Vertical motion of ground water in bed rock induced by earth tide and its influence on a repository for burnt nuclear fuel: a theoretical calculation

G. Rehbinder

Department of Land and Water Resources Engineering, The Royal Institute of Technology, S-100 44 Stockholm, Sweden

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Abstract Earth tide induced by the Moon and the Sun makes the volume of pores or the apertures of fractures of the bed rock vary, which in turn makes the ground water move. The flow equation of ground water, complete with the tide effect, and its solution, shows that the diffusive propagation of pore pressure is decisive for the response of the perturbation of gravity. Moreover the solution shows that diffusive flow dominates in the Fennoscandian Precambrian rock. The analysis shows that a constitutive coefficient relating the varying perturbation gravity and porosity of bed rock can be determined by field measurements of the level of water in boreholes. Water in a cavity in the bed rock is exposed to an oscillatory interaction with the surrounding pore water. If the cavity is a repository for nuclear waste, this interaction deserves attention. The analysis shows that the magnitude of the oscillatory motion of water between a repository for nuclear waste and its. surrounding groundwater is much less than the vertical motion of the phreatic surface.

Keywords Earth tide; ground water

Nomenclature

a_{ij}, b_{ij}, c_{ij}	auxiliary constants
d_c	diameter of cavity
D_s	distance between the Sun and the Earth
D_m	distance between the Moon and the Earth
$\mathbf{F}_s$, $\mathbf{M}_m$	traction forces
$\mathbf{g} = (0, 0, -g)$	gravity acceleration due to the Earth
$\mathbf{g}_s = g\varepsilon_s(\phi_{1s}, \phi_{2s}, \phi_{3s})$	gravity acceleration due to the Sun
$\mathbf{g}_m = g\varepsilon_m(\phi_{1m}, \phi_{2m}, \phi_{3m})$	gravity acceleration due to the Moon
$\mathbf{g}_{tot} = g(\phi_{1p}, \phi_{2p}, \phi_{3p} - 1)$	total gravity acceleration
$\mathcal{G}$	gravitational constant
H, L_1, L_2	vertical and horizontal dimensions of flow domain
h	displacement of the phreatic surface
k_j	complex auxiliary constant
K	hydraulic conductivity
M_s, M_m	masses of the Sun and the Moon, respectively
$\mathbf{M}_s$, $\mathbf{M}_m$	rotation matrices
n, n_0	perturbed and unperturbed porosity
p, p_e	pore pressure and equivalent pore pressure
$p_0(t)$	atmospheric pressure
$\mathbf{q}$	flux of ground water per unit area

q	rotation quaternion
$\hat{\mathbf{Q}}$	rotation axis
r_0	radius of the Earth
$\mathbf{r} = (x, y, z)$	coordinate system on the surface of the Earth
$\mathbf{r}_s = (x_s, y_s, z_s)$	auxiliary coordinate system
$\mathbf{r}_m = (x_m, y_m, z_m)$	auxiliary coordinate system
$\mathbf{R} = (X, Y, Z)$	inertial system
$\mathbf{R}_e$, $\mathbf{R}_p$	position vectors
t	time
$\mathbf{u} = \mathbf{v}/n_0$	pore velocity of ground water
$\mathbf{v}$	displacement of ground water particle
z_0	length scale of depth dependence of conductivity
$\hat{\mathbf{X}}_b(t), \hat{\mathbf{Y}}_b(t), \hat{\mathbf{Z}}_b(t)$	rotating unit vectors
$\hat{\mathbf{x}}_s(t), \hat{\mathbf{y}}_s(t), \hat{\mathbf{z}}_s(t)$	rotating unit vectors
$\hat{\mathbf{x}}_m(t), \hat{\mathbf{y}}_m(t), \hat{\mathbf{z}}_m(t)$	rotating unit vectors
α	compressibility coefficient due to pore pressure
β	compressibility coefficient due to perturbation gravity
ε_s	relative magnitude of solar gravity
ε_m	relative magnitude of lunar gravity
λ	dimensionless storativity coefficient
μ	viscosity of water
ϑ	dimensionless compressibility of fractures
ϕ_{1s}, ϕ_{2s}, ϕ_{3s}	auxiliary functions
ϕ_{1m}, ϕ_{2m}, ϕ_{3m}	auxiliary functions
ϕ_{1p}, ϕ_{2p}, ϕ_{3p}	auxiliary functions
Φ_{tot}, Φ_e, Φ_m, Φ_s	gravitational potentials
ρ	density of ground water
Ψ	hydraulic potential
(ξ, η, ζ)	dimensionless Earth fixed coordinates
τ	dimensionless time
Υ_j	auxiliary functions
Ω_e	rotation vector of the Earth
ω_s	angular velocity of the orbit of the Earth
ω_m	angular velocity of the orbit of the Moon
ω_e	angular velocity of the autorotation of the Earth
ω_j	compound angular velocities

Table of astronomical constants

$G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
$D_s = 1.5 \times 10^{11}\,\mathrm{m}$
$D_m = 3.8 \times 10^{8}\,\mathrm{m}$
$r_0 = 6.4 \times 10^{6}\,\mathrm{m}$
$M_s = 2 \times 10^{30}\,\mathrm{kg}$
$M_m = 7.4 \times 10^{22}\,\mathrm{kg}$
$M_e = 6.0 \times 10^{24}\,\mathrm{kg}$
$\omega_s = 1.991 \times 10^{-7}\,\mathrm{s^{-1}}$
$\omega_m = 2.662 \times 10^{-6}\,\mathrm{s^{-1}}$
$\omega_e = 7.272 \times 10^{-5}\,\mathrm{s^{-1}}$

$\theta_e = 23°$
$\varepsilon_s = 4.268 \times 10^{-8}$
$\varepsilon_m = 4.587 \times 10^{-8}$

Introduction

Tidal forces on the Earth, induced mainly by the Moon but also by the Sun, deform the Earth's crust and make not only the sea but also the ground water move. In contrast with the sea, the ground water has negligible inertia. Consequently, the motion of ground water due to tidal effects is less spectacular than tidal waves in the sea.

There are at least three different ways in which the motion of ground water can be induced. Firstly, a varying sea level can force ground water to move in the vicinity of a coast. Simple applications of this mechanism are presented in advanced textbooks on ground water theory, notably by Todd (1977). Recent studies have been presented by Li and Jiao (2001*a*, *b*), Tang and Jiao (2001) and Jiao and Tang (1999).

Secondly, the varying *direction* of the perturbation gravity tilts the phreatic surface which makes the ground water move. This effect is very weak unless the flow domain is extremely large. The phenomenon does not seem to have attracted great interest. An investigation of the possible tilting of the phreatic surface has been presented by Lewkowicz (1985).

Thirdly, the deformations of the Earth's crust are associated with stresses that deform the pores and fractures of the bed rock. These deformations force the ground water to move. If the porosity is a known function of the stresses in the crust, the flow equation of ground water is completed with a source term which is a *known* function of space coordinates and time. The theoretical calculation of deformation and stresses in the Earth's crust due to the earth tide is very complicated. It was initiated by Love (1944). Due to the viscosity of the mantle under the crust, there is a small phase shift between the rotating gravity field and the corresponding prolate deformation of the crust. A comprehensive presentation of this phenomenon is given by Melchior (1978).

Variation of the level in wells, that undoubtedly is induced by tidal forces, has been observed and reported since the end of the nineteenth century. The amplitude of the variations is 20–100 mm, with the larger values observed at low latitudes. A summary of old observations has been published by Bredehoeft (1967) in the introduction to a theoretical approach to the phenomenon. Observations have been published by Robinson (1939) and Richardson (1956).

As mentioned above, a theoretical approach to the problem of how the earth tide affects the radial flow of water in confined and unconfined aquifers has been published by Bredehoeft (1967). The analysis is elegant and mathematically attractive. However, a theoretical relation between stresses in the Earth's crust and the hydraulic storativity of an aquifer must rest on assumptions that are so far-reaching that they can hardly be trusted. An analysis, similar to Bredehoeft's, and the results of field tests have been published by Gieske and de Vries (1985).

The interpretation of observations of the water level in a *bore hole in rock* is difficult unless a unique, and hopefully simple, relation exists between Love's stresses in the crust and the aperture of the fractures of the rock. As mentioned above, such a relation is not available. The standard method to overcome difficulties of this kind is to relate some relevant *observation* and the perturbation gravity. The observations can be fluctuating pore pressure in the ground or the level of water in a borehole.

The motion of ground water due to compression and expansion of the voids in bed rock can have importance for the transport of contaminants. An example of a source of particularly dangerous contaminants is a leaking repository for radioactive waste in a cavity deep in the rock. The main risk for the transport of possible contaminated water from repositories is, of course, the normal motion of ground water. The residence time of a water particle depends on the conductivity and the hydraulic gradient. Simple closed formulae for the comfortable estimation of such residence times, expressed in terms of the hydraulic coefficients of rock, have been presented by Rehbinder and Isaksson (1998). An extensive investigation of all kinds of properties and coefficients of Swedish bed rock has been carried out by The Swedish Nuclear Fuel and Waste Management Co., particularly at the Äspö Hard Rock Laboratory. A thorough presentation of the programme and results has been made by Rehn *et al.* (1997). There is only one superficial study of the possible tidal variations of the ground water that has been presented in connection with discussions on depositing radioactive waste in Sweden. Nilsson (1968) has published graphs of the level in a well, as a function of time, together with what the author calls the "gravity potential" as a function of time. The perturbation gravity has obviously been calculated and not measured, and it seems likely that one of the graphs does not represent the gravity potential but the vertical component of the gravity perturbation. The graphs are geometrically very similar and show that there is a time lag between them. The observations were made in a well somewhere in Skåne in southern Sweden. They have not been interpreted. It is not clear from the publication if the well was drilled in an open or a closed aquifer or in bed rock. A recent study of the influence of earth tide on a possible repository for nuclear waste in Nevada has been published by Bredehoeft (1997). The study deals with rock that is intersected with faults.

Although the magnitude of the varying gravity is small, the corresponding varying porosity serves as a tiny but tireless pump, pushing water into and out of a possible cavity in the bed rock. This tiny pump does not create any *net flux* from the cavity. Yet the small interaction between the water in the cavity and the surrounding pore water, in combination with the molecular diffusion of possible contaminants, can be dangerous. This phenomenon seems to have attracted little interest but deserves attention. A similar problem has been studied by Neper (2001).

The object of the present study is twofold. Firstly to analyze the possibilities to determine a constitutive coefficient that relates gravity and the porosity of fractured rock. Secondly to estimate the magnitude of the displacement of water between a water filled cavity in a rock and its surrounding ground water.

In the first part of the next section, mathematical expressions for the oscillating perturbation of gravity are derived. For natural reasons this part requires lengthy algebraic calculations. They are thus collected into two appendices. These calculations may seem unnecessary since they belong to classical celestial mechanics. Nevertheless the results, summarized in Figs. 4–6, are of decisive importance for the possibility of applying the theoretical solutions. They are also important for planning experiments, since test occasions and latitudes for test sites are not equally well suited. In the next two sections, the flow equation and its solution are presented. Finally the possibilities of using these solutions for the evaluation of field tests are discussed. Moreover, the magnitude of the displacement of a fluid particle at the boundary of a cavity deep in the bed rock is discussed.

Derivation of the gravity force

The gravity that induces tidal variations of the ground water on the Earth is composed of components from the Sun and the Moon. The general calculation of the components is complicated. It is described in the textbook by Melchior (1978). The formulae are also available in the form of computer codes. The general theory takes into account that the Moon does not move in the ecliptic and that the orbits of both the Earth and the Moon are slightly elliptical. Still, a simplified derivation is useful since special cases can be studied explicitly.

The derivation below of the perturbation gravity is based on several simplifications. They are:

1. The solar system consists of the Sun, the Earth and the Moon.
2. The three actual celestial bodies are homogeneous rigid spheres. Consequently they can, from a gravitational point of view, be replaced by point masses.
3. The orbit of the Earth around the Sun is circular.
4. The orbit of the Moon around the Earth is circular.
5. The orbits of the Moon and the Earth lie in the same plane, the ecliptic.
6. Minor effects such as nutation are ignored.
7. The influences of the Moon and the Sun on the gravity on the Earth are separated, and superimposed.
8. From a *kinematical* point of view, the centre of gravity lies at the centre of the Sun.
9. The *kinematical* motion being for a full Moon at winter solstice. Moreover, it is midnight at a given position on the surface of the Earth.

The *total* gravity force is derived in an inertial coordinate system $\mathbf{R} = (X, Y, Z)$ that has its origin at the centre of the Sun and its Z axis perpendicular to the ecliptic. The distances between the Sun and the Earth and between the Earth and the Moon are D_s and D_m respectively. The angular velocities of the Earth rotating around the Sun and the Moon rotating around the Earth are ω_s and ω_m, respectively. The positions of the Earth and the Moon are $\mathbf{R}_e(t)$ and $\mathbf{R}_m(t)$, respectively. The derivation of the gravitational force requires, apart from the inertial system, two auxiliary systems $\mathbf{r}_s = (x_s, y_s, z_s)$ and $\mathbf{r}_m = (x_m, y_m, z_m)$. The situation is shown in Figs. 1 and 2. The radius of the Earth is r_0. Since $r_0 \ll D_s$ and $r_0 \ll D_m$, the total gravity at a given position on the surface of the Earth can be moved to the centre of the Earth.

The gravity contributions from the Sun and the Moon are expressed separately in the two auxiliary systems, respectively. The derivation of the gravitational potential for a system of two bodies is derived in Appendix 1. The two bodies can be either the Earth and the Sun or the Earth and the Moon. The total potential is the sum of these two. The complicated transformations between the different coordinate systems are derived in Appendix 2. A reader who is not particularly interested in the details is recommended to omit the appendices.

The definition of $t = 0$ implies that the given position at the surface of the Earth is identified by the *latitude* $\pi/2 - \theta_0$ and the time $t > 0$. The situation is explained in Fig. 3.

The total gravitational vector, in the *local* Earth fixed system $\mathbf{r} = (x, y, z)$, is $\mathbf{g}_{tot}(t) = \mathbf{g}_s(t) + \mathbf{g}_m(t) + \mathbf{g}_e$, where the dominating part is $\mathbf{g}\,0_e = (0, 0, -g)$. The indices s, m and e stand for Sun, Moon and Earth, respectively. The scalar $g = \mathcal{G} M_e(1/r_0^2) = 9.81\ \mathrm{m^2/s}$, where $\mathcal{G}$ is the gravitational constant.

The dimensionless gravitational forces due to the Sun and the Moon are given in relation to g. Thus

$$\hat{\mathbf{g}}_{tot} = \begin{bmatrix} \phi_{1p}(t) \\ \phi_{2p}(t) \\ \phi_{3p}(t) - 1 \end{bmatrix} = \begin{bmatrix} \varepsilon_s \phi_{1s}(t) + \varepsilon_m \phi_{1m}(t) \\ \varepsilon_s \phi_{2s}(t) + \varepsilon_m \phi_{2m}(t) \\ \varepsilon_s \phi_{3s}(t) + \varepsilon_m \phi_{3m}(t) - 1 \end{bmatrix} \tag{1}$$

where p in the subscript above stands for perturbation. The dimensionless factors are

$$\varepsilon_s = 4.\mathcal{G} M_s r_0 / D_s^3 g = 1.07 \times 10^{-7} \ll 1$$

$$\varepsilon_m = 4.\mathcal{G} M_m r_0 / D_m^3 g = 2.35 \times 10^{-7} \ll 1. \tag{2}$$

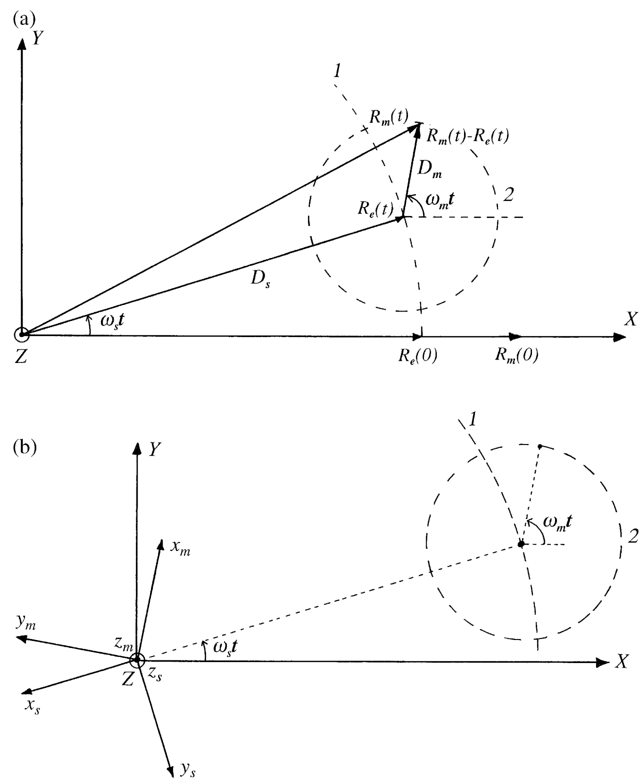

Figure 1 (a) Positions of the Earth and the Moon in the inertial coordinate system (X, Y, Z). 1: Orbit of the Earth around the Sun. 2: Orbit of the Moon around the Earth. (b) The auxiliary coordinate systems in the inertial system. 1: Orbit of the Earth around the Sun. 2: Orbit of the Moon around the Earth

All the contributions to $\mathbf{g}_{ot}$ are conservative. Thus

$$\mathbf{g}_{tot} = -\nabla\Phi_{tot} = -\nabla(\Phi_e + \Phi_s + \Phi_m) \tag{3}$$

where Φ_e, Φ_s and Φ_m are the gravity potentials of the Earth, the Sun and the Moon, respectively. Since $\varepsilon_s \sim \varepsilon_m \ll 1$,

$$\Phi_{tot}(x, y, z, t) = g(x\phi_{1p}(t) + y\phi_{2p} + z). \tag{4}$$

The perturbation gravity $\hat{\mathbf{g}}_s(t) + \hat{\mathbf{g}}_m(t)$ can be decomposed into sums of sine and cosine functions, the arguments of which are the sums and differences of ω_e, ω_m

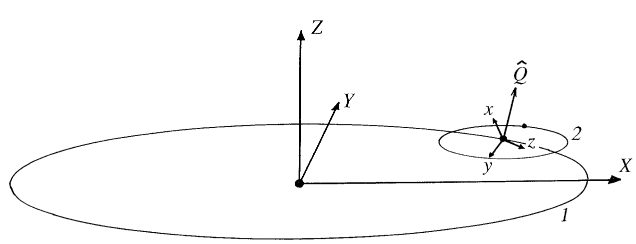

Figure 2 Perspective sketch of the inertial system, showing the rotation axis of the Earth, $\hat{\mathbf{Q}}$, and the local coordinate system $\mathbf{r} = (x, y, z)$ at the Earth. 1: Orbit of the Earth around the Sun. 2: Orbit of the Moon around the Earth

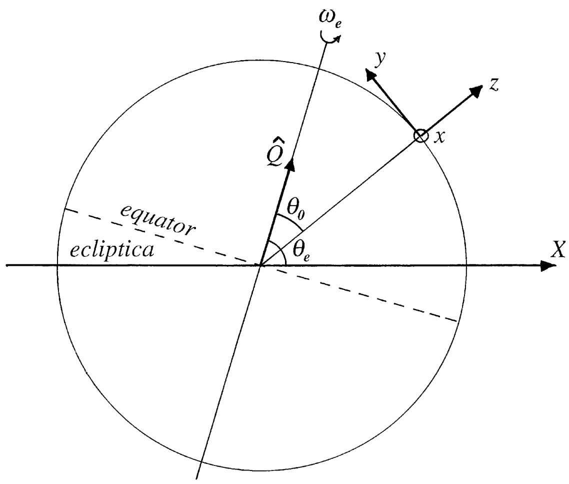

Figure 3 Definition of the local coordinate system $\mathbf{r} = (x, y, z)$ and the angles θ_0 and θ_e. The figure shows the situation for $t = 0$

and ω_s. The decomposition gives complicated expressions. The perturbation gravity can be written as

$$\begin{bmatrix} \phi_{1p}(t) \\ \phi_{2p}(t) \\ \phi_{3p}(t) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{N} a_{1j}(\theta_0)\cos\omega_j t + b_{1j}(\theta_0)\sin\omega_j t + c_1(\theta_0) \\ \sum_{j=1}^{N} a_{2j}(\theta_0)\cos\omega_j t + b_{2j}(\theta_0)\sin\omega_j t + c_2(\theta_0) \\ \sum_{j=1}^{N} a_{3j}(\theta_0)\cos\omega_j t + b_{3j}(\theta_0)\sin\omega_j t + c_3(\theta_0) \end{bmatrix}. \quad (5)$$

The constants $c_1(\theta_0) \ll 1, c_2(\theta_0) \ll 1$ and $c_3(\theta_0) \ll 1$, which imply that they can be n eglected in Eq. (5).

If the analysis included the eccentricity of the orbits, etc., N would be infinite.

The character of the three components of the perturbation gravity is similar, as Fig. 4 shows. Fig. 5 shows that they depend strongly on the latitude.

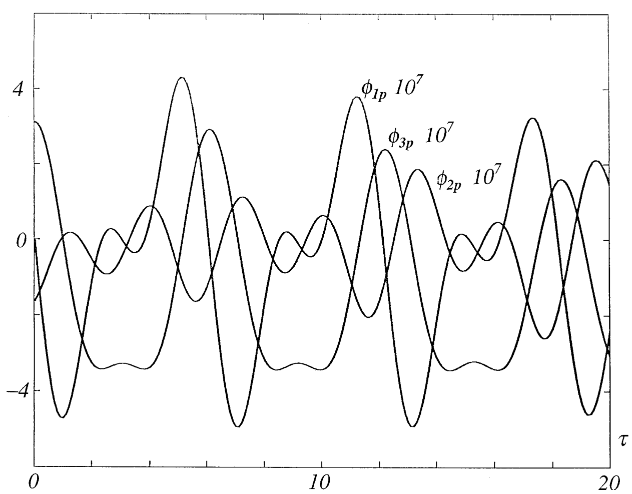

Figure 4 The three components of the dimensionless perturbation gravitation during three days and nights at 60° latitude N ($\theta_0 = 30°$)

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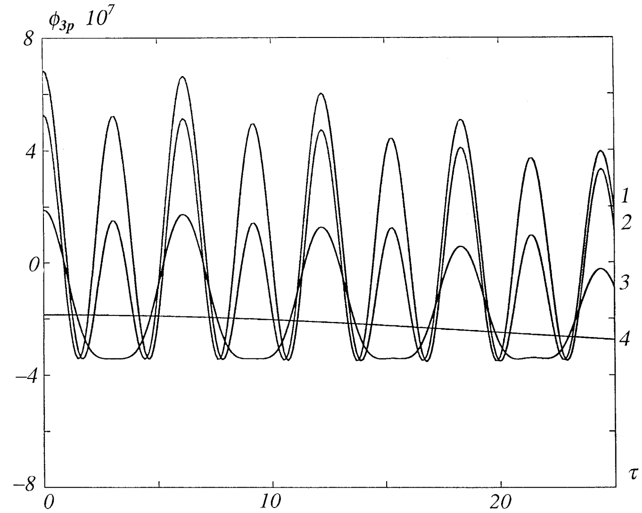

Figure 5 Vertical component of the dimensionless perturbation gravity during four days and nights. 1: the Equator. 2: the Tropic of Cancer. 3: the north Arctic Circle. 4: the north pole

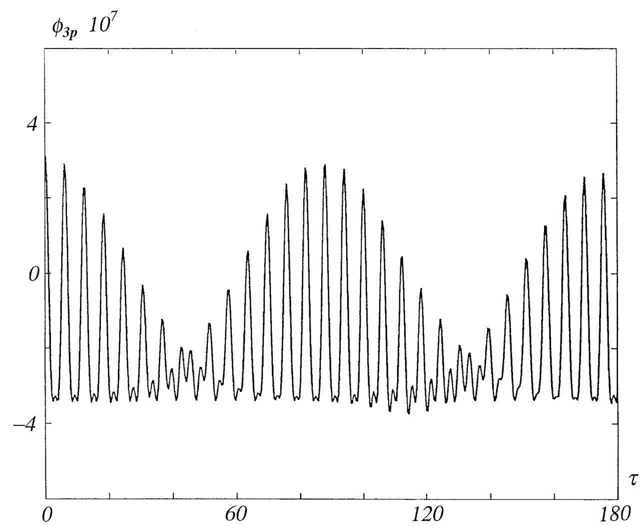

Figure 6 Vertical component of the dimensionless perturbation gravity in 60° latitude n. during one month after full moon and winter solstice

Flow equations in a rotation coordinate system

The analysis of the flow of ground water in rock is based on the theory of creep flow of an incompressible liquid through a permeable continuum. The equations refer to the coordinate system $\mathbf{r} = (x, y, z)$, that is fixed relative to the Earth and rotates with it. The flux per unit area is $\mathbf{q} = (q_1, q_2, q_3)$. The density and viscosity of water are ρ and μ, respectively. The conductivity and porosity of the rock are K and n, respectively. The pore pressure is $p(\mathbf{r}, t)$ and the corresponding hydraulic potential is $\psi(\mathbf{r}, t) = z - (p - p_0)/\rho g$. The atmospheric pressure $p_0(t)$ is assumed to be equal to zero.

The pore pressure and flow pattern are given by the equation of conservation of mass, Darcy's law and a constitutive equation relating the porosity with the gravity and the pore pressure. The mass conservation law is

$$\frac{\partial n}{\partial t} + \nabla \cdot q = 0. \tag{6}$$

The coordinate system $\mathbf{r}$ is rotating, which means that Darcy's law must be completed with Coriolis forces and the centrifugal force. The latter varies with latitude. The background can be found in some advanced textbook on fluid mechanics, notably by Tritton (1992).

Since $\omega_s \ll \omega_e$, the Coriolis force due to the rotation of the Earth around the Sun is negligible compared to the corresponding force due to the autorotation of the Earth. Hence if $\Omega_e(t)$ is the rotation vector of the Earth's rotation, Darcy's law is

$$q = -K\left(\nabla\Psi - \frac{2}{g}\Omega_e \times q\right). \tag{7}$$

The definition of the rotation vector implies that $|\Omega_e| = |\omega_e|$. If the lateral extent of the flow domain is far less than the radius of the Earth, θ_0 is constant and the contribution from the centrifugal force to the pore pressure $p_c = -\rho(|\Omega| r_0 \sin\theta_0)^2/2$ is constant. Moreover it cannot be distinguished experimentally from the pore pressure p. Therefore, the pore pressure is replaced with an equivalent pore pressure, $p \rightarrow p + p_c$.

The vertical perturbation of the gravity expands and compresses the fractures of the bedrock. The porosity, which is a measure of the volume of the fractures, is proportional to the projection of the vertical gravity on the normal of each fracture. This means that, if a fracture is *vertical*, its aperture is constant. Besides this, the rock has conventional storativity, i.e. the porosity varies with pore pressure. Thus

$$n = n_0(1 + \alpha\rho g\Psi + \beta g\phi_{3p}). \tag{8}$$

In (8) n_0 is the undisturbed reference value, α is the compressibility that corresponds to ordinary storativity and β is the compressibility due to the *vertical* perturbation of the gravity. The variation of the *direction* of $\hat{\mathbf{g}}$ implies a lateral tilting of the phreatic surface and will be neglected in this analysis.

The approximation saying that gravity is constant in Darcy's law implies that varying gravity appears only as a source term in the flow equation. This is analogous to Boussinesq's approximation for thermally induced flow. In the conventional theory of ground water flow, the conductivity is constant, or possibly a known function of space coordinates, although the porosity varies with pore pressure. In complete analogy with this, the conductivity is assumed to be constant although the porosity varies with the perturbation gravity.

The pore velocity is given by Dupuit and Forchheimer's equation:

$$\mathbf{u}(\mathbf{r}, t) = (u_1, u_2, u_3) = \mathbf{q}/n_0. \tag{9}$$

The displacement of a fluid particle is

$$\mathbf{v}(\mathbf{r}, t) = (v_1, v_2, v_3) = \int_0^t \mathbf{u}(\mathbf{r}, t')\mathrm{d}t'. \tag{10}$$

The lower integration limit $t' = 0$ is, of course, arbitrary. The flow domain $\mathscr{D}$ is characterized by a vertical length scale H and lateral length scales L_1 and L_2. At the bottom of the domain, $z = -H$, the flux across the boundary is equal to zero. The reason for this assumption will be discussed later. At the lateral boundaries the hydraulic potential is assumed to be undisturbed.

The displacement of the phreatic surface $h(x, y, t)$ is assumed to be small, i.e. $h \ll H$, $h \ll L_1$, $h \ll L_2$. This means that the boundary condition at the top of the domain, $z = 0$, can be linearized, implying that

$$\mathscr{D} = \begin{cases} 0 \le x \le L_1 \\ 0 \le y \le L_2 \\ -H \le z \le 0 \end{cases} \tag{11}$$

is independent of $h(x, y, t)$. Moreover, $H \ll r_0$, implying that the latitude of the domain is constant.

The small displacement of the phreatic surface is estimated by the *vertical* pore velocity through $z = 0$. Hence

$$h(x,y,z) = \int_0^t u_3(0,y,z,t')\mathrm{d}t' = -\frac{K}{n_0}\int_0^t \left(\frac{\partial \Psi(x,y,z,t')}{\partial z}\right)_{z=0} \mathrm{d}t'. \tag{12}$$

The problem (6)–(11) has three length scales H, L_1 and L_2 but at least four time scales. Firstly the imposed ones, namely the diurnal ω_e^{-1}, the monthly ω_m^{-1} and the annual ω_s^{-1}. Secondly H/K, which is related to the flux per unit area. It seems convenient to use the latter in this context as in conventional analysis of groundwater motion. If the dimensionless variables

$$\begin{aligned} &\hat{\mathbf{q}} = \mathbf{q}/K \\ &\hat{\Psi} = \Psi/H \\ &\hat{\mathbf{u}} = \mathbf{u}/K \\ &\hat{\mathbf{v}} = \mathbf{v}/H \\ &(\xi, \eta, \zeta) = (x/H, y/H, z/H) \\ &\tau = Kt/H \\ &\hat{\omega}_j = \omega_j H/K \\ &\hat{n} = n/n_0 \\ &\hat{h} = h/H \\ &\hat{L}_1 = L_1/H \\ &\hat{L}_2 = L_1/H \end{aligned} \tag{13}$$

are introduced in the flow equations, they give

$$\frac{\partial \hat{n}}{\partial \tau} + \nabla \cdot \hat{\mathbf{q}} = 0 \tag{14}$$

$$\hat{\mathbf{q}} = -\nabla\hat{\Psi} + \frac{2}{\mathrm{Ek}}\hat{\Omega} \times \hat{\mathbf{q}} \tag{15}$$

$$\hat{n} = (1 + \alpha\rho g H\hat{\Psi} + \beta g \phi_{3p}). \tag{16}$$

The domain is

$$\mathscr{D} = \begin{cases} 0 \le \xi \le \hat{L}_1 \\ 0 \le \eta \le \hat{L}_2 \\ -1 \le \zeta \le 0 \end{cases} \tag{17}$$

Ekman's number,

$$\mathrm{Ek} = \frac{g}{\omega_e K} \equiv \frac{\mu}{\rho\omega_e(\sqrt{\mu K/\rho g})^2}, \tag{18}$$

in Eq. (15) is based on the length scale of the pores $\sqrt{\mu K/\rho g}$. Since $\mathrm{Ek} \gg 1$ in all practical cases, the influence of the Coriolis force is ignored. If (16) is differentiated with respect to τ and (15) differentiated with respect to the space coordinates and the derivatives are introduced in (14), the flow equation becomes

$$\alpha\rho g H \frac{\partial\hat{\Psi}}{\partial\tau} - \nabla^2\hat{\Psi} = -\beta g \frac{\mathrm{d}\phi_{3p}}{\mathrm{d}\tau}. \tag{19}$$

The problem is thus characterized by dimensionless parameters, viz.

$$\lambda = \alpha\rho g H$$

$$\vartheta = \beta g$$

$$\hat{\omega}_j = \omega_j H/K. \tag{20}$$

If the perturbation of gravity is ignored, Eq. (19) represents the consolidation of pore pressure as in ordinary ground water flow.

As also mentioned previously, the boundary at $\zeta = 0$ is linearized, i.e. the domain $\mathscr{D}$ is undeformed. The dimensionless displacement of the phreatic surface is given by (12) and (13).

Thus

$$\hat{h}(\xi,\eta,\tau) = -\frac{1}{n_0}\int_0^{\tau}\left(\frac{\partial\hat{\Psi}(\xi,\eta,\tau')}{\partial\zeta}\right)_{\zeta=0}\mathrm{d}\tau'. \tag{21}$$

According to the definition of $\tau = 0$, the displacement of the phreatic surface is identically equal to zero at midnight, for the full Moon at winter solstice.

The vertical extent of the domain, $z = -H$, does not exist in the sense that the rock is impermeable at a certain depth. Yet it can be defined according to the well known fact that hydraulic conductivity of Precambrian bed rock decreases with depth according to the "logarithmic law". If

$$K(z) = K_0 \mathrm{e}^{z/z_0} \tag{22}$$

the conductivity is practically vanishing if $|z/z_0| \geq 3$, which means that $H \approx 3z_0$. Extensive measurements in Sweden by Carlsson and Olsson (1977) show that $z_0 \approx 500\,\mathrm{m}$.

Solution of the flow equation

If the lateral length scales are large, i.e. $\hat{L}_1 \gg 1$ and $\hat{L}_2 \gg 1$, the flow is mainly vertical and is induced by the expansion and compression of the fractures. If $\tau \gg 1$, transients have faded. The vertical perturbation gravity in Eq. (5) can be written in complex form:

$$\phi_{3p}(\tau) = \sum_{j=1}^{N} a_{3j}\cos\hat{\omega}_j\tau + b_{3j}\sin\hat{\omega}_j\tau \equiv \mathrm{Re}\left\{\sum_{j=1}^{N}(a_{3j} - \mathrm{i}b_{3j})\mathrm{e}^{\mathrm{i}\hat{\omega}_j\tau}\right\} \tag{23}$$

which implies that the dimensionless displacement of the phreatic surface, Eq. (21), can be derived from Eq. (19) and (23). The derivation is presented in Appendix 3. The displacement is

$$\hat{h}(\tau) = \frac{\vartheta}{n_0}\mathrm{Re}\left\{\sum_{j=1}^{N}\frac{(b_{3j} + \mathrm{i}a_{3j})(1 - \mathrm{e}^{\mathrm{i}\hat{\omega}_j\tau})}{\mathrm{i}}\frac{\tanh k_j}{k_j}\right\}. \tag{24}$$

If the flow is quasi-steady

$$\lim_{\hat{\omega}_j\lambda\to 0}\hat{h}(\tau)=\frac{\vartheta}{n_0}\mathrm{Re}\left\{\sum_{j=1}^{N}(a_{3j}-\mathrm{i}b_{3j})(1-\mathrm{e}^{\mathrm{i}\hat{\omega}_j\tau})\right\}\equiv\frac{\vartheta}{n_0}(\phi_{3p}(\tau)-\phi_{3p}(0)) \tag{25}$$

and if the flow dominates by hydraulic diffusivity

$$\lim_{\hat{\omega}_j\lambda\to\infty}\hat{h}(\tau)=\frac{\vartheta}{n_0}\mathrm{Re}\left\{\sum_{j=1}^{N}\sqrt{\frac{2}{\lambda\hat{\omega}_j}}(1-\mathrm{i})(a_{3j}-\mathrm{i}b_{3j})(1-\mathrm{e}^{\mathrm{i}\hat{\omega}_j\tau})\right\}. \tag{26}$$

Discussion

If a *solution* of the general equation (19) is known *a priori*, its interpretation difficult. The reason is that the damping of the amplitude of $\Psi(\mathbf{r}, t)$ and the phase shift between $\phi_{3p}(t)$ and $\Psi(\mathbf{r}, t)$ depend not only on the coefficient λ but also on the angular velocities ω_j. This means that the perturbation of gravity generates the propagation of pore pressure in a dispersive way. A tempting approach is to assume that two frequencies, namely the diurnal and the semi-diurnal, dominate and that the monthly can be neglected. Unfortunately this is not possible unless the flow is quasi-steady. Gieske and de Vries (1985) have demonstrated this difficulty for a 2D radial problem.

The formulae (24)–(26) in the previous sections will be used for two purposes. The first one is to offer a possibility to evaluate the constitutive coefficient β that describes the relation between the perturbation gravity and the porosity of the bed rock. The second one is to estimate the displacement of a fluid particle of ground water in the vicinity of a cavity due to the perturbation gravity. An example of a cavity is a repository for radioactive waste in the Precambrian bed rock of Sweden.

Evaluation of the constitutive coefficient β

If the levels of the phreatic surface $h(t)$ and $\phi_{3p}(t)$ are measured, the coefficient β can be evaluated by formula (25) or (26). As mentioned above, evaluation is complicated. However, if the contribution from one frequency ω_ν dominates, the situation is less complicated. If $\lambda\hat{\omega}_\nu \gg 1$ the situation is simple and if $\lambda\hat{\omega}_\nu \ll 1$ it is trivial. Fig. 5 shows that there is one case for which the semi-diurnal frequency dominates, namely for a test site at the equator. In this case $N = 1$ and $\omega_1 \approx 2\omega_e$ is an acceptable approximation. Fig. 5 also shows that, if the test site is located north or south of the Arctic circles, respectively, a frequency which is approximately equal to the diurnal dominates. In this case $N = 2$, $\omega_2 \approx 2\omega_e$ and $a_{31} = b_{31} = 0$. This approximation is less accurate than the previous one. At the north and south poles the annual frequency dominates.

An idealized case, in which the contribution from one frequency dominates, can serve as a qualitative demonstration of the importance of the dispersion. Let $\phi_{3p}(\tau) = b_{3\nu}\sin\hat{\omega}_\nu\tau$. Then Eq. (24) yields

$$\hat{h}(\hat{\omega}_\nu\tau)=\frac{\vartheta b_{3\nu}}{n_0}\mathrm{Re}\left\{\mathrm{i}(\mathrm{e}^{\mathrm{i}\hat{\omega}_\nu\tau}-1)\frac{\tanh k_\nu}{k_\nu}\right\}. \tag{27}$$

The practically important limits are,

$$\lim_{\lambda\hat{\omega}_\nu\to 0}\hat{h}(\hat{\omega}_\nu\tau)=\frac{\vartheta b_{3\nu}}{n_0}\sin\hat{\omega}_\nu\tau \tag{28}$$

and

$$\lim_{\lambda\hat{\omega}_\nu\to 0}\hat{h}(\hat{\omega}_\nu\tau)=\frac{\vartheta b_{3\nu}}{n_0}\sqrt{\frac{2}{\lambda\hat{\omega}_\nu}}\left(1+\sqrt{2}\sin(\hat{\omega}_\nu\tau-\pi/4)\right). \tag{29}$$

The damping of the amplitude of $\hat{h}(\hat{\omega}_\nu \tau)$ and the phase shift between $\phi_{3\nu}(\hat{\omega}_\nu \tau)$ and $\hat{h}(\hat{\omega}_\nu \tau)$ are shown in Fig. 6. The graphs show that if the flow is quasi-steady ($\lambda\hat{\omega}_\nu \to 0$), the amplitude has its maximum and there is no phase shift. Increasing λ implies decreasing amplitude and increasing phase shift. The fact that $\hat{h}(\lambda\hat{\omega}_\nu \to \infty) \to 0$ does not indicate that the mass conservation equation is violated; if the time scale of the oscillations is much less than the diffusion time, the compression (or expansion) of the pores due to varying gravity is instantaneously compensated for by expansion (or compression) due to the generated pore pressure.

It is worthwhile noting that, if the perturbation gravity increases, the level decreases. The reason is that gravity is positive upwards, which means that increasing perturbation counteracts the gravitation of the Earth. This in turn relaxes the fractures of the rock.

The above discussion shows that the constitutive coefficient β can be evaluated from experiments with *simple means* in two cases.

The simplest one is, of course, quasi-steady flow, for which there is no need to worry about frequencies or phase shift. Still there is reason to choose the test occasion. At high latitude it seems least unfavourable to perform the test when the Moon is full or when it changes from vane to new, and to run the test for less than three days. This is clear from Fig. 7. The coefficient is given by the flow equation (19):

$$\beta = \frac{n_0(\sup h - \inf h)}{gH(\sup \phi_{3p} - \inf \phi_{3p})} \quad \text{quasi-steady flow, generated by any kind of tide.} \qquad (30)$$

In the other case the flow of ground water is highly diffusive, but generated by *one sinusoidal* perturbation of the gravity. The coefficient is given by the solution (29):

$$\beta \sim \frac{n_0(\sup h - \inf h)}{4gb_{3\nu}} \sqrt{\frac{\alpha\rho g \omega_\nu}{K}} \quad \text{diffusive flow, generated by sinusoidal tide.} \qquad (31)$$

The frequency can be semi-diurnal ($\omega_\nu = 2\omega_e$) or approximately diurnal ($\omega_\nu \approx \omega_e$). If the perturbation of the gravity is approximately sinusoidal, the magnitude of $b_{3\nu}$ can be measured. The relevant properties of the rock must be known *a priori*.

If a mobile gravimeter with sufficient resolution is not available, the difference between the extreme values of ϕ_{3p} can be calculated from the relatively simple formulae in the first section, or from the more exact but complicated formulae given in some advanced textbooks, notably Melchior (1978).

Displacement of a fluid particle at a small cavity

If a cavity is located below the surface, a fluid particle at the boundary of the cavity oscillates through its interface. The general 3D flow equation cannot be solved in closed form, not even for quasi-steady flow. If the diameter of the cavity $d_c \ll H$, the flow is practically vertical and the cavity serves as a small short circuit of the pore pressure. This means that the displacement of a fluid particle at ζ is limited by the displacement of the phreatic surface. Thus

$$0 < |\hat{v}_3(\zeta, \tau)| < \hat{h}(\tau). \qquad (32)$$

The dispersion due to the different frequency contributions complicates the solution. If the contribution from one frequency ω_ν dominates, the solution is simpler like in the previous subsection. Hence Fig. 8 shows the magnitude of the displacement of a fluid particle as a function of depth, divided by the magnitude of the displacement of the phreatic surface. The practical implication of this result is important for two reasons. Firstly, since the displacement decreases rapidly with depth, particularly if the influence of diffusion is important. Secondly since $h(\omega_\nu t)$ can easily be measured in bore holes at the actual site. It is worth observing that in Fig. 8 the vertical axis represents the depth, although the depth is the undependent variable.If the flow is quasi-steady, the solution is

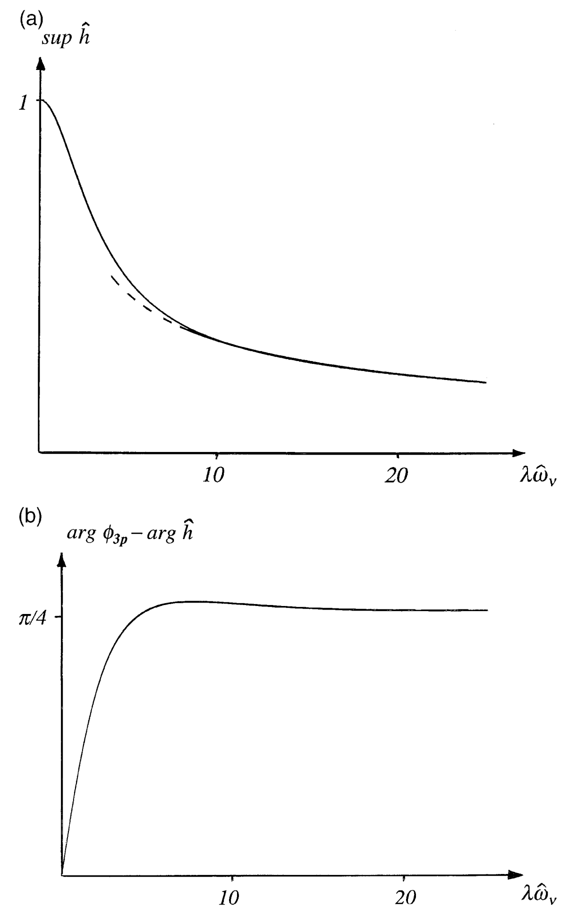

Figure 7 (a) Amplitude of the level of the phreatic surface as a function of dimensionless frequency. The broken line shows the limit for $\lambda\hat{\omega}_v \to \infty$, Eq. (33) (b) Phase shift between the perturbation gravity and the level of the phreatic surface as a function of dimensionless frequency

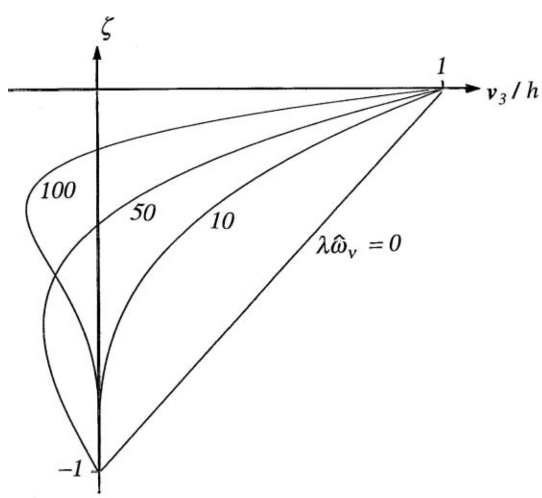

Figure 8 The magnitude of displacement of a fluid particle as a function of the depth. Dimensionless frequency as the parameter

particularly simple:

$$\lim_{\lambda\hat{\omega}_\nu \to 0} \hat{v}_3 \approx (1+\zeta)\hat{h}(\tau). \tag{33}$$

Equation (33) is the straight line in Fig. 8.

Character of tidal flow of ground water in Swedish bed rock

The above analysis shows that the character of the flow, no matter if it is quasi-steady or highly diffusive, is decisive for evaluation. An estimate that indicates the character can be derived from the extensive measurements that have been carried out at possible repository sites in Sweden, particularly at Äspö Hard Rock Laboratory. According to Rehn *et al.* (1997) and Carlsson and Olsson (1977), the following figures are characteristic:

$\alpha \sim 0.5 \times 10^{-9}\,\mathrm{Pa}^{-1}$
$K \sim 2 \times 10^{-9}\,\mathrm{m\,s}^{-1}$
$H \sim 1.5 \times 10^{3}\,\mathrm{m}.$

The semi-diurnal, diurnal, monthly and annual frequencies are

$2\omega_e = 1.454 \times 10^{-4}\,\mathrm{rad\,s}^{-1}$
$\omega_e = 7.272 \times 10^{-5}\,\mathrm{rad\,s}^{-1}$
$\omega_m = 2.662 \times 10^{-6}\,\mathrm{rad\,s}^{-1}$
$\omega_s = 1.991 \times 10^{-7}\,\mathrm{rad\,s}^{-1}$

These figures yield $\lambda \sim 7 \times 10^{-3}$ and consequently $1 \times 10^4 < \lambda\hat{\omega}_\nu < 8 \times 10^5$. Although the uncertainties of K, α and H are large, it is clear that the oscillating flow is highly diffusive. Therefore the phase shift between measured values of the perturbation gravity and the level of the phreatic surface in bed rock is $\pi/4$. Since the dominating frequency at the Arctic circle is approximately diurnal, and not semi-diurnal, it means that the time lag between the perturbation gravity and the level of the phreatic level is $\sim$ 3 h. Since the motion of ground water is highly diffusive, it is concentrated in the vicinity of the phreatic surface. The motion halfway down between the ground and the hydraulic bottom is effectively negligible. Incidentally, it can be noted that $\mathrm{Ek} \sim 3 \times 10^{13} \gg 1$, not surprisingly implying that Coriolis force in the flow equation is negligible.

Sources of errors

The analysis is based on several simplifications which means that their possible influences of the validity of the theoretical results must be examined.

The dominating source of error is violation of the continuum approximation. This approximation is particularly questionable for rock, which is of importance in the present context. The question is if it is worthwhile to discuss it in general terms since it is the main problem everywhere in geohydrology, rock mechanics, seismology, etc. If the purpose of the present theory is to define the constitutive coefficient β, the situation seems less unfavorable since the only length scale H is very large compared to all conceivable length scales of pores, distances between fractures, size of fractures, etc. On the other hand, if the purpose is to estimate the displacement of a fluid particle at a rock cavern, the length scale d_c is not much larger than all conceivable length scales of pores, which means that the continuum approximation is violated as usual.

Another source of error is the assumption that the rock or soil is homogeneous and isotropic. Since the porosity is assumed to depend on gravity it is natural to assume that the conductivity, which is a measure of the cross sectional area of the pores, also depends on the gravity. This violation of *geometry* is identical with that of the common theory of ground water flow, namely that the conductivity is independent of pore pressure. It is worth noticing that for practical cases, i.e. highly diffusive flow, the two effects counteract, thus reducing the influence of the approximation.

Interpretation of the measurement of the level of the phreatic surface can be difficult. Two effects are insidious since they can act within a time scale that can be mistaken for tide. Johansson (1986) has reported that, during hot summer days, the level of ground water in soil varies *diurnally* due to root water uptake. It cannot be excluded that such variations occur in bed rock. The effect is, of course, relatively large if the vegetation consists of deciduous wood. This source of error can probably be ignored in the Precambrian bed rock of Fennoscandia, which is relevant in this context. The reason is that most parts of Fennoscandia is covered by coniferous forests that has low root uptake. Although the summers at 60°N are short and normally cool, the wary scientist avoids field tests during hot summer periods. The greatest impact on the level in wells is due to the variation of the atmospheric pressure. It is important to observe that the measured variation of the level in a well that is due to the atmospheric pressure is not generated by its temporal variation but by its spatial. Field tests at latitude 60°N, such as in Scandinavia, should be avoided during the spring or fall when Rossby waves propagate atmospherical depressions around the globe. Heavy rainfalls affect the level of the phreatic surface but they do not occur diurnally or semi-diurnally. A particular kind of disturbance, very rare and weak in Scandinavia, is an earthquake.

In many practical field tests the problem is to *rule out* possible influence of the perturbation gravity, not to identify it. Good examples of the difficulties of interpreting measurements of the levels in wells, possibly generated by Earth tide, earthquakes and atmospheric pressure, are presented by Matsumoto (1992), Tohjima *et al.* (1994) and Roeloffs and Quilty (1997).

Measurement of the vertical component of the perturbation gravity by means of a gravimeter is fairly accurate. Calculation of all three components is extremely accurate. Even the approximate calculations, the result of which are given in the second section, give good accuracy. Measurement of the level of the phreatic surface in a bore hole or a well can be performed with a capacitive or resistive gauge. The accuracy of such devices is good, but some non-linearity can occur.

Conclusions

The oscillating perturbation gravity induced by the rotation of the Earth and the presence of the Sun and the Moon induces motion of the ground water in the bed rock. The following conclusions can be drawn.

1. The flow is vertical. Since the oscillating perturbation of gravity has its maximum at the Equator, so has the displacement of the free surface of the ground water, provided that the hydraulic coefficients of the rock are the same. North and south of the Arctic circles the magnitude and character of the perturbation of gravity is such that the induced displacement of the free surface and the gravity perturbation as well can be difficult to measure.
2. If the influence of the storativity of the bed rock is negligible, the oscillating flow of ground water is quasi-steady and in phase with the perturbation of gravity. The displacement of the phreatic surface assumes its maximum and is independent of the normal hydraulic coefficients of the rock.
3. If the flow is diffusive, i.e. if the influence of the storativity cannot be neglected, there

is a time lag between the oscillating perturbation gravity and the level of the oscillating phreatic surface. The amplitude of the oscillations of the level of the phreatic surface is less than those corresponding for quasi-steady flow. If the diffusivity of the flow increases is unlimited, the time lag reaches a maximum and the amplitude vanishes. If the flow is highly diffusive it means that the variation of porosity due to the variation of gravity is balanced instantaneously by the corresponding variation of the porosity due to the induced pore pressure.

4. Measurement of the level of the phreatic surface in bore holes and measurement, or calculation, of the vertical perturbation gravity offer a possibility to calculate the constitutive coefficient that relates the gravity and porosity. The evaluation is particularly simple for quasi-steady flow since it is independent of the normal hydraulic coefficients of the bed rock. If the flow is highly diffusive and generated by a tide that is sinusoidal, the evaluation is simple but requires a knowledge of the normal hydraulic coefficient of the bed rock.
5. The motion of the ground water is largest at the surface and decreases with depth. If the flow is diffusive, the decrease with depth is considerable.
6. A repository for burnt nuclear fuel located 600 m below the ground in the Fennoscandian bed rock is exposed mainly to diurnal oscillatory motion of the ground water. Since the flow of ground water is very diffusive, the magnitude of the displacement of a fluid particle is much less than 50% of the corresponding vertical displacement of the phreatic surface. This conclusion can be drawn without a knowledge of the value of the coefficient that relates the perturbation gravity with the porosity.

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References

Altman, S.L. (1986). *Rotations, Quaternions and Double Groups*. Clarendon Press, Oxford.

Bredehoeft, J.D. (1967). Responce of well-aquifer systems to earth tides. *J. Geophys. Res.*, **72**, 3075–3087.

Bredehoeft, J.D. (1997). Fault permeability near Yucca Mountin. *Wat. Res. Res.*, **33**, 2459–2463.

Carlsson, A. and Olsson, T. (1977). Hydraulic properties of Swedish crystalline rock. Hydraulic conductivity and its relation to depth. *Bull. Geol. Inst. Univ. of Uppsala N.S.*, **7**.

Gieske, A. and de Vries, J.J. (1985). An analysis of earth-tide-induced groundwater flow in eastern Botswana. *J. Hydrol.*, **82**, 211–231.

Johansson, P.O. (1986). Diurnal groundwater level fluctuations in sandy till. A model analysis. *J. Hydrol.*, **87**, 125–134.

Jiao, J.J. and Tang, Z. (1999). An analytical solution of groundwater response to tidal fluctuation in a leaky confined aquifer. *Wat. Res. Res.*, **35**(3), 747–751.

Lagally, M. (1944). *Vorlesungen über Vektor-rechnung*. Akademische Verlagsgesellschaft, Leilpzig.

Lamb, H. (1932). *Hydrodynamics*. Cambridge University Press, Cambridge.

Lewkowicz, J.F. (1985). Tilt observations from shallow and deep borehole arrays. *Phys. Earth Planet. Interiors*, **41**(1), 58–65.

Li, H. and Jiao, J.J. (2001a). Tide-induced groundwater fluctuation in a coastal leaky aquifer system extending under the sea. *Wat. Res. Res.*, **37**(5), 1165–1171.

Li, H. and Jiao, J.J. (2001b). Analytical studies of groundwater-head fluctuations in a coastal aquifer overlain by a semi-permeable layer with storage. *Adv. Wat. Res.*, **24**(5), 565–573.

Love, A.E.H. (1944). *A Treatise on the Mathematical Theory of Elasticity*. Dover, New York.

Matsumoto, N. (1992). Regression analysis of anomalous changes of ground water level due to earthquakes. *Geophys. Res. Lett.*, **19**(12), 1193–1196.

Melchior, P. (1978). *The Tides of the Planet Earth*. Pergamon Press, Oxford.

Neper, D.A. (2001). A model of oscillatory transport in granular soils, with application to barometric pumping and earth tide. *J. Contam. Hydrol.*, **48**(3–4), 237–252.
Nilsson, L.Y. (1968). Short-time variation of the ground water and its reasons. *Paper presented at the Int. Symp. on Ground Water Problems in Stockholm, October, 1966*, Pergamon Press, Oxford.
Rehbinder, G. and Isaksson, A. (1998). Large-scale flow of ground water in Swedish bedrock. An analytical calculation. *Adv. Wat. Res.*, **21**, 509–522.
Rehn, I. *et al.* (1997) Äspö HRL – Geoscientific evaluation 1995/5. Models based on site characterization 1986–1995, SKB Technical Report 97–06, Swedish Nuclear Fuel and Waste Management.
Richardson, R.M. (1956). Tidal fluctuations of water level observed in wells in east Tennessee. *Trans. Am. Geophys. Union*, **37**, 461–462.
Robinson, T.W. (1939). Earth-tides shown by fluctuation of water-levels in wells in New Mexico and Iowa. *Trans. Am. Geophys. Union*, **20**, 912–917.
Roeloffs, E. and Quilty, E. (1997). Water level and strain changes preceding and following the August 4, 1995 Kettelman Hills, California, earthquake. *Pure Appl. Geophys.*, **149**, 21–60.
Todd, D.K. (1977). *Groundwater Hydrology*. John Wiley and Sons, New York.
Tang, Z. and Jiao, J.J. (2001). A two-dimensional analytical solution for groundwater flow in a leaky confined aquifer system near open tide water. *Hydrol. Process.*, **15**(4), 573–585.
Tohjima, Y., Igarashi, G. and Wakita, H. (1994). Variations in ground water flow rate at the KSM well, a most sensitive well to earthquake occurrences. *J. Phys. Earth*, **42**(3), 221–235.
Tritton, D.J. (1992). *Physical Fluid Dynamics*. Oxford Science Publications, Oxford.

Appendix 1. Derivation of the potential field for a rotating system of two mass points

Consider two mass points, 1 and 2, rotating in circular orbits around their common centre of gravity. The two masses are M_1 and M_2, respectively, and the distance between them is D. Fig. A1 shows the two mass points and two coordinate systems (x', y', z') and (r', ϕ', θ'). The circle represents a section in the plane through a sphere around point 2, which is located at ($x' = 0$, $y' = 0$, $z' = 0$). Mass point 1, which is located at ($x' = D$, $y' = 0$, $z' = 0$), induces an acceleration at P. The potential of a unit mass at P, relative to the moving point 1, is a combination of two components:

$$\Phi(P) = -\frac{\mathscr{G}M_1 1}{\sqrt{D^2 - 2r'D\cos\vartheta + (r')^2}} + \frac{\mathscr{G}M_1 1}{D^2} r'\cos\vartheta$$

where $\mathscr{G}$ is the gravitational constant. The auxiliary angle ϑ is the angle between the P vector and the x' axis.

The relation between Cartesian and spherical coordinates is the common one:

$$x' = r'\sin\theta'\cos\phi'$$
$$y' = r'\sin\theta'\sin\phi'$$
$$z' = r'\cos\theta'.$$

Hence

$$\cos\vartheta = (\sin\theta'\cos\phi',\ \sin\theta'\sin\phi', \cos\theta')\cdot(1,0,0) = \sin\theta'\cos\phi'.$$

If $r' \ll D$,

$$\Phi(P) = \frac{\mathscr{G}M_1 1}{D}\left(2\left(\frac{r'}{D}\right)^2 (1 - 3\sin^2\theta'\cos^2\phi') - 1\right).$$

The force (r', ϕ' and θ' components) acting at a unit mass at the surface of the sphere $r' = r_0$ is given by the spherical gradient of Φ. Thus

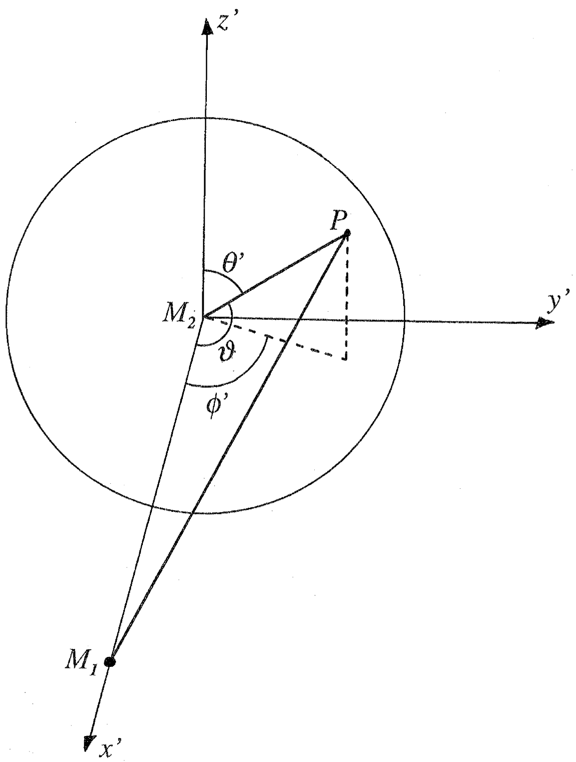

Figure A1 Definition of parameters

$$\mathbf{F} = \frac{4\mathcal{G}M_1 1 r_0}{D^3}\begin{bmatrix} 3\sin^2\theta'\cos^2\phi' - 1 \\ -3\sin\theta'\cos\phi'\sin\phi' \\ 3\sin\theta'\cos\theta'\cos^2\phi' \end{bmatrix}.$$

The above equations are sketched for $z' = 0$ in the celebrated textbook by Lamb (1932). In the x', y' and z' system

$$\Phi(P) = \frac{2M_1\mathcal{G}1}{D^3}(-2(x')^2 + (y')^2 + (z')^2)$$

and consequently

$$\mathbf{F} = -\nabla\Phi = \frac{4\mathcal{G}M_1 1}{D^3}\begin{bmatrix} 2x' \\ -y' \\ -z' \end{bmatrix}.$$

The force is symmetric in the sense that $\mathbf{F}(x', y', z') = \mathbf{F}(x', \sqrt{(y')^2 + (z')^2})$ and antisymmetric in the sense that $\mathbf{F}(x', y', z') = -\mathbf{F}(-x', y', z')$.

Appendix 2. Derivation of g_{tot}

The derivation of the potential field for a rotating system of two mass points in Appendix 1 deals with dynamics. The two potential fields between the Earth and the Sun and the Earth and the Moon can be added. The remaining part of the derivation of g_{tot} in the Earth fixed coordinate system is pure kinematics. It is not trivial despite the geometrical simplifications that are listed in the second section. The coordinate systems are also defined there.

The forces acting on a unit mass at the given position on the surface of the Earth in the two auxiliary system $\mathbf{r}_m = (x_m, y_m, z_m)$ and $\mathbf{r}_s = (x_s, y_s, z_s)$ are

$$\mathbf{F}_s = \frac{4\mathcal{G}M_s}{D_s^3}\begin{bmatrix} 2x_s \\ -y_s \\ -z_s \end{bmatrix}$$

and

$$\mathbf{F}_m = \frac{4\mathcal{G}M_m}{D_m^3}\begin{bmatrix} 2x_m \\ -y_m \\ -z_m \end{bmatrix}$$

where **G** is the gravitational constant.

The two auxiliary coordinate systems rotate in the inertial system and so does an Earth fixed coordinate system $\mathbf{r} = (x, y, z)$ that has its z axis perpendicular to the surface of the Earth. The latter system is used for the flow equations in the third section.

The different rotations, two of which are simple and one is complicated, must start with a given configuration, which is arbitrary. They are all derived in the inertial system.

The definition of $t = 0$ implies that the given position at the surface of the Earth is identified by the *latitude* $\pi/2 - \theta_0$ and the time $t > 0$. The situation is explained in Fig. 3.

The relations between the inertial system and the two auxiliary systems are

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -\cos\omega_s t & \sin\omega_s t & 0 \\ -\sin\omega_s t & -\cos\omega_s t & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix}$$

and

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos\omega_m t & -\sin\omega_m t & 0 \\ \sin\omega_m t & \cos\omega_m t & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_m \\ y_m \\ z_m \end{bmatrix}$$

respectively. For convenience the two transformation matrices above are denoted $\mathbf{M}_s$ and $\mathbf{M}_m$, respectively. The base vectors of the two auxiliary systems in the inertial system are

$$\hat{\mathbf{x}}_s(t) = -\begin{bmatrix} \cos\omega_s t \\ \sin\omega_s t \\ 0 \end{bmatrix}$$

$$\mathbf{y}_s(t) = -\begin{bmatrix} -\sin\omega_s t \\ \cos\omega_s t \\ 0 \end{bmatrix}$$

$$\hat{\mathbf{z}}_s(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{\mathbf{x}}_m(t) = \begin{bmatrix} \cos \omega_m t \\ \sin \omega_m t \\ 0 \end{bmatrix}$$

$$\hat{\mathbf{y}}_m(t) = \begin{bmatrix} -\sin \omega_m t \\ \cos \omega_m t \\ 0 \end{bmatrix}$$

$$\hat{\mathbf{z}}_m(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The rotations of the coordinate system $\mathbf{r}$ and the vector of the given position at the surface of the Earth around the Earth's axis are treated separately since their character is more complicated than the representation of the plane circular orbits of the Earth and the Moon in the ecliptic. The angle between the rotation axis of the Earth, $\hat{\mathbf{Q}}$, and the ecliptic is θ_e. The angular velocity of the Earth rotating around $\hat{\mathbf{Q}}$ is ω_e. The X axis is defined such that the normalized rotation vector of the Earth has no Y component, i.e. $\hat{\mathbf{Q}} = (\cos \theta_e, 0, \sin \theta_e)$. Let the base vectors of $\mathbf{r}$ be $\hat{\mathbf{X}}_b(t)$, $\hat{\mathbf{Y}}_b(t)$ and $\hat{\mathbf{Z}}_b(t)$ in the *inertial* system. The definition of $t = 0$ gives

$$\hat{\mathbf{X}}_b(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\hat{\mathbf{Y}}_b(0) = \begin{bmatrix} -\sin(\theta_e - \theta_0) \\ 0 \\ \cos(\theta_e - \theta_0) \end{bmatrix}$$

$$\hat{\mathbf{Z}}_b(0) = \begin{bmatrix} \cos(\theta_e - \theta_0) \\ 0 \\ \sin(\theta_e - \theta_0) \end{bmatrix}$$

$$\mathbf{R}_p(0) = r_0 \hat{\mathbf{Z}}_b(0).$$

The above vectors are most *conveniently* calculated by means Cayley's transformation in quaternion form. It is interesting to note that there is not only a convenience reason, but also a *fundamental* reason, for using this form. If the rotation of the vectors are made by means of Euler's angles, the transformation is not unique for all possible angles. Details on this subject is presented by Altman (1986).

Let the normalized rotation quaternion and its conjugate, in the inertial system, be

$$q(t) = \cos^2 \frac{\omega_e t}{2} + \hat{\mathbf{Q}} \sin^2 \frac{\omega_e t}{2}$$

and

$$q'(t) = \cos^2\frac{\omega_e t}{2} - \hat{\mathbf{Q}}\sin^2\frac{\omega_e t}{2}$$

respectively. Then

$$\hat{\mathbf{X}}_b(t) = q(t)\circ\hat{\mathbf{X}}_b(0)\circ q'(t)$$

$$\hat{\mathbf{Y}}_b(t) = q(t)\circ\hat{\mathbf{Y}}_b(0)\circ q'(t)$$

$$\hat{\mathbf{Z}}_b(t) = q(t)\circ\hat{\mathbf{Z}}_b(0)\circ q'(t).$$

The $\circ$ operator represents quaternion multiplication. The algebraic rules for calculation with quaternions and their applications to Cayley transformations are given in textbooks on advanced vector calculus, notably by Lagally (1944). The expansion of the quaternion triple products yields

$$\hat{\mathbf{X}}_b(t) = \cos^2\frac{\omega_e t}{2}\hat{\mathbf{X}}_b(t=0) - \sin^2\frac{\omega_e t}{2}\hat{\mathbf{Q}}\times\hat{\mathbf{X}}_b(t=0)\times\hat{\mathbf{Q}} + \sin\omega_e t\hat{\mathbf{Q}}\times\hat{\mathbf{X}}_b(t=0) + (\hat{\mathbf{Q}}\bullet\hat{\mathbf{X}}_b(t=0))\sin^2\frac{\omega_e t}{2}\hat{\mathbf{Q}}$$

$$\hat{\mathbf{Y}}_b(t) = \cos^2\frac{\omega_e t}{2}\hat{\mathbf{Y}}_b(t=0) - \sin^2\frac{\omega_e t}{2}\hat{\boldsymbol{Q}}\times\hat{\mathbf{Y}}_b(t=0)\times\hat{\mathbf{Q}} + \sin\omega_e t\hat{\mathbf{Q}}\times\hat{\mathbf{Y}}_b(t=0) + (\hat{\mathbf{Q}}\bullet\hat{\mathbf{Y}}_b(t=0))\sin^2\frac{\omega_e t}{2}\hat{\mathbf{Q}}$$

$$\hat{\mathbf{Z}}_b(t) = \cos^2\frac{\omega_e t}{2}\hat{\mathbf{Z}}_b(t=0) - \sin^2\frac{\omega_e t}{2}\hat{\mathbf{Q}}\times\hat{\mathbf{Z}}_b(t=0)\times\hat{\mathbf{Q}} + \sin\omega_e t\hat{\mathbf{Q}}\times\hat{\mathbf{Z}}_b(t=0) + (\hat{\mathbf{Q}}\bullet\hat{\mathbf{Z}}_b(t=0))\sin^2\frac{\omega_e t}{2}\hat{\mathbf{Q}}.$$

The vector products in the expressions for the rotating base vectors are

$$\hat{\mathbf{Q}}\times\hat{\mathbf{X}}_b(t=0) = \begin{bmatrix}-\sin\theta_e\\0\\\cos\theta_e\end{bmatrix}$$

$$\hat{\mathbf{Q}}\times\hat{\mathbf{X}}_b(t=0)\times\hat{\mathbf{Q}} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\hat{\mathbf{Q}}\bullet\hat{\mathbf{X}}_b(t=0) = 0$$

$$\hat{\mathbf{Q}}\times\hat{\mathbf{Y}}_b(t=0) = \begin{bmatrix}0\\-\cos\theta_0\\0\end{bmatrix}$$

$$\hat{\mathbf{Q}}\times\hat{\mathbf{Y}}_b(t=0)\times\hat{\mathbf{Q}} = \begin{bmatrix}-\sin\theta_e\cos\theta_0\\0\\\cos\theta_e\cos\theta_0\end{bmatrix}$$

$$\hat{\mathbf{Q}} \bullet \hat{\mathbf{Y}}_b(t=0) = \sin\theta_0$$

$$\hat{\mathbf{Q}} \times \hat{\mathbf{Z}}_b(t=0) = \begin{bmatrix} 0 \\ \sin\theta_0 \\ 0 \end{bmatrix}$$

$$\hat{\mathbf{Q}} \times \hat{\mathbf{Z}}_b(t=0) \times \hat{\mathbf{Q}} = \begin{bmatrix} \sin\theta_e \cos\theta_0 \\ 0 \\ -\cos\theta_e \sin\theta_0 \end{bmatrix}$$

$$\hat{\mathbf{Q}} \bullet \hat{\mathbf{Z}}_b(t=0) = \cos\theta_0.$$

Algebraic calculations yield the base vectors. Hence

$$\hat{\mathbf{X}}_b(t) = \begin{bmatrix} -\sin\theta_e \sin\omega_e t \\ \cos\omega_e t \\ \cos\theta_e \sin\omega_e t \end{bmatrix}$$

$$\hat{\mathbf{Y}}_b(t) = \begin{bmatrix} -\cos^2\frac{\omega_e t}{2}\sin(\theta_e-\theta_0) + \sin^2\frac{\omega_e t}{2}\sin(\theta_e+\theta_0) \\ -\sin\omega_e t\cos\theta_0 \\ \cos^2\frac{\omega_e t}{2}\cos(\theta_e-\theta_0) - \sin^2\frac{\omega_e t}{2}\cos(\theta_e+\theta_0) \end{bmatrix}$$

$$\hat{\mathbf{Z}}_b(t) = \begin{bmatrix} \cos^2\frac{\omega_e t}{2}\cos(\theta_e-\theta_0) + \sin^2\frac{\omega_e t}{2}\cos(\theta_e+\theta_0) \\ \sin\omega_e t\sin\theta_0 \\ \cos^2\frac{\omega_e t}{2}\sin(\theta_e-\theta_0) + \sin^2\frac{\omega_e t}{2}\sin(\theta_e+\theta_0) \end{bmatrix}.$$

Since

$$\cos^2\frac{\omega_e t}{2} = \frac{1}{2}(1+\cos\omega_e t)$$
$$\sin^2\frac{\omega_e t}{2} = \frac{1}{2}(1-\cos\omega_e t)$$

it follows that

$$\hat{\mathbf{X}}_b(t) = \begin{bmatrix} -\sin\theta_e \sin\omega_e t \\ \cos\omega_e t \\ \cos\theta_e \sin\omega_e t \end{bmatrix}$$

$$\hat{\mathbf{Y}}_b(t) = \begin{bmatrix} \cos\theta_e\sin\theta_0 - \cos\omega_e t\sin\theta_e\cos\theta_0 \\ -\sin\omega_e t\cos\theta_0 \\ \sin\theta_e\sin\theta_0 + \cos\omega_e t\cos\theta_e\cos\theta_0 \end{bmatrix}$$

$$\hat{\mathbf{Z}}_b(t) = \begin{bmatrix} \cos\theta_e\cos\theta_0 + \cos\omega_e t\sin\theta_e\sin\theta_0 \\ \sin\omega_e t\sin\theta_0 \\ \sin\theta_e\cos\theta_0 - \cos\omega_e t\cos\theta_e\sin\theta_0 \end{bmatrix}$$

and

$$\mathbf{R}_p(t) = r_0\hat{\mathbf{Z}}_b(t).$$

The derivation of the gravitational forces from the Sun and the Moon acting on a unit mass at a given position on the surface of the Earth in the **r** system is performed in steps. The first two equations in this appendix show that the forces are simply related to the base vectors of $\mathbf{r}_s$ and $\mathbf{r}_m$. Let

$$\mathbf{E}_g = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then the forces are acting on a unit mass at the surface of the Earth in the *auxiliary systems* $\mathbf{r}_s$ and $\mathbf{r}_m$. Hence

$$\mathbf{F}_s(t) = \frac{4\mathcal{G}M_s r_0}{D_s^3}\begin{bmatrix} (\mathbf{E}_g\hat{\mathbf{Z}}_b)\bullet\hat{\mathbf{x}}_s \\ (\mathbf{E}_g\hat{\mathbf{Z}}_b)\bullet\hat{\mathbf{y}}_s \\ (\mathbf{E}_g\hat{\mathbf{Z}}_b)\bullet\hat{\mathbf{z}}_s \end{bmatrix}$$

$$\mathbf{F}_m(t) = \frac{4\mathcal{G}M_m r_0}{D_m^3}\begin{bmatrix} (\mathbf{E}_g\hat{\mathbf{Z}}_b)\bullet\hat{\mathbf{x}}_m \\ (\mathbf{E}_g\hat{\mathbf{Z}}_b)\bullet\hat{\mathbf{y}}_m \\ (\mathbf{E}_g\hat{\mathbf{Z}}_b)\bullet\hat{\mathbf{z}}_m \end{bmatrix}$$

where

$$\mathbf{E}_g\hat{\mathbf{Z}}_b = \begin{bmatrix} 2\cos\theta_e\cos\theta_0 + 2\cos\omega_e t\sin\theta_e\sin\theta_0 \\ -\sin\omega_e t\sin\theta_0 \\ -\sin\theta_e\cos\theta_0 + \cos\omega_e t\cos\theta_e\sin\theta_0 \end{bmatrix}.$$

The above forces are transformed to the *inertial system*, and after that the latter vectors are projected on the base vectors of the **r** system. Hence

$$\mathbf{g}_s(t) = \begin{bmatrix} (\mathbf{M}_s^{-1}\mathbf{F}_s)\bullet\hat{\mathbf{X}}_b \\ (\mathbf{M}_s^{-1}\mathbf{F}_s)\bullet\hat{\mathbf{Y}}_b \\ (\mathbf{M}_s^{-1}\mathbf{F}_s)\bullet\hat{\mathbf{Z}}_b \end{bmatrix} \equiv \frac{4\mathcal{G}M_s r_0}{D_s^3}\begin{bmatrix} \phi_{s1}(t) \\ \phi_{s2}(t) \\ \phi_{s3}(t) \end{bmatrix}$$

$$\mathbf{g}_m(t) = \begin{bmatrix} (\mathbf{M}_m^{-1}\mathbf{F}_m)\bullet\hat{\mathbf{X}}_b \\ (\mathbf{M}_m^{-1}\mathbf{F}_m)\bullet\hat{\mathbf{Y}}_b \\ (\mathbf{M}_m^{-1}\mathbf{F}_m)\bullet\hat{\mathbf{Z}}_b \end{bmatrix} \equiv \frac{4\mathcal{G}M_m r_0}{D_m^3}\begin{bmatrix} \phi_{m1}(t) \\ \phi_{m2}(t) \\ \phi_{m3}(t) \end{bmatrix}$$

express the gravitational forces in the *local system* **r**, in which they will be used for calculation of the flow of ground water in the following sections. Inversion of the transformation matrices $\mathbf{M}_s$ and $\mathbf{M}_m$ is trivial. Thus

$$\mathbf{M}_s^{-1} = \begin{bmatrix} -\cos\omega_s t & -\sin\omega_s t & 0 \\ \sin\omega_s t & -\cos\omega_s t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_m^{-1} = \begin{bmatrix} \cos\omega_m t & \sin\omega_m t & 0 \\ -\sin\omega_m t & \cos\omega_m t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Appendix 3. Solution of the flow equation

The vertical perturbation gravity (23) is written in complex form:

$$\phi_{3p}(\tau) = \sum_{j=1}^{N} a_{3j}\cos\hat{\omega}_j\tau + b_{3j}\sin\hat{\omega}_j\tau \equiv \mathrm{Re}\left\{\sum_{j=1}^{N}(a_{3j} - \mathrm{i}b_{3j})\mathrm{e}^{\mathrm{i}\hat{\omega}_j\tau}\right\}.$$

If

$$\hat{\Psi}(\zeta,\tau) = \mathrm{Re}\left\{\sum_{j=0}^{N}\Upsilon_j(\zeta)\mathrm{e}^{\mathrm{i}\hat{\omega}_j\tau}\right\}$$

and (23) are introduced into (19), the amplitude functions $\Upsilon_j(\zeta)$ satisfy

$$\frac{\mathrm{d}^2\Upsilon_j}{\mathrm{d}\zeta^2} - k_j^2\Upsilon_j = \vartheta\hat{\omega}_j(b_{3j} + ia_{3j})$$

where $k_j = \sqrt{\mathrm{i}\hat{\omega}_j\lambda}$ and $j = 0\ldots N$.

The corresponding boundary values above are linearized. This means that the potential is undisturbed at the phreatic surface, $\zeta = 0$, and that there is no flux at the hydraulic bottom, $\zeta = -1$.

$$\Upsilon_j(\zeta = 0) = 0$$

$$\left(\frac{\mathrm{d}\Upsilon_j}{\mathrm{d}\zeta}\right)_{\zeta=-1} = 0.$$

The solutions of the boundary value problems above are

$$\Upsilon_j(\zeta) = \frac{\hat{\omega}_j\vartheta(b_{3j} + \mathrm{i}a_{3j})}{k_j^2}(-1 + \cosh k_j\zeta + \tanh k_j\sinh k_j\zeta).$$

The pore velocity and displacement of a fluid particle are

$$\hat{u}_3(\zeta,\tau) = -\frac{\vartheta}{n_0}\mathrm{Re}\left\{\sum_{j=1}^{N}\hat{\omega}_j\mathrm{e}^{\mathrm{i}\hat{\omega}_j\tau}(b_{3j} + \mathrm{i}a_{3j})\frac{\sinh k_j\zeta + \tanh k_j\cosh k_j\zeta}{k_j}\right\}$$

and

$$\hat{v}_3(\zeta,\tau) = \frac{\vartheta}{n_0}\mathrm{Re}\sum_{j=1}^{N}\left\{\frac{(b_{3j} + \mathrm{i}a_{3j})(1 - \mathrm{e}^{\mathrm{i}\hat{\omega}_j\tau})}{\mathrm{i}}\frac{\sinh k_j\zeta + \tanh k_j\cosh k_j\zeta}{k_j}\right\}$$

which implies that Eq. (21) yields the displacement of the phreatic surface. Thus

$$\hat{h}(\tau) = \frac{\vartheta}{n_0}\mathrm{Re}\left\{\sum_{j=1}^{N}\frac{(b_{3j} + \mathrm{i}a_{3j})(1 - \mathrm{e}^{\mathrm{i}\hat{\omega}_j\tau})}{\mathrm{i}}\frac{\tanh k_j}{k_j}\right\}.$$

If the flow is quasi-steady

$$\lim_{\hat{\omega}_j\lambda\to 0}\hat{h}(\tau)=\frac{\vartheta}{n_0}\mathrm{Re}\left\{\sum_{j=1}^{N}(a_{3j}-\mathrm{i}b_{3j})(1-\mathrm{e}^{\mathrm{i}\hat{\omega}_j\tau})\right\}\equiv\frac{\vartheta}{n_0}(\phi_{3p}(\tau)-\phi_{3p}(0))$$

and if the flow dominates by hydraulic diffusivity,

$$\lim_{\hat{\omega}_j\lambda\to\infty}\hat{h}(\tau)=\frac{\vartheta}{n_0}\mathrm{Re}\left\{\sum_{j=1}^{N}\sqrt{\frac{2}{\lambda\hat{\omega}_j}}(1-\mathrm{i})(a_{3j}-\mathrm{i}b_{3j})(1-\mathrm{e}^{\mathrm{i}\hat{\omega}_j\tau})\right\}.$$