening transformations. In order to accommodate the reflectivity method for spherical applications, an earth-flattening transformation (EFT) is invoked which transforms the system of ordinary differential equations (SODE) for the sphere into the corresponding SODE for a layered half-space. For $SH$ waves, the EFT already in use is exact (Biswas & Knopoff 1970) and an EFT for $P-SV$ waves in an isotropic earth has been obtained recently (Arora, Bhattacharya & Goga 1996). The transformation found here can be used as the EFT in the reflectivity method for Rayleigh waves in a transversely isotropic earth.

On the other hand, the Thomson–Haskell (T–H) matrix method, which has been successful in studying surface waves for a flat layered earth since 1960, cannot be applied in a spherical medium, particularly for Rayleigh waves. For a layered spherical isotropic earth where each layer (shell) is homogeneous, solutions are obtained in terms of Bessel functions (Gilbert & Macdonald 1960; Gaulon et al. 1970; Bhattacharya 1978). However, evaluation of amplitude response, group velocity and other properties of Rayleigh waves with Bessel functions is not as convenient as with exponential functions. Bhattacharya (1976) and Arora et al. (1996) developed a useful technique by considering a specific inhomogeneity within each layer, and obtained solutions in terms of exponential functions.

Until now, analytical solutions have not been available for spherical anisotropic media, even in terms of Bessel functions. Thus, the T–H matrix method for studying various properties of Rayleigh waves could not be extended to such a medium. Furthermore, it is now recognized that effective seismic anisotropy is a comparatively common phenomenon in many parts of the crust and mantle. It seems that general neglect of anisotropy is an oversimplification. The deepest level at which anisotropy has been suggested is immediately above the 650 km discontinuity (Kosarev, Makeyeva & Vinnik 1984; Crampin 1978; Montagner & Anderson 1989). Dhewonski & Anderson (1981) inverted a large data set of approximately 100 normal-mode periods, 500 traveltime summaries and 100 $Q$ values to determine a Preliminary Reference Earth Model (PREM). An important finding of this experiment is that transverse isotropy in the upper 200 km of mantle gave a significantly better fit to the normal-mode data, teleseismic traveltimes and Rayleigh and Love-wave dispersion periods of more than 70 s, than was possible with isotropic models. Montagner & Anderson (1989) preferred transverse isotropy down to 400 km in the upper mantle.

According to Backus (1962), transverse isotropy corresponds to the most general kind of anisotropy that can be given to a radially symmetric earth. Moreover, since it involves no azimuthal dependence, transverse isotropy can be incorporated into standard methods to calculate normal modes of a layered earth (Crampin 1978). There is no a priori reason to believe that this type of anisotropy is dominant over azimuthal anisotropy on a regional scale. In fact, much of the evidence that we have for anisotropy comes from the azimuthal variation, which can reach several per cent. Out of the 21 elastic constants that define the most general kind of anisotropy, however, surface waves are affected only by a subset of their independent combinations — six for Love waves and 12 for Rayleigh waves. If the most general kind of anisotropy is considered, the average of surface-wave velocities over all azimuths depends on five combinations of the elastic coefficients (Smith & Dahlen 1973). These combinations reduce the actual medium to a transversely isotropic medium whose equivalent coefficients, as calculated by Nataf, Nakaniishi & Anderson (1986), are:

$$A = \frac{3}{8}(C_{11} + C_{33}) + \frac{1}{4}C_{12} + \frac{1}{2}C_{66},$$

$$C = C_{33},$$

$$F = \frac{1}{2}(C_{13} + C_{33}),$$

$$L = \frac{1}{2}(C_{44} + C_{55}),$$

$$N = \frac{1}{8}(C_{11} + C_{22}) - \frac{1}{4}C_{12} + \frac{1}{2}C_{66},$$

where $C_{ij}$ are the nine elastic coefficients of the actual anisotropic medium.

In view of the observations made above, we felt it necessary to find analytical solutions and to extend the Thomson–Haskell method to a layered spherical, transversely isotropic earth.

We consider (1) the medium as transversely isotropic, and (2) each of the five elastic parameters to be proportional to $r^p$ and the density to be proportional to $r^{p-2}$, where $p$ is an arbitrary constant.

To obtain solutions, we do not resort to the conventional method of representing the displacement field by potential functions and looking for a set of differential equations, each containing a single potential function as in Bhattacharya (1976), but instead we follow Anderson (1965). With a simple transformation, it is seen that the two radial functions involved in the displacement components satisfy a fourth-order homogeneous linear differential equation which is a quadratic in the squared differential operator and has solutions in terms of exponential functions. We have not considered the effect of gravity. A comparison among Rayleigh waves in a flat earth, a non-gravitating spherical earth and a gravitating spherical earth shows that the effect of curvature of the earth’s layering is an order of magnitude larger than the effect of gravity. The effect of curvature appears to exist even down to a period of 20 s (Takeuchi & Saito 1972). Moreover, it is negligible compared to the error in the observation of the dispersion data. For low-degree modes, which are sensitive to gravity, the present approach is not applicable, and the approach of earlier authors should be used.

Sections 2 and 3 deal with the basic equations and their solutions. In Section 4, a displacement–stress matrix is formed, which is used to obtain the dispersion equation in Section 5. The expression for the dispersion equation with liquid layers at the top of the model is given in Section 6. In Section 7, a simple heterogeneity in each shell is considered, and numerical results are presented in Section 8. For stability of results at shorter periods, a compound matrix (delta matrix) approach involving the formulation and integration of second-order minors of the elements of the corresponding layer matrix is recommended (Takeuchi & Saito 1972; Woodhouse 1981; Friederich & Dalkolmo 1995). The explicit expressions for the elements of the layer matrix presented here for a TI spherical earth can be used to calculate the delta matrix, as is done in Bhattacharya (1986).

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2 BASIC EQUATIONS

We assume that the earth is an elastic sphere with radius $a$ and that the density and elastic parameters vary only with $r$, the distance from the centre of the earth.

We seek the solution for the equation of motion in a spherical $(r, \theta, \phi)$ coordinate system in the form

$$
\hat{U} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{n,m=0}^{\infty} \left[ U(r)S_{n,m}^{s}(r, \theta, \phi) + V(r)S_{n,m}^{t}(r, \theta, \phi) \right] \exp(i\omega t) \, dr,
$$

(1)

where $U(r)$ and $V(r)$ are the unknown functions to be determined, $S_{n,m}^{s}(r, \theta, \phi)$ and $S_{n,m}^{t}(r, \theta, \phi)$ are the radial functions of degree $n$ and order $m$.

3 SOLUTIONS OF THE EQUATION OF MOTION

Let

$$
[A, C, N, L, F, \rho] = [A_0, C_0, N_0, L_0, F_0, \rho_0(r/a)^{-2}](r/a)^9.
$$

(5)

Making use of (5), eqs (4) reduce to

$$
\rho_0 r^2 \frac{d^2 U}{dr^2} + \frac{dU}{dr} + (1 + p) \frac{dU}{dr} - \frac{\Omega}{C_0} (L_0 + F_0) r \frac{dV}{dr} - \frac{\Omega V}{C_0} [1 + (1 + p) F_0 - L_0 - 2(A_0 - N_0)] = 0,
$$

and

$$
\rho_0 r^2 \frac{d^2 V}{dr^2} + \frac{dV}{dr} + (1 + p) \frac{dV}{dr} + \frac{1}{L_0} (L_0 + F_0) \frac{dU}{dr} + \frac{V}{L_0} [2(A_0 - N_0) + (2 + p) L_0] = 0.
$$

(6)

Substituting $r = a \exp(z/a)$ in (6), we obtain

$$
\rho_0 a^2 \frac{d^2 U}{dz^2} + a(1 + p) \frac{dU}{dz} - \frac{\Omega}{C_0} (L_0 + F_0) a \frac{dV}{dz} - \frac{\Omega V}{C_0} [1 + (1 + p) F_0 - L_0 - 2(A_0 - N_0)] = 0,
$$

and

$$
\rho_0 a^2 \frac{d^2 V}{dz^2} + a(1 + p) \frac{dV}{dz} + \frac{1}{L_0} (L_0 + F_0) \frac{dU}{dz} + \frac{V}{L_0} [2(A_0 - N_0) + (2 + p) L_0] = 0.
$$

(7)

Now, considering

$$
U = u \exp \left(- \frac{p + 1}{2} \frac{z}{a} \right) = u \exp(qz/a),
$$

$$
V = v \exp \left(- \frac{p + 1}{2} \frac{z}{a} \right) = v \exp(qz/a),
$$

(8)

where $q = -(p + 1)/2$, eqs (7) become

$$
a^2 \rho_0 a^2 \frac{d^2 u}{dz^2} + \frac{u}{C_0} [a^2 \omega^2 \rho_0 + 2(A_0 - N_0 + q F_0) - \Omega L_0 - q^2 C_0] = 0,
$$

and

$$
a^2 \rho_0 a^2 \frac{d^2 v}{dz^2} + \frac{v}{L_0} [a^2 \omega^2 \rho_0 + 2N_0 - \Omega A_0 - (1 - q)^2 L_0] = 0.
$$

(9)
The above equations are of the form
\[ \frac{d^2u}{dz^2} + c_1 u + b_1 \frac{du}{dz} + d_1 v = 0 \]
and
\[ \frac{d^2v}{dz^2} + c_2 v + b_2 \frac{dv}{dz} + d_2 u = 0, \]
(10)
where
\[ c_1 = \frac{1}{a^2} \left[ a^2 \omega^2 \rho_0 - 4(A_0 - N_0 + qF_0) - \Omega L_0 - a^2 C_0 \right], \]
\[ c_2 = \frac{1}{a^2} \left[ a^2 \omega^2 \rho_0 + 2N_0 - \Omega A_0 - (1 - q)\Omega L_0 \right], \]
\[ b_1 = \frac{\Omega}{aC_0} [L_0 + F_0], \quad b_2 = [L_0 + F_0], \]
\[ d_1 = \frac{\Omega}{aC_0} [2(A_0 - N_0) + qF_0 + (1 - q)L_0], \]
\[ d_2 = \frac{1}{a^2} \left[ 2(A_0 - N_0) + qF_0 + (1 - q)L_0 \right]. \]
Since, \( b_1 d_2 + b_2 d_1 = 0 \) in eqs (10), each of \( u \) and \( v \) satisfy the biquadratic equation

\[ \frac{d^4Y}{dz^4} + \frac{d^2Y}{dz^2} \left[ c_1 + c_2 - b_1 b_2 + [c_1 c_2 - d_1 d_2] \right] Y = 0, \]
i.e.

\[ \frac{d^4Y}{dz^4} - \frac{1}{a^2} \left[ M_1 M_2 - M_3 \right] \frac{d^2Y}{dz^2} + \frac{1}{a^2} \left[ M_1 M_2 - M_4 \right] Y = 0, \]
(11)
where
\[ M_1 = \frac{4(A_0 - N_0) + \Omega L_0 + 4qF_0 + q^2 - a^2 \omega^2 \rho_0}{C_0} = S_1 + \frac{L_0}{C_0} \Omega, \]
\[ M_2 = \frac{\Omega A_0 - 2N_0}{L_0} + (1 - q)^2 - a^2 \omega^2 \rho_0}{L_0} = S_2 + \frac{A_0}{C_0} \Omega, \]
\[ M_3 = \frac{\Omega}{C_0} \left[ C_0 \left( L_0 + F_0 \right) \right], \]
\[ M_4 = \frac{\Omega}{C_0} \left[ 2(A_0 - N_0) + qF_0 + (1 - q)L_0 \right]^2, \]
(12)
where \( S_1, S_2 \) are the parts of \( M_1, M_2 \) respectively that are independent of \( \Omega \).

The solutions of eqs (9) are
\[ u = A_{11} e^{\nu_1 z} + A_{12} e^{-\nu_1 z} + A_{21} e^{\nu_2 z} + A_{22} e^{-\nu_2 z}, \]
(13)
\[ v = A_{11} e^{\nu_1 z} + A_{12} e^{-\nu_1 z} + A_{21} e^{\nu_2 z} + A_{22} e^{-\nu_2 z}, \]
(14)
where \( \nu_i \) \((i = 1, 2)\) are given by

\[ \nu_1^2 = \frac{1}{2a^2} \left[ (M_1 + M_2 - M_3) \pm \sqrt{(M_1 + M_2 - M_3)^2 - 4(M_1 M_2 - M_4)^{1/2}} \right]. \]
(15)
For a given \( \nu_1 \), the displacement amplitude ratios \( (A/A') \), are obtained from (10) as

\[ \frac{A}{A'} = \frac{-b_1 \nu_1 + d_1}{\nu_1^2 + c_1} = -\frac{-b_2 \nu_1}{\nu_1^2 + d_2}. \]
Thus
\[ \left( \frac{A}{A'} \right)_{\nu_1} = \frac{A_{11}}{A_{11}'} = \phi_{11} + \nu_1 \phi_{12} = \gamma_1, \quad \text{say}, \]
\[ \left( \frac{A}{A'} \right)_{\nu_2} = \frac{A_{12}}{A_{12}'} = \phi_{11} - \nu_1 \phi_{12} = \gamma_2, \quad \text{say}, \]
\[ \left( \frac{A}{A'} \right)_{\nu_2} = \frac{A_{21}}{A_{21}'} = \phi_{21} + \nu_2 \phi_{22} = \gamma_2, \quad \text{say}, \]
where
\[ \phi_{11} = \Omega \left[(q - 1)L_0 - qF_0 - 2(A_0 - N_0)\right]/D_1, \]
\[ \phi_{12} = \Omega\left[L_0 + F_0\right]/D_1, \]
\[ \phi_{21} = \Omega\left[L_0(q - 1) - 2(A_0 - N_0) - qF_0\right]/D_2, \]
\[ \phi_{22} = \left[L_0 + F_0\right]/D_2, \]
\[ D_1 = C_0 \left[ a^2 \nu_1^2 - M_1 \right] \]
and
\[ D_2 = C_0 \left[ a^2 \nu_2^2 - M_2 \right]. \]
(16)

The solutions (13) and (14) become
\[ u = \gamma_1 A_{11} e^{\nu_1 z} + \gamma_1 A_{12} e^{-\nu_1 z} + A_{21} e^{\nu_2 z} + A_{22} e^{-\nu_2 z}, \]
(17)
\[ v = A_{11} e^{\nu_1 z} + A_{12} e^{-\nu_1 z} + A_{21} e^{\nu_2 z} + A_{22} e^{-\nu_2 z}. \]
(18)

From (3) the radial functions \( \sigma(r), \tau(r) \) can be written as
\[ \sigma(r) = C \frac{dU}{dr} - \frac{F}{r} (2U - \Omega V), \]
\[ \tau(r) = L \left[ \frac{dV}{dr} - \frac{1}{r} (V - U) \right]. \]
Using (5), we obtain
\[ \sigma(r) = C_0 \left( \frac{r}{a} \right)^{\nu_1} \frac{dU}{dr} + F_0 \left( \frac{r}{a} \right)^{\nu_1} \left( 2U - \Omega V \right), \]
\[ \tau(r) = L_0 \left( \frac{r}{a} \right)^{\nu_2} \left[ \frac{dV}{dr} - \frac{1}{r} (V - U) \right]. \]

Using relation (8) in the above equations we obtain
\[ \exp \left( \frac{3 - p}{2} \frac{z}{a} \right) \sigma(r) = C_0 \frac{du}{dz} + \frac{1}{2} (2F_0 + qC_0) u - \frac{\Omega}{a} F_0 v, \]
\[ \exp \left( \frac{3 - p}{2} \frac{z}{a} \right) \tau(r) = L_0 \frac{dv}{dz} - \frac{1}{a} (1 - q) v + \frac{L_0}{a} u. \]
(19)

4 DISPLACEMENT-STRESS MATRIX
We have from relation (8) that
\[ u = U \exp \left( \frac{p + 1}{2} \frac{z}{a} \right), \quad v = V \exp \left( \frac{p + 1}{2} \frac{z}{a} \right). \]
We define
\[ \theta = \exp \left( \frac{3 - p}{2} \frac{z}{a} \right) \sigma(r), \]
\[ \tau = \sqrt{\Omega} \exp \left( \frac{3 - p}{2} \frac{z}{a} \right) \tau(r). \]
We know that $V$, $U$, $\sigma$ and $\tau$ are continuous. However, following Bhattacharya (1976), we shall consider the continuity of $\sqrt{\Omega}$, $\alpha$, $\beta$ and $-\tau$. Using solutions (18) in (19), we obtain

\[
\begin{align*}
\dot{\gamma} &= \frac{1}{a} \left\{ [C_0 \gamma_1 \alpha_1 v_1 + (2F_0 + C_0 q)\gamma_1] - \Omega F_0 \right\} A_{11} e^{\nu_1} + [C_0 \alpha_2 v_2 + 2F_0 + C_0 q - \Omega F_0 \gamma_2] A_{21} e^{\nu_2} + [-C_0 \gamma_1 \alpha_1 v_1 + (2F_0 + C_0 q)\gamma_1 - \Omega F_0] A_{12} e^{-\nu_1} + [-C_0 \alpha_2 v_2 + 2F_0 + C_0 q - \Omega F_0 \gamma_2] A_{22} e^{-\nu_2} \right\}, \\
-\tau &= - \sqrt{\frac{\Omega L_0}{a}} \left\{ [\alpha_1 + q - 1 + \gamma_1] A_{11} e^{\nu_1} + [\alpha_2 \gamma_2 + (q - 1)\gamma_2 + 1] A_{21} e^{\nu_2} + [-\alpha_1 + q - 1 + \gamma_1] A_{12} e^{-\nu_1} + [-\alpha_2 \gamma_2 + (q - 1)\gamma_2 + 1] A_{22} e^{-\nu_2} \right\}.
\end{align*}
\]

We define the stress–displacement matrix $X$ as

\[
X = [\sqrt{\Omega} \alpha, \alpha, \beta, -\tau]^T = D \cdot E(z) \cdot R,
\]

where the superscript $T$ denotes, as usual, the transpose of the matrix.

\[
R = [aA_{11}, aA_{21}, aA_{12}, aA_{22}]^T,
\]

$E(z) = \text{diag}[e^{\nu_1}, e^{\nu_2}, e^{-\nu_1}, e^{-\nu_2}]$,

and

\[
D = \begin{bmatrix}
\sqrt{\Omega} & \gamma_1 & \gamma_2 & -\sqrt{\sqrt{\Omega} L_0} \\
\gamma_1 & \sqrt{\Omega} & 1 & \frac{1}{a} [C_0 \gamma_1 \alpha_1 v_1 + (2F_0 + C_0 q)\gamma_1 - \Omega F_0] \\
\gamma_2 & 1 & \sqrt{\Omega} & \frac{1}{a} \left\{ [-C_0 \gamma_1 \alpha_1 v_1 + (2F_0 + C_0 q)\gamma_1 - \Omega F_0] \right\} \\
-\sqrt{\sqrt{\Omega} L_0} & \frac{1}{a} [\alpha_2 \gamma_2 + (q - 1)\gamma_2 + 1] & \frac{1}{a} [\alpha_2 \gamma_2 + (q - 1)\gamma_2 + 1] & \frac{1}{a} [\alpha_2 \gamma_2 + (q - 1)\gamma_2 + 1]
\end{bmatrix}.
\]

The displacement–stress matrix $X$ is the matrix solution of the equations of motion (3) and depends on the material properties of the layer and the four arbitrary constants that appear in $R$. We observe that the elements of the third and fourth columns in $D$ are obtained, respectively, from the first column by changing $v_1$ to $-v_1$ and the second column by changing $v_2$ to $-v_2$. Eq. (21) can be written as

\[
X = (DF)(F^{-1} \cdot E(z) \cdot F)(F^{-1} R) = G \cdot Q(z) \cdot L,
\]

where

\[
F = \begin{bmatrix} 1 & -v_1 & 0 & 0 \\ 0 & 0 & -v_2 & 1 \\ 1 & v_1 & 0 & 0 \\ 0 & 0 & v_2 & 1 \end{bmatrix},
\]

\[
L = F^{-1} \cdot R = \begin{bmatrix} a(A_{11} + A_{12}), & a(A_{11} + A_{12}) \\ a(A_{21} + A_{22}), & a(A_{21} + A_{22}) \end{bmatrix},
\]

\[
Q(z) = F^{-1} E(z) F = \begin{bmatrix} \cosh v_1 z & -v_1 \sinh v_1 z & 0 & 0 \\ -v_1 \sinh v_1 z & \cosh v_1 z & 0 & 0 \\ 0 & 0 & \cosh v_2 z & -\sinh v_2 z \\ 0 & 0 & -v_2 \sinh v_2 z & \cosh v_2 z \end{bmatrix}.
\]

\[
G = \begin{bmatrix} \sqrt{\Omega} & \frac{2}{a} \phi_{11} & 1 & \frac{1}{a^2} [C_0 \alpha_1 \phi_{12}] \\ 0 & -2v_1^2 \phi_{12} & \frac{-2v_1 [C_0 \phi_{11}] + (2F_0 + C_0 q)\phi_{12}] }{a} \\ 2a \sqrt{\Omega} L_0 \phi_{22} & 0 & \frac{-2v_2 [C_0 + \Omega F_0 \phi_{22}]}{a} \\ 2 \sqrt{\Omega} L_0 \phi_{21} & 2 & \frac{2}{a} (2F_0 + C_0 - \Omega F_0 \phi_{21}) \\ \frac{-2 \sqrt{\Omega} L_0 (q - 1) \phi_{11}}{a^2} \\ \frac{-2v_1}{a} \sqrt{\Omega} (1 + \phi_{11}) \\ 2 \sqrt{\Omega} L_0 v_2 [\phi_{21} - (q - 1) \phi_{22}] \\ \frac{-2 \sqrt{\Omega} L_0 [a^2 \phi_{22} + (q - 1) \phi_{21} + 1]}{a} \end{bmatrix}.
\]

In the above formulation, we have expressed $\gamma_i$ in terms of $\phi_i$.

### 5 Rayleigh Waves on a Layered Spherical Earth

We consider an $M$-layered model of the earth. The layers are concentric shells bounded by concentric spherical surfaces which we shall refer to as interfaces, for example the $m$th shell is bounded by the spheres $r = r^{m-1}$ at the bottom and $r = r^m$ at the top. We take $m$ to vary from $M$ to 1 towards the free surface, the $m$th shell being a complete sphere. In all the shells, including the complete sphere, the material parameters vary...
as in eq. (5), i.e. in the mth shell
\[ [A_m, C_m, N_m, L_m, F_m, \rho_m] = [A_{\text{top}}, C_{\text{top}}, N_{\text{top}}, L_{\text{top}}, F_{\text{top}}, \rho_{\text{top}}(r/a)^{-2}] (r/a)^p. \]

It is known that at a certain depth, determined by \( w \) and \( n \), the solution switches from an oscillatory to an exponential behaviour. Thus, at some level below this depth, the earth model becomes immaterial for the solutions obtained. Thus we have assumed the Mth shell to be an isotropic sphere below this depth in our numerical scheme without affecting the result (Takeuchi & Saito 1972). Let \( X'^{\text{th}} \) denote the value of \( X \) at the mth interface, i.e. at \( r = r^{(m)} \) or \( z = z^{(m)} \). Eq. (23) for the mth shell is
\[ X_m = G_m Q(z) L_m. \]

Shifting the origin of reference to the \( (m-1) \)th interface, i.e. \( z = z^{(m-1)} \), and by writing \( (z - z^{(m-1)}) \) for \( z \), we have
\[ X_m = G_m Q(-h_m) L_m, \]
\[ X_m = G_m Q(-h_m) G_m^{-1} X^{(m-1)} \]
(27)
where \( B_m \) is the layer matrix and is given by
\[ B_m = G_m Q(-h_m) G_m^{-1}. \]

Since the displacements and stresses are continuous across interfaces,
\[ X_m^{(m-1)} = X_m^{(m-1)}. \]
(28a)
Using (27) and (28a) from the \( (M-1) \)th interface successively up to the top layer we get
\[ X_M^{(M-1)} = B_{M-1} B_{M-2} B_{M-3} \ldots B_1 X_1^{(0)}, \]
(28b)
where \( B_{M+i} \) is the product of \( (M-1) \) layer matrices, each of this order. The boundary condition of a stress-free surface implies that
\[ X_1^{(0)} = \begin{bmatrix} \sqrt{\Omega} u^{(0)}, \psi^{(0)}, 0, 0 \end{bmatrix}^T, \]
Hence
\[ X_M^{(M-1)} = G_M L_M - B \begin{bmatrix} \sqrt{\Omega} u^{(0)}, \psi^{(0)}, 0, 0 \end{bmatrix}^T, \]
where, from (24)
\[ L_M = \frac{1}{2} \left[ a(A_{11} + A_{12}), -a(A_{11} - A_{12})/v_1, \right. \]
\[ \left. -a(A_{21} - A_{22})/v_2, (A_{21} + A_{22}) \right]^T. \]

For surface waves, the requirement of displacements dying out at large distances from the surface requires the terms containing \( e^{-z_2} \) and \( e^{-z_2} \) to be absent as \( z \) becomes large and negative towards the centre, since as \( r \to 0, z \to -\infty \). Thus we must choose \( A_{12} = 0 \) and \( A_{22} = 0 \).

Thus (29) reduces to
\[ \frac{1}{2} \begin{bmatrix} aA_{11}, -aA_{11}/v_1, -A_{21}/v_2, A_{21} \end{bmatrix}_M, \]
\[ = G_M^{-1} B \begin{bmatrix} \sqrt{\Omega} u^{(0)}, \psi^{(0)}, 0, 0 \end{bmatrix}^T, \]
\[ = J \begin{bmatrix} \psi^{(0)}, \psi^{(0)}, 0, 0 \end{bmatrix}^T, \]
where \( J = G_M^{-1} B \). Pre-multiplying both sides by
\[ Y = \begin{bmatrix} 1 & v_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_M \]
we have
\[ [0, 0] = J \begin{bmatrix} \psi^{(0)}, \psi^{(0)}, 0, 0 \end{bmatrix}^T, \]
(30)
where
\[ J = Y J_H \]
(31)
so that
\[ H = YG_M^{-1}. \]
(32)
Eliminating \( u^{(0)}, \psi^{(0)} \) from (30), we obtain
\[ J_{11} J_{22} - J_{12} J_{21} = 0 \]
(33)
as the dispersion equation, which gives a relation between \( n \) and \( \omega \) (or period \( T \), where \( T = 2\pi/\omega \) in normal-mode oscillations. In (33), \( J_{ij} \) are the elements of \( J \).

6 INCLUSION OF LIQUID LAYERS AT THE TOP

The modified equations for a liquid layer as given by Takeuchi & Saito (1972) for the non-gravitating isotropic case are
\[ \frac{dU}{dr} = \frac{1}{\lambda} \sigma - \frac{1}{r} (2U - \Omega V), \]
(34a)
\[ \frac{d\sigma}{dr} = -\omega^2 \rho U, \]
(34b)
\[ V = -\frac{\sigma}{\rho \omega^2 r}. \]
(34c)

We eliminate \( V \) in (34a) using (34c) and obtain
\[ \sigma \left[ 1 - \frac{\Omega \lambda}{r^2 \omega^2 \rho} \right] \lambda \left[ \frac{dU}{dr} + \frac{2U}{r} \right] = \lambda \frac{dU}{dr}, \]
(35)
For a liquid layer with constant density we take \( \rho = \rho_0 \) and \( \lambda = \lambda_0 (r/a)^2 \), so that (35) becomes
\[ \sigma \left[ 1 - \frac{\Omega \lambda_0}{a^2 \omega^2 \rho_0} \right] = \lambda_0 (r/a)^2 \left[ \frac{dU}{dr} + \frac{2U}{r} \right]. \]
(36)
On differentiating it and using (34b) we obtain
\[ r^2 \frac{d^2 U}{dr^2} + 4r \frac{dU}{dr} + \left( \frac{a^2 \omega^2 \rho_0}{\lambda_0} + 2 - \Omega \right) U = 0. \]

Let us put \( r = a \exp(z/a) \) and \( U = u \exp(-3z/2a) \).

Thus we have
\[ a^2 \frac{d^2 u}{dz^2} - \left( \frac{1}{4} + \Omega - \frac{a^2 \omega^2}{\lambda_0} \right) u = 0. \]
The solution therefore is $u = A_{11} e^{\mu z} + A_{12} e^{-\mu z}$, where $\mu$ is given by

$$\alpha^2 = \left[ \frac{1}{4} + \frac{\alpha^2 \omega^2}{\omega^2} \right]^{1/2}. $$

Also,

$$(r/a)^{1/2} \sigma = (r/a)^{1/2} \frac{\partial u}{\partial z} + \frac{u}{2a} = \sigma \ (\text{say}),$$

and

$$ (r/a)^{-2} u = \bar{U} \ (\text{say}).$$

Here

$$D_0 = \left( 1 - \frac{\omega^2}{\omega^2} \right). $$

Using continuity of $\sigma, \bar{U}$, the displacement-stress matrix $X' = [\bar{U}, \sigma]^T$ is

$$ X' = \left[ e^{2\mu a} \quad e^{-2\mu a} \right] \left[ \begin{array}{cc} \frac{\mu}{2a} & 0 \\ 0 & \frac{\mu}{2a} \end{array} \right] \left[ \begin{array}{c} e^{2\mu a} \\ 0 \end{array} \right]. $$

and following Bhattacharya (1976) the layer matrix for the $m$th liquid layer is obtained as

$$ \bar{A}_m = \left[ \begin{array}{c} \cosh P_m - \sinh P_m \frac{2a}{\mu} \frac{1}{\mu} \left( \frac{e^{2\mu a} - 1}{4a^2} \right) \\ -\sinh P_m \frac{2a}{\mu} \frac{1}{\mu} \left( \frac{e^{2\mu a} - 1}{4a^2} \right) \end{array} \right], $$

where

$$ \mu_m = \frac{\exp \left( 2\mu_m / a \right)}{\exp \left( 2\mu_m / a \right)} = \exp(-2\mu_m / a). $$

has been used in reducing the layer matrix into this form.

If there are $M'$ liquid layers above the topmost solid layer, using the continuity of $\bar{U}$ and $\bar{a}$ at the interface of liquid layers and at the liquid boundary, and the free liquid surface boundary condition of $\bar{a}^{(0)} = 0$, we have

$$ \bar{a}_1(0) = \bar{a}_n(0), $$

where $\bar{a}_n = L_m / L_{m-1}$ and $L = A_{m-1} \cdot A_{m-1} \cdot \ldots \cdot A_1$, the product of liquid layer matrices. Following Bhattacharya (1976), the dispersion equation is

$$ J_{11}(J_{22} + \pi_m \bar{J}_{23}) - (J_{12} + \pi_m \bar{J}_{13}) \bar{J}_{21} = 0, $$

where $J$ is given by (31).

### 7 Particular Cases at $p = -2$, i.e. $q = 1/2$

Until this point, our development has been valid for any value of $p$. The formidable-looking elements of matrices become quite manageable and symmetrical for the particular value $p = -2$ or $q = 1/2$. Here from (15) we have

$$ v_1^2 = \frac{1}{2a^2} \left[ (M_1 + M_2 - M_3) \pm \Omega \right]. $$

where

$$ \Omega = \frac{\Omega}{C_0} \left( Y_0 - 4X_0 \right) \left( S_0 + \frac{A_0}{C_0} \right). $$

We have defined

$$ X_0 = L_0 + F_0, $$

$$ Y_0 = L_0 + F_0 + 4(A_0 - N_0). $$

With $p = -2$ we note that, from (16), $\phi_{11} = \phi_{22} = \phi_{12}$. The matrix $G$ can be obtained from (25). Furthermore, $|G|$ is given by

$$ |G| = \frac{2a^4 v_1^2 \Omega_C L_0}{a^2} F_1 F_2, $$

which is of the type

$$ x(v_1^2 + v_2^2) + x^2 v_1^2 v_2^2 - xy(v_1^2 + v_2^2) + (1 - y)^2 $$

or

$$ (1 - y)^2 + x(v_1^2 + v_2^2)(1 - y) + x^2 v_1^2 v_2^2, $$

which factors into

$$ (1 - y + xv_1^2)(1 - y + xv_2^2). $$

Thus

$$ |G| = \frac{2a^4 v_1^2 \Omega_C L_0}{a^2} F_1 F_2, $$

where

$$ F_1 = 1 - \frac{\Omega}{4D_1 D_2} \left( Y_0^2 - 4a^2 v_1^2 X_0 \right) $$

$$ F_2 = 1 - \frac{\Omega}{4D_1 D_2} \left( Y_0^2 - 4a^2 v_2^2 X_0 \right) $$

We define

$$ k_0 = 1 + \frac{4F_0}{C_0}, \quad k_1 = 1 + \frac{\Omega X_0 D}{D_1 D_2 F_2}, $$

$$ k_2 = \frac{\Omega D X_0}{D_1 D_2 F_2}, \quad k_3 = 1 + \frac{\Omega X_0 F_0}{C_0 D_2}, $$

$$ k_4 = 1 + \frac{\Omega X_0 X_0}{D_1}, \quad k_5 = \frac{Y_0}{4} - a^2 v_1^2 X_0, $$

$$ k_6 = Y_0 - X_0, $$

$$ k_7 = 1 + \frac{\Omega X_0}{D_1}, \quad k_8 = 1 + \frac{\Omega Y_0}{D_1}. $$
and obtain the elements of the inverse matrix $G^{-1} = [g_{ij}]_{4 \times 4}$ as

$$g_{11}^{-1} = \frac{a k_4}{(2 \sqrt{\Omega} D_1)},$$
$$g_{12}^{-1} = (-k_4 + k_2 k_6)/(4 \sqrt{\Omega} D_1),$$
$$g_{13}^{-1} = \frac{\sqrt{\Omega}}{2 a v_4 D_1} \left\{ a^2 v_4^2 x_0 - \frac{F_0 D_1}{C_0} - \frac{Y_0}{4} X_0 \right\},$$
$$g_{14}^{-1} = \frac{\sqrt{\Omega}}{2 a v_4 D_1} \left\{ -X_0 \left[ 1 - \frac{\Omega X_0}{4} \left( \frac{Y_0}{4} - a^2 v_4^2 x_0 \right) \right] \right\},$$
$$g_{21}^{-1} = \sqrt{\Omega} k_3 k_6/(4 D_1 F_1),$$
$$g_{22}^{-1} = a(Y_0 - X_0 k_6)/(4 D_2 F_1),$$
$$g_{23}^{-1} = \left[ D_2 + \frac{Y_0}{4} k_0 - a^2 v_4^2 x_0 + X_0 D k_2/F_2 \right]/(2 v_4 D_2 F_1),$$
$$g_{24}^{-1} = \left[ k_0 + k_3 - 1 + Y_0 k_2 k_6 \right]/(4 a v_4^2 F_1),$$
$$g_{31}^{-1} = k_3 k_6/(2 F_1),$$
$$g_{32}^{-1} = a^2 X_0/(2 C_0 v_4^2 D_2 F_1),$$
$$g_{33}^{-1} = -a Y_0/(4 C_0 v_4^2 D_2 F_1),$$
$$g_{34}^{-1} = -1/(2 C_0 v_4 F_1),$$
$$g_{41}^{-1} = 0,$$
$$g_{42}^{-1} = 0,$$
$$g_{43}^{-1} = \sqrt{\Omega} Y_0 k_3/(4 v_4^2 L_0 D_1 F_1),$$
$$g_{44}^{-1} = a \sqrt{\Omega} X_0 k_3/(2 L_0 D_0 F_1).$$

Observe that $(F_1/2)$ is a factor in all elements of $G^{-1}$. Now, the layer matrix

$$B_m = G_m Q(-h_m) G_m^{-1} \frac{1}{F_1} [a_{ij}]_{4 \times 4}.$$  

We use the following notation for writing the elements $a_{ij}$ of the layer matrix:

$$P_m = -v_{1m} h_m,$$
$$Q_m = -v_{2m} h_m,$$
$$S_p = \sinh P_m,$$
$$S_q = \sinh Q_m, \quad S' = (S_p - S_q)$$
$$C_p = \cosh P_m,$$
$$C_q = \cosh Q_m, \quad C' = (C_p - C_q).$$

Thus we obtain

$$a_{11} = k_3 C' + \frac{1}{2} (k_4 - k_2 k_6) S' + F_1 \left( C_p + \frac{1}{2} S_q \right),$$
$$a_{12} = \sqrt{\Omega} \frac{D X_0 k_6}{2 D_2} \left\{ (Y_0 - X_0 k_6) C' - 2 \left( D_2 + k_3 + \frac{F_0}{C_0} Y_0 + \frac{D X_0 k_6}{F_2} \right) S' \right\} - 2 D_2 F_1 S_q,$$
$$a_{13} = \frac{a \sqrt{\Omega}}{C_0 D_2} \left\{ X_0 C' + \frac{1}{2} Y_0 S\right\},$$
$$a_{14} = -\frac{a k_3}{L_0} [S' + F_2 S_q],$$
$$a_{21} = -\frac{\sqrt{\Omega} L_0}{D_1} \left\{ \frac{1}{2} k_1 k_6 C' + \left( k_5 + k_2 k_6 \frac{Y_0}{4} + \frac{F_0}{C_0} D_1 \right) S' \right\} + \frac{F_0}{C_0} D_1 F_1 S_p,$$
$$a_{22} = -k_1 k_6 C' + \frac{1}{2} \left( k_5 + k_2 k_6 + \frac{\Omega}{D_1} \right) S' + F_1 \left( C_p - \frac{1}{2} k_6 S_p \right),$$
$$a_{23} = \frac{a_0}{C_0} \left\{ F_1 S_p - S' \right\},$$
$$a_{24} = \frac{a \sqrt{\Omega}}{2 D_1} \left\{ Y_0 S' - 2 X_0 C' \right\},$$
$$a_{31} = \frac{C_0}{a} \left\{ C' (Y_0 - k_0 X_0) \frac{\Omega}{D_1} - \frac{\Omega F_0}{C_0 D_2} + k_2 k_6 \right\}$$
$$+ \left\{ \frac{k_3}{2} + \left( \frac{1}{D_1} + \frac{F_0}{C_0 D_2} \right) \frac{k_0}{4} \right\} \left\{ a^2 v_4^2 - k_3^2 + \frac{\Omega X_0}{D_1} \right\}$$
$$+ k_2 k_3 k_5 + \frac{F_0}{C_0} (k_2 k_3 Y_0 + D_1 + 1)$$
$$+ F_1 \left\{ \left( a^2 v_4^2 - \frac{k_3^2}{4} \right) S_p + \left( k_0 \frac{k_6}{C_0} - k_2 S_p \right) \right\},$$
$$a_{32} = \frac{C_0}{a} \left\{ \frac{1}{2} C' (Y_0 - k_0 X_0) \frac{\Omega}{D_1} - \frac{\Omega F_0}{C_0 D_2} + k_2 k_6 \right\}$$
$$+ \left\{ \frac{k_3}{2} + \left( \frac{1}{D_1} + \frac{F_0}{C_0 D_2} \right) \frac{k_0}{4} \right\} \left\{ a^2 v_4^2 - k_3^2 + \frac{\Omega X_0}{D_1} \right\}$$
$$+ k_2 k_3 k_5 + \frac{F_0}{C_0} (k_2 k_3 Y_0 + D_1 + 1)$$
$$+ F_1 \left\{ \left( a^2 v_4^2 - \frac{k_3^2}{4} \right) S_p + \left( k_0 \frac{k_6}{C_0} - k_2 S_p \right) \right\},$$
$$a_{33} = - \left( k_3 C' + \left( \frac{k_3}{4} + \frac{4 F_0}{C_0} \right) S' - F_1 \left( C_p + \frac{1}{2} k_6 S_p \right) \right),$$
$$a_{34} = \frac{C_0}{L_0} \frac{\Omega}{D_1} \left\{ \frac{1}{2} (Y_0 - k_0 X_0) C' \right\} - \left( X_0 D - \frac{F_0}{C_0} D_1 F_2 \right) S_q$$
$$+ \left( k_3 + \frac{F_0}{C_0} Y_0 + \frac{F_0}{C_0} D_1 \right) S',$$
$$a_{41} = \frac{L_0}{a} \left\{ \frac{1}{2} k_5 \left( \frac{\Omega}{D_1} - \frac{F_0}{C_0 D_2} + k_2 k_6 \right) C' \right\}$$
$$+ \left( \frac{1}{4} - a^2 v_4^2 \right) k_1 + \frac{k_6}{C_0} \left( \frac{1}{D_1} + \frac{F_0}{C_0 D_2} \right)$$
$$+ \Omega \frac{F_0}{C_0} (1 - F_1) \frac{1}{4} k_2 k_3 k_6 S'$$
$$+ (k_3 D + F_1) \left( a^2 v_4^2 + \frac{F_0}{C_0} \Omega \frac{1}{4} \right) S_p.$$
The elements of the layer matrix (40) simplify further and satisfy the relation
\[ \tilde{B}_{ij}(h_m) = \frac{1}{r_{ij}} \tilde{B}_{15-15-0}(-h_m). \]

9 NUMERICAL RESULTS

A computer program has been prepared to determine the period \( T \) for a given value of spherical harmonic degree \( n \), for a transversely isotropic layered earth with inhomogeneity in each layer as in eq. (5) with \( p = -2 \) (Section 7). Following Bhattacharya (1976), we choose \( A_0, \ C_0, \ N_0, \ L_0, \ F_0, \ \rho_0 \) in such a way that the averages of \( A, \ C, \ N, \ L, \ F, \ \rho \) are each equal to the corresponding constant value of the layer in a given layered earth model where each layer is homogeneous.

We know that in a transversely isotropic model the medium velocities are given by
\[ \frac{\kappa}{\sqrt{\Omega}} = \frac{Z}{\sqrt{\Omega}}, \quad \frac{\varphi}{\sqrt{\Omega}} = \frac{P}{\sqrt{\Omega}}, \quad \sigma = \frac{\kappa}{\sqrt{\Omega}} \] (Dziewonski & Anderson 1981).

In an isotropic model \( \varphi = \sigma = \alpha \), the P-wave velocity, and \( \beta_0 = \beta_\nu = \beta \), the S-wave velocity. We define
\[ \phi = C/A, \quad \xi = N/L, \quad \eta = F/(A - 2L). \]

(43)
To convert an isotropic model to a transversely isotropic one we assume that\[ \varphi = \frac{\alpha}{\sqrt{\phi}}, \quad \beta_\nu = \frac{\beta}{\sqrt{\xi}}. \]

(44)
To verify the present approach through numerical results, we first consider an isotropic model by taking \( \lambda_0 = \gamma_0 = \lambda_1 = \gamma_1 = \lambda_2 = \gamma_2 = 1 \), because there are many results available for isotropic models with and without considering gravity and also with homogeneous layers. Table 1 shows the present results for a fundamental-mode Rayleigh wave for the Gutenberg-Bullen A (GBA) model (Takeuchi, Dorman & Saito 1964) for \( n = 25 \). The present values of \( T \) completely agree to four or five significant figures with those of Takeuchi et al. (1964) and Bhattacharya (1976). Abe (1970) obtained results for this model considering gravitational perturbations. Table 1 also shows the difference in \( T \) considering gravitational perturbations. This difference is much smaller than the difference due to transverse isotropy for \( n > 40 \).

We have also considered the transversely isotropic GBA model by taking \( \phi = 0.94, \xi = 1.02 \) and \( \eta = 0.95 \) in the upper mantle between the depths of 38 and 220 km. At other depths, the model is isotropic. This transverse isotropy is based on Dziewonski & Anderson (1981) and on Montanger & Anderson (1989). The result for this (transformed) transversely isotropic GBA model is given in Table 1. For a given value of \( n \), transverse isotropy causes a slight increase in the value of \( T \), and this increase has a maximum at \( n = 80 \).

Similarly, results for the first higher mode, \( M_{21} \), of the Rayleigh wave are given in Table 2. Our results for an isotropic medium agree with those of Bhattacharya (1976) to four decimal places. In the transversely isotropic GBA model, the value of \( T \) increases for a given value of \( n \). The increase in \( T \) is a maximum at \( n = 75 \).

GBA is a continental model. For oceanic model 7-5-0 (Saito...
Table 1. Results for the Rayleigh fundamental mode for the Gutenberg-Bullen A model obtained by the present method. Results obtained by other authors using different methods are also presented for comparison. The period $T$ is in seconds.

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Table 2. Results for the Rayleigh $M_{21}$ mode for the Gutenberg-Bullen A model. The period $T$ is in seconds and gravity has been neglected.

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<td>41.052</td>
<td>41.364</td>
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</tr>
<tr>
<td>250</td>
<td>33.656</td>
<td>33.657</td>
<td>33.937</td>
<td>0.280</td>
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<tr>
<td>375</td>
<td>23.163</td>
<td>23.162</td>
<td>23.383</td>
<td>0.221</td>
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<tr>
<td>450</td>
<td>19.485</td>
<td>19.484</td>
<td>19.672</td>
<td>0.188</td>
</tr>
<tr>
<td>500</td>
<td>17.609</td>
<td>17.609</td>
<td>17.779</td>
<td>0.170</td>
</tr>
</tbody>
</table>

Table 3. Results for the 7-5-0 model (oceanic) for the Rayleigh fundamental mode. The period $T$ is in seconds, and gravity has been neglected.

<table>
<thead>
<tr>
<th>n</th>
<th>Saito &amp; Takeuchi (1966)</th>
<th>Bhattacharya (1976)</th>
<th>Present method Isotropic</th>
<th>Tran-Iso</th>
<th>Difference due to anisotropy</th>
</tr>
</thead>
<tbody>
<tr>
<td>150</td>
<td>6686</td>
<td>66857</td>
<td>66857</td>
<td>67999</td>
<td>0.742</td>
</tr>
<tr>
<td>200</td>
<td>5073</td>
<td>50722</td>
<td>50722</td>
<td>51287</td>
<td>0.562</td>
</tr>
<tr>
<td>250</td>
<td>4087</td>
<td>40863</td>
<td>40863</td>
<td>41309</td>
<td>0.443</td>
</tr>
<tr>
<td>300</td>
<td>3424</td>
<td>34241</td>
<td>34243</td>
<td>34605</td>
<td>0.362</td>
</tr>
<tr>
<td>350</td>
<td>2950</td>
<td>29501</td>
<td>29502</td>
<td>29804</td>
<td>0.302</td>
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<tr>
<td>400</td>
<td>2595</td>
<td>25953</td>
<td>25954</td>
<td>26210</td>
<td>0.256</td>
</tr>
<tr>
<td>500</td>
<td>2104</td>
<td>21036</td>
<td>21036</td>
<td>21223</td>
<td>0.187</td>
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<tr>
<td>600</td>
<td>1784</td>
<td>17839</td>
<td>17838</td>
<td>17975</td>
<td>0.137</td>
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</table>
Table 4. Results for the transversely isotropic model 'PREM' for the Rayleigh fundamental mode. The period $T$ is in seconds.

<table>
<thead>
<tr>
<th></th>
<th>Dziwonski et al. (1981) (considering gravity)</th>
<th>Present method (neglecting gravity)</th>
<th>Difference due to gravity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$T$</td>
<td>$T$</td>
<td>Diff.</td>
</tr>
<tr>
<td>25</td>
<td>297.68</td>
<td>296.97</td>
<td>0.71</td>
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<tr>
<td>30</td>
<td>262.09</td>
<td>261.73</td>
<td>0.36</td>
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<tr>
<td>35</td>
<td>234.57</td>
<td>234.37</td>
<td>0.20</td>
</tr>
<tr>
<td>40</td>
<td>212.32</td>
<td>212.19</td>
<td>0.13</td>
</tr>
<tr>
<td>60</td>
<td>153.33</td>
<td>153.27</td>
<td>0.06</td>
</tr>
<tr>
<td>80</td>
<td>119.47</td>
<td>119.42</td>
<td>0.05</td>
</tr>
<tr>
<td>100</td>
<td>97.63</td>
<td>97.57</td>
<td>0.04</td>
</tr>
<tr>
<td>120</td>
<td>82.42</td>
<td>82.36</td>
<td>0.060</td>
</tr>
<tr>
<td>140</td>
<td>71.23</td>
<td>71.17</td>
<td>0.052</td>
</tr>
<tr>
<td>160</td>
<td>62.68</td>
<td>62.63</td>
<td>0.049</td>
</tr>
</tbody>
</table>

Andersson (1981). The results are compared in Table 4 with those of Dziwonski & Andersson (1981). In PREM we have considered transverse isotropy, sphericity and dissipation but have neglected gravity; Dziwonski & Andersson (1981), however, have considered gravity. The small difference in the values of $T$ is due to the neglect of gravity; such a difference in the values of $T$ due to gravity has already been noted in Table 1.

We have also determined the phase velocity $c_i$ for different values of the period $T$ and have observed the effect of transverse isotropy on the phase velocity. Fig. 1 shows the phase velocity $c_i$ for a transversely isotropic medium, and $c_i - c_c$, where $c_c$ is the phase velocity for the corresponding isotropic medium. In Fig. 1, we have considered both continental (GBA) and oceanic (7-5-0) models. The transverse isotropy in these models has been considered only in the upper part of the upper mantle, as described earlier in this section. The effect of transverse isotropy is seen to decrease the phase velocity.

In fact, from (44)

$$\alpha = \sqrt{0.094 < z} \quad \text{and} \quad \beta = \beta/\sqrt{1.02 < \beta},$$

whereas $\alpha_i$ and $\beta_i$ remain the same as $\alpha$ and $\beta$, respectively, of the isotropic model. The lower values of the TI layer velocities cause the decrease in phase velocity in the corresponding TI model.

Fig. 1 shows that for a continental model the difference between the phase velocities in the isotropic and transversely isotropic models rises sharply from $T = 20 \text{s}$; the difference reaches its maximum at about $T = 70 \text{s}$ and decreases slowly. The maximum difference is about 1.1 per cent of the phase velocity. For the oceanic model the difference is approximately 0.045 km s$^{-1}$ in the period range 40 to 110 s. The difference remains higher than that in the continental model except for periods greater than 220 s. The maximum difference in the oceanic model is 1.2 per cent of the phase velocity and occurs at $T = 90 \text{s}$.

10 CONCLUSIONS

(1) The dispersion equation for Rayleigh waves in a layered, transversely isotropic, spherical earth has been obtained by considering density (radial distance) and the five elastic parameters in each shell to vary as an arbitrary constant power of the radial distance (eq. 5).

(2) The work of Bhattacharya (1976) in considering inhomogeneity in each layer has been extended and improved by taking

(i) the medium as transversely isotropic;

(ii) the index of the power as arbitrary and not necessarily a particular function of the layer velocities. To obtain the dispersion function, we have considered the index to remain the same in each layer.

(3) A potential-function representation of the displacement field and decoupling of the differential equations involving the two potential functions for the $P-SV$ motion have been avoided. Here, the radial functions in the displacement components have been seen to satisfy a fourth-order homogeneous linear differential equation which has solutions in terms of exponential functions.

(4) Although solutions of equations of motion for spheroidal oscillations in a spherical isotropic layer were obtained either in terms of Bessel functions (Gaulon et al. 1970; Bhattacharya 1978) or in terms of exponential functions (Bhattacharya 1976; Arora et al. 1996), similar solutions in a transversely isotropic layer have not been available until now. Because of the lack of such solutions, the Thomson–Haskell matrix method in a transversely isotropic layered spherical earth could not be applied. Here we obtain the solutions in such a layer in terms of exponential functions and extend the Thomson–Haskell method to a transversely isotropic earth.

(5) The availability of exponential function solutions, as obtained in this work, makes the dispersion function for spherical layers as easy to work on as in the corresponding case of a flat earth.

(6) It is seen that the dispersion function is further simplified when $p = -2$. Here the elastic constants are proportional to $r^{-2}$. Thus, they increase with depth and have a small gradient, which is generally true inside the Earth. In the parts where they decrease, finer layering may be necessary. The numerical
results of period versus $n$ with this value of $p$ show good agreement with results obtained by other methods.

(7) Gravity has been neglected because its effect decreases rapidly with decreasing period. Furthermore, its effect is much smaller than that due to transverse isotropy for periods less than 200 s.

(8) The analytical method developed here is comprehensive and exact. The assumption of the variation of elastic parameters and density in each layer is justified because even the well-accepted layering with homogeneous layers to simplify the problem is an approximation of the real Earth. The effect due to variation can be reduced by subdivision of the layers.

(9) For numerical illustration, isotropic models are converted to transversely isotropic models by taking horizontally polarized $P$- and $S$-wave velocities in a transversely isotropic layer to be the same as $P_0$ and $S$-wave velocities in an isotropic medium. We take normal values of $\phi$, $\zeta$ and $\eta$ (eq. 43) in the transversely isotropic mantle between 38 and 220 km for the continental model (GBA) and between 23 and 225 km for the oceanic model (7-5-0). At other depths the models are isotropic. Numerical results show that, for a given period, the phase velocity decreases compared with that in the corresponding isotropic media. The decrease is as large as 1 per cent in the period range 45 to 110 s for the continental model and in the period range 30 to 130 s for the oceanic model.

(10) Numerical results for the fundamental mode obtained with the present approach agree well with previous results obtained using other methods. For computations at short periods/lower modes, it is necessary to formulate a compound matrix from the layer matrix presented here. This may be taken up as a subject of further study, as has been done by Bhattacharya (1986) for the isotropic case.

REFERENCES


