Black Hole Coalescence and Mergers: Review, Status, and “Where are We Heading?”

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(Received December 3, 1999)

I review recent progress in 3D numerical relativity, focusing on simulations involving black holes evolved with singularity avoiding slicings. After a long series of axisymmetric and perturbative studies of distorted black holes and black hole collisions, similar studies were carried out with full 3D codes. The results show that such black hole simulations can be carried out extremely accurately, although instabilities plague the simulation at uncomfortably early times. However, new formulations of Einstein’s equations allow much more stable 3D evolutions than ever before, enabling the first studies of 3D gravitational collapse to a black hole. With these new formulations, for example, it has become possible to perform the first detailed simulations of 3D grazing collisions of black holes with unequal mass and spin, and with orbital angular momentum. I discuss the 3D black hole physics that can now be studied, and prospects for the future. Such studies may be able to provide information about the final plunge of two black holes, which is relevant to gravitational wave astronomy, and will be very useful as a foundation for future studies when advanced techniques like black hole excision mature to the point that they permit full orbital coalescence simulations.

§1. Introduction

In the last few years, a major focus of the “traditional” 3D numerical relativity community has been in roughly four areas: (1) the development of improved formulations of the equations, (2) evolutions of black holes, especially the development of black hole excision techniques, and evolutions of full 3D black hole systems, from distorted black holes to collisions with increasingly general initial conditions, (3) the evolution of pure gravitational waves, and (4) the evolution of full 3D GR hydro, applied primarily to neutron star binaries. By “traditional” 3D numerical relativity, I mean 3D evolutions of the full Einstein equations carried out using 3+1 evolution of Cauchy initial data on spacelike slices using finite difference techniques. There are a number of important alternatives that have made very significant progress recently, such as the characteristic approach to black hole evolution, 1) the conformal field equation approach 2)-7) and others that I will not be able to discuss here. Further, I will restrict discussion to vacuum spacetimes, since there are other articles in this volume covering the recent progress on hydrodynamics. Generally speaking, in spite of the perception that progress is rather slow in this field (and OK, it is), I want to share some of the excitement that many of us in the numerical community are feeling with progress that has been made. Serious problems remain, especially in the area of long term evolution of 3D black hole spacetimes. But as I show below, in the last few years the community has developed far more stable evolution schemes.

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than ever before, and it has been shown that highly distorted 3D black holes and limited black hole collisions can be very accurately performed, with very accurate waveform extraction even for very low energy waves \((10^{-7}M)\) and that the collapse of pure waves to form black holes and 3D grazing collisions of spinning black holes can be simulated, as long as the evolution times are fairly short (less than \(50M\)). It has also made major progress in developing black hole excision techniques that will likely be the key to extending these evolutions to much longer time periods. I see this as significant and exciting progress, even though the ultimate goal of evolving binary black hole mergers for a number of orbits may not be achieved for some years.

§2. Why are black holes so difficult? A brief history

Numerical evolutions of Einstein’s equations in 3D are extremely difficult, as evidenced by the relatively slow progress over the years in spite of the efforts of a great many people. Even in vacuum 3D spacetimes without black holes, where there are no complications due to evolution of hydrodynamics, equations of state, and so on, the evolution of pure gravitational waves has been very difficult, \(^{8) - 10)\) (see sections below). When black holes are considered, the presence of singularities makes evolution even much more problematic. One must simultaneously deal with singularities inside them, follow the highly non-linear regime near the horizons, and calculate the linear regime in the radiation zone where the waves represent a very small perturbation on the background spacetime metric. In axisymmetry this has been achieved, for example, for stellar collapse, \(^{11)\) rotating collisionless matter, \(^{12)\) distorted vacuum black holes with rotation \(^{13)\) and without, \(^{14)\) and for equal mass colliding black holes, \(^{15),16)\) but with difficulty. These 2D evolutions can be carried out to roughly \(t = 100 - 150M\), where \(M\) is the ADM mass of the spacetime, but beyond this point large gradients related to singularity avoiding slicings usually cause the codes to become very inaccurate and crash. This is one of the fundamental problems associated with black hole evolutions: if one uses the gauge freedom in the Einstein equations to bend time slices up and around the singularities, one ends up with pathological behavior in metric functions describing the warped slices that eventually leads to numerical instabilities.

In 3D the problems are even more severe with this traditional, singularity avoiding time slicing approach. To simulate the coalescence of two black holes in 3D, evolutions of time scales \(t \approx 10^2 - 10^3M\) will be required. Traditional slicing approaches, coupled with the standard ADM-like formulations of the Einstein equations, can presently carry evolutions only to about \(t = 50M\) or less. (But see below for major improvements enabled by recent variations in the formulations.) However, in spite of these difficulties, great progress is being made on several fronts. Alternative approaches to standard numerical evolution of black holes, such as apparent horizon boundary conditions (also known as black hole excision), characteristic evolution, and possibly evolution on hyperboloidal slices, \(^{2) - 7)\) promise much longer evolutions. Apparent horizon conditions cut away the causally disconnected region interior to the black hole horizon, allowing better behaved slicings. These have been well developed in 1D, spherically symmetric studies \(^{17) - 20)\) and to a lesser extent, in full
3D evolutions, \(^{21,22}\) Most recently, the Penn State/Pitt/Texas collaboration reports success in evolving the merger of two black holes, with unequal masses and momenta on each hole, through the point where a common horizon appears. \(^{23}\) Characteristic evolution with ingoing null slices have very recently been successful in evolving 3D single black holes for essentially unlimited times, even with distortions away from spherical or axisymmetry. \(^{24}\) These alternate approaches look promising, but will take time to develop into general approaches to the two black hole coalescence problem.

In the rest of this paper, I will focus on results obtained to date using singularity avoiding slicings to evolve black hole and gravitational wave spacetimes. Although these techniques are limited, they have so far enabled many investigations of the physics of highly distorted black holes, rotating holes, and black hole collisions, in axisymmetry and full 3D. The emphasis here is on what physics one can do now. This work will provide a foundation for future studies with advanced excision, characteristic, or other techniques, once they are perfected.

### 2.1. Axisymmetric black hole simulations

In axisymmetry, even with singularity avoiding slicings, it is possible to perform accurate and long term \((t \approx 150M)\) simulations of several classes of black hole systems that teach us much about the full problem that we ultimately wish to solve. A class of highly distorted black holes was developed and studied numerically, \(^{25-27,14,28,29}\) showing that even highly nonlinear black hole evolutions can be very cleanly studied. The initial data typically have a three-metric with the form \(d\ell^2 = \tilde{\psi}^4 \left(e^{2q} (d\eta^2 + d\theta^2) + \sin^2 \theta d\phi^2\right),\) where \(\eta\) is a radial coordinate related to Cartesian coordinates by \(\sqrt{x^2 + y^2 + z^2} = e^\eta.\) \(^{25}\) Given a choice for the “Brill wave” function \(q,\) the Hamiltonian constraint leads to an elliptic equation for the conformal factor \(\tilde{\psi}.\) The function \(q\) represents the wave surrounding the BH, and is chosen to be \(q(\eta, \theta, \phi) = a \sin^n \theta \left( e^{-\left(\frac{a+b}{2}\right)^2} + e^{-\left(\frac{a-b}{2}\right)^2}\right) (1 + c \cos^2 \phi).\) (Here I have given the full 3D generalization of the original axisymmetric data; axisymmetric data sets are recovered for \(c = 0.\) ) If the amplitude \(a\) vanishes, the undistorted Schwarzschild solution results, while small values of \(a\) correspond to a perturbed BH. Depending on the choice of Brill wave parameters and extrinsic curvature, these black holes can represent highly distorted rotating black holes mimicking those that are formed during the spiralling merger of two spinning black holes. These studies proved to be very rich in black hole physics, for example allowing dynamics of apparent and event horizons of black holes to be studied, \(^{30}\) and waveforms to be extracted, from the numerical evolutions.

These simulations were extended to axisymmetric black hole collisions, including equal mass BHs, \(^{15,16}\) boosted BHs, and unequal mass BHs. The black hole collision work led to the revival of black hole perturbation theory, reviewed by Pullin in this volume, which turns out to be an essential tool in both interpreting and confirming numerical simulation in the right regimes. First, if the two holes are so close together initially that they have actually already merged into one, they might be considered as a single perturbed Schwarzschild hole (the so-called “close limit”). Price and
Pullin and others\textsuperscript{31}--\textsuperscript{35} used this technique to produce waveforms for colliding black holes in the Misner\textsuperscript{36} and Brill and Lindquist\textsuperscript{37} black hole initial data. Secondly, when the holes are very far apart, one can consider one black hole as a test particle falling into the other. Then one rescales the answer obtained by formally allowing the “test particle” to be a black hole with the same mass as the one it is falling into.\textsuperscript{15,16,32,38}

The details of this success of the synergistic perturbative/numerical program has provided insights into the nature of collisions of holes, and should also apply to many systems of dynamical black holes. The waveforms and energies agree remarkably well with numerical simulations. Moreover, second order perturbation theory\textsuperscript{35} spectacularly improved the agreement between the close limit and full numerical results for even larger distances between the holes, although ultimately beyond a certain limit the approximation is simply inappropriate and breaks down.

Together this large body of work provides a detailed and very well understood picture of axisymmetric black hole interactions, which also provides an excellent test bed for studies in 3D cartesian coordinates of the very same system. If one is unable to reproduce the results of this large body of axisymmetric spacetime studies with general 3D codes, one will not be able to go on the general problem of orbiting, spinning, and coalescing black holes.

Most of this perturbative work has been based on the gauge-invariant perturbation formalism developed by Moncrief,\textsuperscript{39} but it is now being extended to the more general case of the curvature based Teukolsky approach, which naturally handles rotating black holes. A recent application of this approach to distorted black holes with both even- and odd-parity black hole distortions has been very successful.\textsuperscript{40} Following on this success, a much more ambitious project ongoing in Potsdam (the so-called “Lazarus” project) aims to treat the problem of evolving colliding or distorted, rotating black hole spacetimes, such as those that will be created from binary black hole inspiral, is underway. Lazarus will be able to perturbatively evolve binary or distorted black hole initial data, in an appropriate close limit, to predict the waveforms, or alternatively, to take binary merger data that has been evolved to a close limit state, and to continue the evolution via perturbative means even if the full numerical simulation might run into difficulty.

2.2. 3D testbeds

Armed with robust and well understood axisymmetric black hole codes, we now consider the 3D evolution of axisymmetric distorted black hole initial data. I first discuss results obtained with standard 3+1 ADM formulations, and later move to more recent formulations. These same axisymmetric initial data sets can be ported into a 3D code in cartesian coordinates, evolved in 3D, and the results can be compared with those obtained with the 2D, axisymmetric code discussed above. For example, we studied the evolution of the nonlinear distorted single black hole axisymmetric initial data set \((a, b, w, n, c) = (0.5, 0, 1, 2, 0)\). In Fig. 1(a) we show the result of the 3D evolution, focusing on the \(\ell = 2\) gauge-invariant wave function extracted at a radius \(r = 8.7M\) as a function of time. Superimposed on this plot is the same function computed during the evolution of the same initial data set with a 2D nonlinear
Fig. 1. We show the (a) $\ell = 2$ and (b) $\ell = 4$ Zerilli functions vs time, extracted during 2D and 3D evolutions of the data set $(a, b, w, n, c) = (0.5, 0, 1, 2, 0)$. The functions were extracted at a radius of $8.7M$. The 2D data were obtained with $202 \times 54$ grid points, giving a resolution of $\Delta\eta = \Delta\theta = 0.03$. The 3D data were obtained using $300^3$ grid points and a resolution of $\Delta x = 0.0816M$.

code, based on the one described in detail in Refs. 14) and 41). It is important to note that the 2D results were computed with a different slicing (maximal), different coordinate system, and a different spatial gauge. Yet the physical results obtained by these two different numerical codes, as measured by the waveforms, are remarkably similar (as one would hope). These results have been reported in much more detail in Refs. 26) and 25).

In a more advanced example, we take a fully 3D distorted black hole data set, for which there is no axisymmetric testbed, and compute its evolution in 3D cartesian coordinates. In this case, if the amplitude of the Brill wave is low enough, we may also compute the evolution perturbatively as shown in Refs. 42) and 43), and compare with the full 3D evolution in cartesian coordinates. The non-axisymmetric initial data set $(a = -0.1, b = 0, c = 0.5, w = 1, n = 4)$ was evolved in full 3D numerical relativity, with cartesian coordinates, maximal slicing, and zero shift, waveforms were extracted, and the results were compared with perturbative evolution. In Figs. 2,

Fig. 2. We show the waveform for the $\ell = 2, m = 0$ mode, extracted from the linear and nonlinear evolution codes. The dotted (solid) line shows the linear (nonlinear) evolution.
Fig. 3. Waveforms are shown for the $\ell = 4, m = 0$ extracted from the linear and nonlinear evolution codes. The dotted (solid) line shows the linear (nonlinear) evolution.

Fig. 4. Waveforms are shown for the $\ell = 4, m = 2$ mode, extracted from the linear and nonlinear evolution codes. The dotted (solid) line shows the linear (nonlinear) evolution.

3 and 4 we show results for a large part of the BH spectrum of modes excited. The waveforms for the linear and nonlinear evolutions are each plotted on the same graphs, extracted at $r = 12.6M$. The total energy radiated for these modes runs from $E \sim 3 \times 10^{-4}M$ for the $\ell = 2, m = 0$ mode, to $E \sim 3 \times 10^{-7}M$ for the $\ell = 4, m = 2$ mode. The agreement between these two completely independent treatments is remarkable, giving complete confidence in the reliability of these waveforms. Not only do the perturbative results confirm those of full 3D numerical relativity, but the 3D results confirm the perturbative treatment. These results show that now it is possible in full 3D numerical relativity, in cartesian coordinates, to study the evolution and waveforms emitted from highly distorted black holes, even when the final waves leaving the system carry a very small amount of energy.

Similar comparisons were carried out with full 3D evolutions of axisymmetric colliding black holes, studied as described above with fully nonlinear axisymmetric codes and perturbation theory. The first 3D simulations of colliding black holes were carried out by the NCSA/WashU collaboration in 1994-1995. Following on
the success of axisymmetric calculations of Misner data by the same group, and by
the beautiful perturbative results pioneered by Price and Pullin and collaborators,
Misner data provides a useful test case for the two black hole problem in full 3D. Re-
sults of the 3D simulations agree very well with those carried out in 2D and through
perturbation theory in the appropriate regimes, including comparisons of wave en-
ergies and event horizon studies. Together, these studies of distorted and colliding
black holes in 3D, with the perturbative and nonlinear axisymmetric comparisons,
form a very solid basis for believing that full 3D numerical codes are able to produce
very accurate, if limited, 3D back hole physics results. However, as I emphasized,
these 3D simulations, all obtained with the standard 3+1 ADM formalism, gener-
ally developed numerical instabilities before $t = 40M$ and crash. We will return to
this point below, where new formulations will prove their worth in stabilizing these
simulations. But first I discuss a new community infrastructure that, I hope, will
help accelerate efforts of the community, by allowing it to easily explore different
techniques, and to share each other’s work, to help break through these barriers and
to speed progress.

§3. Cactus: A new community code for relativity and astrophysics

The computational and collaborative needs of numerical relativity are clearly
immense. To develop a basic 3D code with all the different modules, including par-
alleling layers, adaptive mesh refinement, elliptic solvers, initial value solvers, gauge
conditions, black hole excision modules, analysis tools, wave extraction, hydrody-
namics modules, visualization tools, etc., require dozens of person years of effort
from many different disciplines (in fact, such a feat has still not been done by the en-
tire community!). Different groups often needlessly repeat each other’s effort, further
slowing the progress of the field. The NSF Black Hole Grand Challenge was a first
attempt to address this problem, and an outgrowth of that effort led to the develop-
ment of the “Cactus” Computational Toolkit (CCTK), developed by the Potsdam
group, in collaboration first with NCSA and Washington University, and now with
a growing number of international collaborators in various disciplines. Originally
designed to solve Einstein’s equations, the CCTK has grown into a general purpose
parallel environment for solving complex PDE’s\textsuperscript{45, 46} that is being picked up by
various communities in computational science. Here I focus on its application to
Einstein’s equations.

Cactus is designed to minimize barriers to the community development and
use of the code, including the complexity associated with both the code itself and
the networked supercomputer environments in which simulations and data analysis
are performed. This complexity is particularly noticeable in large multidisciplinary
simulations such as ours, because of the range of disciplines that must contribute to
code development (relativity, hydrodynamics, astrophysics, numerics, and computer
science) and because of the geographical distribution of the people and computer
resources involved in simulation and data analysis.

The collaborative technologies that we are developing within Cactus include the
following:
• A modular code structure and associated code development tools. Cactus defines coding rules that allow one, with only a working knowledge of Fortran or C, to write new code modules that are easily plugged in as “thorns” to the main Cactus code (the “flesh”). The “flesh” contains basic computational infrastructure needed to develop and connect parallel modules into a working code. It allows one to plug in not only different physics modules, such as the basic Einstein solver, other formulations of the equations, analysis routines, etc., but also different parallel domain decomposition modules, I/O modules, utilities, elliptic solvers, and so forth. A thorn may be any code that the user wants, in order to provide different initial data, different matter fields, different gauge conditions, visualization modules, etc. Thorns need not have anything to do with relativity: the flesh can be used as basic infrastructure for any set of PDE’s, from Newtonian hydrodynamics equations to Yang Mills equations, that are coded as thorns. The user inserts the hook to their thorn into the flesh code in a way that the thorn will not be compiled unless it is designated to be active. We have developed a makefile and perl-based thorn management system that, through the use of preprocessor macros that are appropriately expanded to the arguments of the flesh and additional arguments defined by each thorn, is able to create a code which can configure itself to have a variety of different modules. This ensures that only those variables needed for a particular simulation are actually used, and that no conflicts can be created in subroutine argument calling lists.

• A consistency test suite library. An increased number of thorns makes the code more attractive to its community but also increases the risk of incompatibilities. Hence, we provide technology that allows each developer to create a test/validation suite for their own thorn. These tests are run prior to any check in of code to the repository, ensuring that new code reproduces results consistent with previous one, and hence cannot compromise the work of other developers relying on a given thorn.

So, how does a user use the code? A detailed user guide is available with the code (see http://www.cactuscode.org), but in a nutshell, one specifies which physics modules, and which computational/parallelism modules, are desired in a configuration file, and makes the code on the desired architecture, which can be any one of a number of machines from SGI/Cray Origin or T3E, Dec Alpha, Linux workstations or clusters, NT clusters, and others. The make system automatically detects the architecture and configures the code appropriately. Control of run parameters is then provided through an input file. Additional modules can be selected from a large community-developed library, or new modules may be written and used in conjunction with the pre-developed modules.

Our experiences with Cactus up to now suggest that these techniques are effective. It allows a code of many tens of thousands of lines, but with a compact flesh that is possible to maintain despite the large number of people contributing to it. The common code base has enhanced the collaborative process, having important and beneficial effects on the flow of ideas between remote groups. This flexible, open code architecture allows, for example, a relativity expert to contribute to the code without knowing the details of, say, the computational layers (e.g., message passing or AMR libraries) or other components (e.g., hydrodynamics). We encourage users from throughout the relativity and astrophysics communities to make use of this
freely downloadable code infrastructure and physics modules, either for their own use, or as a collaborative tool to work with other groups in the community.

§4. New formulations: Towards a stable evolution system

With this collaborative Cactus infrastructure, it is easy to insert alternate variations on, e.g., evolution equations. In what follows I report on new results obtained with Cactus, using the “BSSN” formulations, in the last year.

As discussed above, the 3D evolution of Einstein’s equations has proved very difficult, with instabilities developing on rather short time scales, even in cases of weakly gravitating, vacuum systems, such as low amplitude gravitational waves, as summarized in an important paper by Baumgarte and Shapiro.\(^8\) In this work, it was shown how one can achieve highly improved stability by making a few key changes to the formulation of the ADM equations, most notably through a conformal decomposition and by rewriting certain terms in the 3D Ricci tensor to eliminate terms that spoil its elliptic nature. In fact, essentially the same tricks were already noticed a few years earlier by Shibata and Nakamura.\(^10\) Hence I refer to these formulations collectively as “BSSN” after the four authors. These subtle changes to the standard ADM formalism have a very powerful stabilizing effect on the evolutions. Evolutions of weak waves that would develop instabilities and crash with the standard ADM formulation run much longer with the new system, and as shown in Alcubierre et al.,\(^{47}\) the new system and variations allow for the first time the successful evolution of highly nonlinear gravitational waves to form a black hole in 3D while the standard ADM treatment would fail long time before the black hole formation. Further work by the Palma group, showed the deep connection between the BSSN formulations and the Bona-Massó family of formulations,\(^{48}\) leading to the possibility of a fully hyperbolic, very stable formulation that shares advantages from many sides.

The Palma, Potsdam and WashU groups also showed that these new formulations lead to much more stable black hole evolutions as well. While standard ADM formulations can evolve black holes very accurately for a short period of time, as described above, large peaks in metric functions cause by so-called “grid-stretching” develop instabilities, which cause the codes to crash caused too soon to study orbits of black holes. The new formulations can significantly extend the evolution times (by factors of 2 or much more) that can be achieved. In all cases, the evolutions are convergent, but seem to have larger error than standard ADM or Bona-Massó system. These effects were recently analyzed in a paper by Alcubierre et al.\(^{49}\) We are now in the process of applying these new formulations to a series of interesting spacetimes, including pure gravitational waves, black holes, and neutron stars, some results of which are reported below.

4.1. Evolving pure gravitational waves

With these new formulations, we are now able to study the nonlinear dynamics of pure gravitational waves with much more stability than ever before. This allows us to use numerical relativity to probe general relativity in highly nonlinear regime. Can one form a black hole in full 3D from pure gravitational waves? Does one see
critical phenomena in full 3D? These inherently nonlinear phenomena have been investigated in 1D and 2D studies, but little is known about generic 3D behavior.

In our investigations, we take as initial data a pure Brill type gravitational wave, later studied by Eppley and others. The metric takes the form

$$ds^2 = \Psi^4 \left[ e^{2q} \left( d\rho^2 + dz^2 \right) + \rho^2 d\phi^2 \right] = \Psi^4 ds^2,$$

where $q$ is a free function subject to certain boundary conditions. Following Refs. 43), 26), 54), we choose $q$ of the form

$$q = a \rho^2 e^{-r^2} \left[ 1 + c \frac{\rho^2}{(1+\rho^2)} \cos^2(n\phi) \right],$$

where $a, c$ are constants, $r^2 = \rho^2 + z^2$ and $n$ is an integer. For $c = 0$, these data sets reduce to the Holz axisymmetric form, recently studied in full 3D Cartesian coordinates. Taking this form for $q$, we impose the condition of time-symmetry, and solve the Hamiltonian constraint numerically in Cartesian coordinates. An initial data set is thus characterized only by the parameters $(a, c, n)$. For the case $(a, 0, 0)$, we found in Ref. 55) that no AH exists in initial data for $a < 11.8$, and we also studied the appearance of an AH for other values of $c$ and $n$.

We have surveyed a large range of this parameter space, but here I discuss two cases of interest: (i) a subcritical (but highly nonlinear) case where after a violent collapse of the self-gravitating waves, there is a subsequent rebound and after a few oscillations the waves all disperse, and (ii) a supercritical case where the waves collapse in on themselves and immediately form a black hole.

The subcritical case studied in Ref. 47) has parameters $(a=4, c=0, n=0)$ in the notation above. It is a rather strong axisymmetric Brill wave (BW). The evolution of this data set shows that some part of the wave propagates outward while some other part implodes, re-expanding after passing through the origin. However, due to the nonlinear self-gravity, not all of it immediately disperses out to infinity; again some

![Fig. 5.](https://academic.oup.com/ptps/article-abstract/doi/10.1143/PTPS.136.87/1820723) (a) Evolution of the log of the lapse $\alpha$ at $r=0$ for the axisymmetric data (4,0,0). The dashed/dotted/solid lines represent simulations at low/medium/high resolution. (b) Evolution of the Riemann invariant $J$ at $r=0$. The wave disperses after dynamic evolution, leaving flat space behind.
fraction of it re-collapses and bounces again. After a few collapses and bounces the wave completely disperses out to infinity. This behavior is shown in Fig. 5(a), where the evolution of the central value of the lapse is given for simulations with three different grid sizes: $\Delta x=\Delta y=\Delta z=0.16$ (low resolution), 0.08 (medium resolution) and 0.04 (high resolution), using $32^3$, $64^3$ and $128^3$ grid points respectively. At late times, the lapse returns to 1 (the log returns to 0). Figure 5(b) shows the evolution of the log of the central value of the Riemann invariant $J$ for the same resolutions. At late times $J$ settles on a constant value that converges rapidly to zero as we refine the grid. With these results, and direct verification that the metric functions become stationary at late times, we conclude that spacetime returns to flat (in non-trivial spatial coordinates; the metric is decidedly non-flat in appearance!).

The same simulation carried out with the standard ADM systems crashes far earlier than in the present case with the BSSN systems, which essentially run forever. With this experience, we next try the case of an even stronger amplitude wave, which in this case will actually collapse on itself and form a black hole. In Fig. 6, we show the development of the data set $(a=6, c=0.2, n=1)$, a full 3D data set, leading to formation of a black hole by gravitational collapse. (This is the first such 3D
simulation.) The figure also compares this black hole formation to results obtained with an axisymmetric data set. The system clearly collapses on itself and rapidly forms a black hole. The waveform extraction shows that the newly formed hole then rings at its quasinormal mode frequency. High quality images and movies of these simulations can be found at [http://jean-luc.aei-potsdam.mpg.de].

These results are exciting examples of how numerical relativity can act as a laboratory to probe the nonlinear aspects of Einstein’s equations. Pure gravitational waves are clearly a rich and exciting research area that allow one to study Einstein's equations as a nonlinear theory of physics. With these new capabilities of accurate 3D evolution that can follow the implosion of waves to a black hole, there is much more physics to study, including the structure of horizons, full 3D studies of critical phenomena, and much more. Further, this study of pure vacuum waves has helped us to understand the importance of developing and testing new formulations of Einstein’s equations for numerical purposes. Without the new formulations, these results simply could not be obtained. Further, we have run literally hundreds of simulations like these in order to determine which variation on the “BSSN” families of formulations perform best. With this new knowledge, we move back to the problem of 3D black holes.

§5. Black holes

Having tested these new formulations of Einstein’s equations on the problem of pure gravitational waves, we now apply what we have learned to the considerably more complex problem of black hole evolutions. We first applied these new formulations to black hole spacetimes that have been very carefully tested in axisymmetry and with 3D nonlinear numerical codes, but with standard 3+1 formulations, as well as with perturbative methods. In summary, these new formulations are able to extend the evolutions by a considerable amount, since instabilities that develop during grid stretching caused by singularity avoiding slicings are highly suppressed. 3D black hole evolutions that crashed previously by $t = 20 - 30 M$ now routinely extend several times longer. However, at such a late time pathological peaks in metric functions associated with singularity avoiding slicings cannot be resolved, and then of course evolutions become very inaccurate. This is actually a major improvement! Previously, with traditional formulations, instabilities would develop as large gradients were created near the black hole, causing the codes to crash prematurely. As discussed above and shown in many papers,\cite{42,43,25,27} the evolutions can be actually very accurate — allowing the extraction of very delicate waveforms from a large spectrum of modes — and remain so until the code crashes. With the new formulations, we find that we can break far through the former crash barrier, but as features become underresolved the results naturally become less accurate. For a fuller discussion of these results, and mathematical analysis giving insight into the improved behavior of the new formulations, please see Ref. 49.

As an example of what we are now able to do with these new formulations, we now turn to an advanced application of a fully 3D “grazing collision” of two black holes.
5.1. True 3D grazing black hole collision

In the previous sections I have shown that using the standard ADM formulations of Einstein’s equations in 3D cartesian coordinates, it is possible to perform very accurate evolutions of gravitational wave and black hole spacetimes. But with these formulations, there are instabilities that appear when large gradients develop, leading to premature crashing of the code. On the other hand, the new “BSSN” family of formulations are much more robust. Having been allowed to break through the barriers seen in evolving pure nonlinear gravitational waves, we now apply these formulations to full 3D grazing collisions of black holes of the type originally considered by Brügmann a few years ago.

The initial data sets we use here for binary BH systems were developed originally by Brandt and Brügmann. They are very convenient, since no isometry is needed and hence the elliptic solver can be applied on standard cartesian grid without the need to apply boundary conditions on strangely shaped (e.g., non-planar!) surfaces.

A few of these data sets were first evolved by Brügmann using the standard ADM formulation. This was a first pioneering attempt to go beyond the highly symmetric black hole collisions that had been studied previously, combining for the first time unequal mass, spinning black holes with linear and orbital angular momentum. Brügmann was able to use nested grids to provide reasonable resolution near the holes, while putting the boundary reasonably far away. The result was that for selected data sets he was able to carry out the evolution far enough to observe a merger of the two apparent horizons. However, the difficulties of the ADM formulation, discussed above, coupled with poor resolution achievable at that time limited these evolutions to about $t = 7M$, and it was not possible to extract detailed physics, such as horizon masses, waveforms, energies, spins, etc.

Now we apply all we have learned in the last few years, together with the advanced computational infrastructure developed in Cactus and the accessibility of much larger computers, and revisit this same problem. This work is still in progress, and calculations are presently underway to refine the waveforms and the energy accounting, but I can report the following preliminary results.

For initial data of the type described in Ref. 57), we follow Brügmann and choose individual mass parameters $M_1 = 1.5$, and $M_2 = 1.0$, and linear and spin momenta on each hole such that the overall mass and angular momentum of the initial slice are measured from asymptotic properties to be

$$M_{\text{ADM}} = 3.1,$$  \hspace{1cm} (5.1)

$$J = 6.7 \text{ so that } a/m = J/M_{\text{ADM}}^2 = 0.70.$$  \hspace{1cm} (5.2)

On a 256 processor Origin 2000 machine at NCSA we are able to run simulations of $387^3$, which take roughly 100GB of memory. Still, with sufficient resolution to carry out long term evolutions, the boundaries are still rather close, at roughly $x = 12M$. We use the “BSSN” formulations to carry out the evolutions, coupled with vanishing shift and either maximal or algebraic slicings (of the “1+log” family) and with a 3-step Crank-Nicholson method. Further details are in preparation for publication. Under these conditions, we find that we are able to evolve the black
hole merger far beyond the time at which the horizons merge, beyond $t = 30M$, at which time the simulations become fairly inaccurate. (I must point out that we have to date only studied the apparent horizons. The event horizons can also be located by techniques developed in Ref. 30). At present it is not known whether a single event horizon is present on the initial slice in this data set.) Depending on computational parameters, the simulations can be carried out far beyond this time without crashing, in stark contrast to earlier attempts which were doomed to crash far earlier.

Of course, the “time to crash” is not a measure of success of a code! What we are really interested in is whether we are able to extract meaningful physics from such simulations. We are in the process of analyzing such simulations in great detail, and the results are very encouraging. First, for the example discussed above, I begin with qualitative measurements of the physics we extract. In Fig. 7, I show a sequence of visualizations of simulations near the time just before, during, and after the merger of the two holes. The coordinate locations of the apparent horizons (AH) are shown as colored surfaces. The colormap represents the local gaussian curvature of the surface, computed from the induced 2-metric on the horizon. As the holes approach each other and merge, a global AH develops. Meanwhile, a burst of gravitational waves, indicated by the colored wisps emanating from the BH system develops and propagates away. The Newman-Penrose quantity $\Psi_4$, computed fully nonlinearly, is used to indicate the gravitational waves. As this system has no symmetries and includes rotation, all $(\ell, m)$-modes and both even- and odd-parity polarizations of the waves are present, leading to a much more complex structure in the wave patterns than one is used to see in such simulations. But this is now moving much closer to what one expects to see in nature, and it, too, will be rather complicated! A full multipolar analysis of the waves is in progress, and it is clear that quasinormal mode ringing of the final BH is present, as expected.

We can also make a quantitative analysis of the simulation to extract detailed black hole physics. Again, this analysis is presently underway, but preliminary results indicate that we are indeed able to extract rather accurate physics from such simulations. For example, as shown in Ref. 58), the AH contains a wealth of information about the black hole. By making various geometric measurements of the AH, one can estimate the BH mass, the spin, and the orientation of the spin axis. For analytic Kerr black holes, it is known from early work of Smarr that there is a unique relation between the spin parameter $a/m$ and shape of the horizon, which in axisymmetry can be parameterized by the ratio of the polar circumference ($C_p$) to the equatorial circumference ($C_e$), which we designate as $C_r$. For a Schwarzschild BH, the AH surface is spherical, and hence $C_r = 1$. For a rotating BH, the horizon geometry bulges out along the equator, and hence this parameter sinks below unity. For a distorted rotating black hole, as shown in Ref. 60), after an initial transient the horizon shape oscillates about its equilibrium shape provided by the Kerr geometry.

For a full 3D simulation, there is no single parameter to describe the shape. However, we can make multiple measurements of the horizon and determine an effective equatorial circumference and multiple polar circumferences going through the final spin axis of the black hole. This spin axis, assuming that little angular
Fig. 7. We show a sequence of visualizations of the merger of two black holes with unequal mass and spin. The apparent horizons are shown as the surfaces at the center of the image, and the colors represent the gaussian curvature. The waves, shown emanating from the merger, are visualizations of the real part of the Newman-Penrose quantity $\Psi_4$. The top left panel shows the system just before the merger, while the bottom right shows the system much later.

momentum is radiated away, should line up closely with the total angular momentum vector of the initial data. By doing so, we should be able to estimate the angular spin and orientation of the final BH formed in the merger, and see if it is consistent with completely independent asymptotic measurements made from the initial data.

The results are shown in Fig. 8. The dashed line shows the expected AH shape value for a Kerr BH with spin parameters determined from the initial data. The solid lines show actual measurements of the horizon found dynamically during the evolution. We make two different computations of the AH shape by making measurements along independent directions along the horizon surface. The two polar circumferences are measured orthogonally through the spin axis of the final BH.
Fig. 8. We show the shape functions of the final spinning black hole formed in a merger, as discussed in the text. A spinning black hole has unique ratio of polar to equatorial circumferences, determined by its spin parameter $a/m$. If distorted, the black hole shape will oscillate about this value as it settles down to its equilibrium shape. The dashed line shows the expected value for this system, determined from asymptotic measurements, while the solid lines show two independent measurements of the shape of merged hole, showing remarkable consistency of these completely different measurements.

These results remarkably confirm the overall consistency of the simulation, by comparing completely different measures of the system in both the asymptotic and near, strong field zone at the horizon. Further, they are the first demonstration that such physical calculations can be done now in full 3D numerical relativity simulations.

Another calculation involving energies can be done, to assess the overall energy accounting and the accuracy of the emitted energy computed through waves in the far zone. In Fig. 9, we show the result of the mass calculation of the horizon, determined through the formulae related the so-called irreducible mass $M_{\text{ir}}$ and total horizon mass $M_{\text{AH}}$:

$$M_{\text{ir}} = \frac{\text{Area}_{\text{AH}}}{16\pi}, \quad (5.3)$$

$$M_{\text{AH}}^2 = (M_{\text{ir}})^2 + J^2/(4(M_{\text{ir}})^2). \quad (5.4)$$

After the horizons merge, from earlier studies of merging black holes in axisymmetry
and 3D, we expect the horizon area to quickly settle down as only a small amount of wave energy will be swallowed by final hole. However, due to the numerical effects of grid stretching, the numerically computed area of the horizon (which lives just near the extreme peaks in radial metric functions) will begin to become inaccurate as errors mount. This picture is very familiar from numerous such studies of simpler BH simulations in 2D and 3D. However, the “plateau” in the area curve, just after the merger, is clear, which allows one to estimate the final mass of the black hole. Higher resolution studies show a similar but less pronounced effect, with a delay in the onset of the spurious growth in the horizon area. Until this grid stretching is eliminated with suitable excision techniques and/or gauge conditions, this problem is unavoidable.

Using this technique, we can estimate the final mass of the BH to be $M_{\text{BH}} = 3.08$ in this simulation. Comparing this with the initial mass of the spacetime, $M_{\text{ADM}} = 3.11$, we again find consistency in the overall energy accounting from very different physical measurements in the asymptotic, weak field and near, strong field zones. As a further test of the accuracy of this simulation, we can integrate the energy carried
by outgoing waves through a 2-sphere surrounding the interior, and we find that slightly less than 1% of the total mass of the spacetime is emitted, again, perfectly consistent with the other asymptotic and near zone measurements!

These results are preliminary, but indicate that for the first time we are indeed now able to simulate the late merger stages of two black holes colliding, with rather general spin, mass, and momenta, and that we can now begin to study the fine details of the physics. A full analysis, with more detailed computer simulations, and with full convergence tests, is underway and will be reported elsewhere.

Without more advanced techniques, such as black hole excision, these simulations will be limited to the final merger phase of black hole coalescence. Hence, it is important that the community continue to focus on this long term solution, and recently there has been excellent progress, as evidenced by the results from the PSU/Texas/Pitt collaboration. But while that is under development, we can take advantage of our capabilities and explore this phase of the inspiral now. Our goal is several fold: (a) to explore new black hole physics of the “final plunge” phase of the binary BH merger, (b) to try to determine some useful information relevant for gravitational wave astronomy, and (c) to provide a strong foundation of knowledge for this process that will be useful when more advanced techniques, such a black hole excision, are fully developed. Before using these techniques to extend the ability of the community to handle the earlier orbital phase, it will be important to undertake detailed investigations of this most violent phase in advance, because it will provide a testbed to ensure that results are correct, and because the understanding we gain through such investigations may be useful in devising appropriate techniques for longer term evolution.

§ 6. Conclusion

Numerical relativity in 3D continues to move forward, as evidenced by new and exciting results in pure gravitational waves, fully relativistic neutron star mergers, grazing collisions of unequal mass, spinning black holes, and the development of more stable formulations of Einstein’s equations. I have focused here on recent progress in vacuum black hole evolutions with traditional 3+1 evolutions with singularity avoiding slicings, because at present these are the most mature techniques that are already yielding physics in full 3D simulations. In particular, the collapse of 3D gravitational waves to form black holes offers a laboratory for studying nonlinear aspects of Einstein’s theory, including generic 3D critical phenomena, without complications due to hydrodynamics or singularities present from the beginning. The grazing merger of two black holes with unequal mass and spin can also be accomplished now, although for limited evolution times (up to about $t = 50M$), but this time is long enough for us to study important physical processes of this final plunge that simply cannot be obtained without simulations. When alternate techniques that promise to extend the evolutions, such as black hole excision, and the conformal hyperbolic approach, are developed, they may provide much more powerful approaches in the future. But the physical simulations we are able to do on a shorter time scale will be important testbeds for these future approaches, and we hope they will be able to provide some
urgently needed information about gravitational wave astronomy.

Many images and movies of the simulations reported here are available at [http://jean-luc.aei-potsdam.mpg.de or http://jean-luc.ncsa.uiuc.edu].

Acknowledgements

The work presented in this paper has been the result of a collaborative effort from many talented people with whom I am privileged to work. In particular, I thank Miguel Alcubierre, Gabrielle Allen, Pete Anninos, Toni Arbona, John Baker, Werner Benger, Carles Bona, Steve Brandt, Bernd Brügmann, Karen Camarda, Manuela Campanelli, Changxue Deng, Ed Evans, Tom Goodale, Christian Hege, Daniel Holz, Gerd Lanfermann, Carlos Lousto, Joan Massó, Andre Merzky, Mark Miller, Lars Nerger, Thomas Radke, John Shalf, Joan Stela, Wai-Mo Suen, Ryoji Takahashi and Malcolm Tobias. I am especially grateful to Lars Nerger and Ryoji Takahashi for help in preparing the final manuscript. The work has been supported by the AEI, NCSA, by the NSF grant PHY 96-00507, and NASA HPCC Grand Challenge Award NCCS5-153.

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