DEMOGRAPHY© Volume 15, Number 1 February 1978

THE STABLE BIRTHS RESULTING FROM A TIME DEPENDENT CHANGE BETWEEN TWO NET MATERNITY FUNCTIONS

P. Cerone
A. Keane
Department of Mathematics, The University of Wollongong, Wollongong, Australia

Abstract—The asymptotic birth rate for a one-sex population in which the net maternity function changes to one of bare replacement was first discussed by Keyfitz and has since been studied by several authors. The present generalization allows for a time dependent transition from any net maternity function to another and, thus, includes all previous models.

INTRODUCTION

Let the subscript 1 refer to a one-sex population prior to $t = 0$ and the subscript 2 for $t > 0$. Let the net maternity function be $\phi(x)$, the total births in the corresponding stable population $B(t) = Be^{rt}$, and the net reproductive rate R.

With the net maternity function $\phi_1(x)$ dropping immediately to the level of bare replacement, so that $\phi_2(x) = \phi_1(x)/R_1$, Keyfitz (1971) derived the asymptotic total births as

$$B_2 = \frac{B_1(R_1 - 1)}{r_1 R_1 \kappa}, \tag{1}$$

where r_1 is the intrinsic rate of growth in the initial population, and $\kappa = \int_0^\infty x\phi_2(x)\,dx$ is the average age of childbearing in the subsequent population.

Since this pioneer work was published, a number of extensions and generalizations have appeared in the literature.

Firstly, Frauenthal (1975) produced a gradual change by assuming that the individuals born before $t = 0$ continued with the net maternity function $\phi_1(x)$, while all new individuals produced after $t = 0$ adopted the new net maternity function. The effect of these assumptions is to change (1) to

$$B_2 = \frac{B_1(R_1 - 1)}{r_1 \kappa}. \tag{2}$$

An extension by Tognetti (1976) assumed that $\phi_2(x) = \frac{R_2}{R_1}\phi_1(x)$ instantaneously, thus, eventually resulting in a stable rather than a stationary population with $B(t) = B_2 e^{r_2 t}$, where

$$B_2 = \frac{B_1(R_1 - R_2)}{(r_1 - r_2)R_1\kappa}, \tag{3}$$

and

$$\kappa = \int_0^\infty e^{-r_2 x}\, x\phi_2(x)\,dx. \tag{4}$$

In an adjacent paper, Mitra (1976) generalized the Keyfitz model to allow the net maternity function to change instantaneously from $\phi_1(x)$ to $\phi_2(x)$, where $\phi_2{}^*(0) = 1$, and * denotes the one-sided Laplace transform. In our notation, he shows

$$B_2 = \frac{B_1}{r_1\kappa}[1 - \phi_2{}^*(r_1)]. \tag{5}$$

More recently, Cerone and Keane (1978) incorporated time dependence into the Keyfitz model, where the net maternity function varied from $\phi_1(x)$ to $\phi_1(x)/R_1$ by assuming a time dependent net maternity function given by

$$\Phi(x, t) = \left[\frac{1}{R_1} + \left(1 - \frac{1}{R_1}\right)e^{-\lambda t}\right]\phi_1(x). \tag{6}$$

There is no need to specify $\phi_2(x)$ in terms of $\phi_1(x)$, as shown by Mitra (1976). Thus, we can take

$$\Phi(x, t) = \phi_2(x) + e^{-\lambda t}[\phi_1(x) - \phi_2(x)], \quad (7)$$

in which change with both time and age are incorporated. The subscript 2 in the time dependent model specifies the ultimate value of a parameter after the change.

Note, the simplest change in the net maternity function is achieved by varying the birth rate as indicated by Mitra (1976), since time dependence of the survivor function raises some difficulties.

THE SOLUTION OF THE GENERAL MODEL

Note that, for general net maternity functions, where the initial net reproductive rate is R_1 and the final is R_2,

$$\phi_1^*(r_1) = 1, \quad \phi_1^*(0) = R_1,$$
$$\phi_2^*(r_2) = 1, \quad \phi_2^*(0) = R_2. \quad (8)$$

Inserting (7) into the generalized renewal equation

$$B(t) = B_1 \int_0^\infty e^{-r_1 x}\, \Phi(x + t, t)\, dx + \int_0^t B(t - x)\Phi(x, t)\, dx, \quad (9)$$

where $B(t)$ is the total number of births at time t, gives the model to be solved.

The Laplace transform of (9) yields

$$B^*(p) = B_1 \left[\frac{\phi_2^*(r_1) - \phi_2^*(p)}{p - r_1} + \frac{1 - \phi_1^*(p + \lambda)}{p + \lambda - r_1} - \frac{\phi_2^*(r_1) - \phi_2^*(p + \lambda)}{p + \lambda - r_1} \right] + B^*(p)\phi_2^*(p) + B^*(p + \lambda) \cdot [\phi_1^*(p + \lambda) - \phi_2^*(p + \lambda)]. \quad (10)$$

In equation (10), we let $p \to r_2$ and use the Taeuberian result that

$$\lim_{p \to r_2} (p - r_2)B^*(p) = B_2.$$

Hence, on noting that

$$\lim_{p \to r_2} \frac{1 - \phi_2^*(p)}{p - r_2} = \kappa,$$

where κ, the average age of childbearing in the ultimate population, is given by equation (4), we obtain

$$B_2\kappa = B_1 \left[\frac{\phi_2^*(r_1) - 1}{r_2 - r_1} + \frac{1 - \phi_1^*(r_2 + \lambda)}{\lambda + r_2 - r_1} - \frac{\phi_2^*(r_1) - \phi_2^*(r_2 + \lambda)}{\lambda + r_2 - r_1} \right] + B^*(r_2 + \lambda)[\phi_1^*(r_2 + \lambda) - \phi_2^*(r_2 + \lambda)]. \quad (11)$$

If we let $\lambda \to \infty$ in (11), so that the change in the net maternity function occurs instantaneously at $t = 0$, we obtain a generalization of equation (5) as

$$B_2 = \frac{B_1}{\kappa(r_1 - r_2)} [1 - \phi_2^*(r_1)].$$

$B^*(r_2 + \lambda)$ can be found by a simple extension of the technique developed by Cerone and Keane (1978), which involves setting up a backward recurrence relation from equation (10).

FURTHER GENERALIZATIONS

Consider the model for the total number of births $B(t)$, given by

$$B(t) = B_1 \int_0^\infty e^{-r_1 x}\, \Phi_\nu(x + t, t)\, dx + \int_0^t B(t - x)\Phi_\lambda(x, t)\, dx, \quad (12)$$

where

$$\Phi_k(x, t) = \phi_2(x) + e^{-kt}[\phi_1(x) - \phi_2(x)], \quad (13)$$

so that the females born before the origin adopt a transition rate ν and those after a rate λ. Thus, the total births $B(t)$ will eventually be of the form $B(t) = B_2 e^{r_2 t}$, where B_2 is given by (11) with λ replaced by ν in the terms arising from the initial population.

While particular choices of ν and λ will reproduce the models previously discussed, the above extension provides for a further variety of possibilities.

REFERENCES

Cerone, P., and A. Keane. 1978. The Momentum of Population Growth with Time Dependent Net Maternity Function. Demography 15:131–134.

Frauenthal, J. C. 1975. Birth Trajectory Under Changing Fertility Conditions. Demography 12:447–454.

Keyfitz, N. 1971. On the Momentum of Population Growth. Demography 8:71–80.

Mitra, S. 1976. Influence of Instantaneous Fertility Decline to Replacement Level on Population Growth: An Alternative Model. Demography 13:513–519.

Tognetti, K. P. 1976. Some Extensions of the Keyfitz Momentum Relationship. Demography 13:507–512.