Numerical modelling of the geodynamo: a systematic parameter study

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SUMMARY
We analyse ~50 3-D numerical calculations of hydrodynamic dynamos driven by convection in a spherical shell. We examine rigid and stress-free boundaries, with Prandtl number 1, magnetic Prandtl numbers in the range 0.5–5, Ekman numbers $E \approx 10^{-3}$–$10^{-4}$ and Rayleigh numbers to 15 times critical. No parametrizations such as hyperviscosities are used. Successful dynamos are compared with non-magnetic convection solutions. Results for various spectral truncations suggest that the calculations are well resolved when the kinetic and magnetic energy drops by more than two orders of magnitude from the spectral peak to the cut-off, although the basic features are still captured at lower resolution. With few exceptions we obtain dipole-dominated magnetic fields. The dynamos operate in the strong-field regime where Lorentz and Coriolis forces are of similar order. The critical magnetic Reynolds number for self-sustained dynamos is of order $R_m \approx 50$. However, we also find that the field can die away when $R_m$ is too large. The minimum magnetic Prandtl number at which we find dynamo action depends on the Ekman number to the 3/4 power. Dynamos at $E \approx 10^{-3}$ are subcritical whereas those at $E = 10^{-4}$ are generally supercritical. The presence of the magnetic field tends to break the equatorial symmetry of the flow and favours convection inside the inner core tangent cylinder. With stress-free boundaries, dynamo action suppresses the axisymmetric azimuthal wind that dominates in non-magnetic convection. The field morphology is broadly similar for both kinds of boundary conditions. When low-pass filtered, several models exhibit field structures that resemble the geomagnetic field at the core–mantle boundary to a surprising degree.

Key words: dynamo theory, Earth’s core, geomagnetic field, magnetohydrodynamics.

1 INTRODUCTION
The last few years have seen substantial progress in our understanding of the mechanisms that generate the Earth’s magnetic field in the liquid outer core. 3-D model calculations of convection-driven magnetohydrodynamic dynamos in a spherical shell have shown that dipole-dominated magnetic fields, comparable in strength to the Earth’s magnetic field, are generated by this process (Glatzmaier & Roberts 1995a,b, 1996, 1997; Kageyama & Sato 1997; Kuang & Bloxham 1997; Busse et al. 1998); see Fearn (1998) for a recent review. The morphology of the non-dipole part of the field in some of these models is qualitatively similar to that of the present geomagnetic field (Christensen et al. 1998).

This agreement is perhaps remarkable, given that some model parameters in the simulations have values that are, because of computational constraints, far from their values in the Earth’s core. Specifically, the Ekman number, which measures the importance of viscous forces on the flow relative to Coriolis forces, is in the range $10^{-3}$–$10^{-2}$, many orders of magnitude larger than in the core, and the magnetic Prandtl number, the ratio of fluid viscosity to magnetic diffusivity, is of order one or larger in the models compared to $10^{-6}$ for liquid metals. In order to reach somewhat more Earth-like parameter values, both Glatzmaier & Roberts and Kuang & Bloxham have used so-called hyperdiffusivites, in which the viscosity, thermal diffusivity and magnetic diffusivity are taken as increasing functions of the angular wavenumber. Hyperdiffusivity damps small-scale structures in the solutions and is meant to represent a scale-dependent turbulent mixing caused by the interaction of the large-scale flow with unresolved (subgrid scale) eddies. However, it is not clear how turbulent mixing should be parametrized in dynamo models. Zhang & Jones (1997) argue that hyperdiffusivity leads to an effective Ekman number which is larger than its nominal value. In addition to hyperdiffusion, these dynamo calculations neglect fluid inertia except in the axisymmetric parts of the flow, in order to allow for larger time steps. Even so, only a small
number of convective dynamo solutions have been examined, and little is known about how the control parameters affect the field morphology, intensity, and its variation with time.

Another reason for a systematic study is to establish the critical conditions for the onset of dynamo action. For the most part, previous dynamos have been found by trial and error. Although it is clear that the magnetic Reynolds number, which measures magnetic field induction and advection relative to ohmic losses, must exceed some critical value, we have little idea of how the critical condition for self-sustained dynamo action depends on the various control parameters. Dynamo action can be subcritical if non-magnetic convection solutions coexist at the same parameters and are stable against small magnetic perturbations, or supercritical if a small seed field will grow. Alternatively, the term subcritical is sometimes used when dynamo action occurs below the critical Rayleigh number for the onset of non-magnetic convection. In this situation, the magnetic field is a necessary ingredient for fluid motion to occur. St. Pierre (1993) has found one such dynamo in Cartesian geometry.

An important question in dynamo theory is how the poloidal part of the field is produced from the toroidal part and vice versa (e.g. Moffatt 1978). Most previous numerical dynamos have been described as being of the $2\omega$-type, in which the poloidal field is created by an $\alpha$-effect, that is, through the shearing and twisting of toroidal field lines in a helical flow (Moffatt 1978; Roberts & Gubbins 1987), and the toroidal field by an $\omega$-effect, that is, by the shearing of the poloidal field by differential rotation. In a preceding paper (Olson et al. 1999, hereafter referred to as Paper I) we identified two regimes among our dynamo solutions. In the strongly columnar regime, convection occurs only inside the inner core tangent cylinder, an imaginary cylinder parallel to the rotation axis that touches the inner core at the equator. Convection in this regime takes the form of Taylor columns aligned with the rotation axis. The conversion process between the two field components occurs through an $\alpha$-effect in both directions ($\alpha^2$-dynamo). In the so-called fully developed regime, the region inside the tangent cylinder is also convecting, typically with rising flow near the polar axis. In this regime the axisymmetric toroidal field has the opposite polarity inside and outside the tangent cylinder and is created by an $\alpha$-effect outside the tangent cylinder and by an $\omega$-effect inside ($\alpha^2\omega$-dynamo). The dynamo described by Glatzmaier & Roberts (1995a, 1996) is of the same type. In contrast, Kuang & Bloxham (1997) obtained fairly uniform and intense axisymmetric toroidal fields due to a dominant $\omega$-effect. They attributed this difference to the stress-free boundary conditions used in their models. Although the natural velocity boundary condition at the core–mantle boundary (CMB) is rigid (no slip), Kuang & Bloxham (1997) argued that model calculations strongly overestimate the influence of viscous friction and a stress-free boundary would better represent the mechanical conditions at the CMB. In this paper we investigate the effect of different velocity boundary conditions over a range of the control parameters.

Most of the numerical dynamos exhibit dipolar symmetry of the poloidal magnetic field, with the axial dipole dominating the field at the surface. Exceptions are the quadrupolar dynamos obtained by Grote et al. (1999) for stress-free boundaries. It is not clear under what conditions the dipole symmetry is preferred over the quadrupole symmetry (e.g. Gubbins 1994). Some earlier severely truncated dynamo models had magnetic field structures dominated by higher-order multipoles (e.g. Wicht & Busse 1997). Another question is how the dominant wavelength of the flow is influenced by the magnetic field. Previous studies on magnetocoonvection, in which the flow is subject to an imposed magnetic field, suggest that the dominant short wavelengths in non-magnetic convection at low Ekman number give way to much larger convection cells when the Lorentz force is of the same order as the Coriolis force (Eltayeb & Kumar 1977; Fearn 1979; Cardin & Olson 1995). However, the imposed magnetic field in those calculations has a simple large-scale structure. It is not clear if the more complex fields created by a self-sustained dynamo will have the same effect. We do know that viscous forces must be small compared to the rotational forces for the magnetic field to affect the wavelength of the flow. It is not entirely clear how low values of Ekman numbers need to be to reach that regime (see Dormy et al. 1998).

In this paper we report a systematic study of dynamo behaviour, covering Ekman numbers $E = 10^{-3}$–$10^{-4}$. Rayleigh numbers up to 15 times critical and magnetic Prandtl numbers $Prm = 0.5$–$5$ at a fixed Prandtl number $Pr = 1$ for both rigid and stress-free boundary conditions. Low magnetic Prandtl numbers are geophysically more realistic, but self-sustained dynamo action is more difficult to achieve in this parameter range (e.g. Busse et al. 1998). Accordingly, one of our goals is to determine the conditions for low-$Prm$ dynamos. Another issue is adequate numerical resolution. Some earlier dynamo models suffered from poor resolution (e.g. Hirschberg & Busse 1995), as indicated by the strong sensitivity of the result on slight changes in the level of spectral truncation. Other models alleviated the resolution problem using hyperdiffusion. We therefore include a systematic resolution study of three dynamos to establish some criteria for convergence and also to determine how under-resolution affects the results.

2 MODEL SET-UP

We study time-dependent 3-D thermal convection and magnetic field generation in an electrically conducting Boussinesq fluid enclosed in a rotating spherical shell. The ratio of the inner shell radius to the outer radius is $r_i/r_o = 0.35075$. We assume fixed temperatures on both boundaries and treat the regions exterior to the fluid shell as electrically insulating. Gravity varies linearly with the radius. Mechanical boundary conditions are either rigid or stress-free; in the former case both boundaries are assumed to corotate.

2.1 Governing equations

We solve the following dimensionless equations:

$$
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \tau + 2 \mathbf{e} \times \mathbf{u} + \mathbf{V} \mathbf{P} = \mathbf{R} \mathbf{a} \tau + Prm^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B}, \quad (1)
$$

$$
\nabla \cdot \mathbf{u} = 0, \quad (2)
$$

$$
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = Pr^{-1} \nabla^2 T, \quad (3)
$$

$$
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + Prm^{-1} \nabla^2 \mathbf{B}, \quad (4)
$$

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where $\mathbf{B}$, $\mathbf{u}$, $P$ and $T$ are magnetic induction, velocity, pressure perturbation and temperature, respectively. The motion is measured with respect to a spherical coordinate system $(r, \theta, \phi)$ rotating with uniform angular velocity $\Omega \hat{z}$. The fundamental length scale is the shell thickness $D$, time is scaled by $D^2/\nu$, with $\nu$ the kinematic viscosity, and temperature is scaled by the imposed difference $\Delta T$ between inner and outer boundaries. The magnetic field is scaled by $(\mu_0 \mu F) \Omega / \Delta T$, with $\mu$ the magnetic permeability and $\eta$ the magnetic diffusivity. With this scaling, the dimensionless outer radius is $r_0 = 1.5398$ and the inner radius is $r_1 = 0.5398$.

Aside from the choice of boundary conditions, the system is controlled by four dimensionless parameters: the Ekman number, $E = v / \Omega D^2$, the Prandtl number, $Pr = \nu / \kappa$, the (modified) Rayleigh number, $Ra = 2g \Delta T D / \Omega^2$, and the magnetic Prandtl number, $Pm = \nu / \eta$, where $\kappa$ is thermal diffusivity, $\alpha$ is thermal expansivity and $g$ is gravity at the outer boundary. Because a search through the 4-D parameter space involves too many calculations, we fix the Prandtl number $Pr = 1$ in all calculations. Based on molecular transport properties, the Prandtl number in the Earth’s outer core is thought to be of the order 0.1. According to the Prandtl hypothesis, this parameter should approach one when turbulent diffusivities are employed. Using $Pr = 1$ implies the magnetic Prandtl number $Pm$ and the Roberts number $q = \kappa / \eta$ are identical, so that they can be used interchangeably in this paper. Our scaling for the magnetic field implies that the non-dimensional value of $B$ is equivalent to the square root of the (local) Elsasser number $\Lambda$. In terms of dimensional magnetic field $B$, $\Lambda = B^2 / (\mu_0 \mu F)$ and measures the influence of Lorentz forces compared to Coriolis forces. The scaled kinetic and magnetic energy densities, $E_{\text{kin}}$ and $E_{\text{mag}}$, are obtained as the average values inside the spherical shells of $u^2/2$ and $(PmE)^{-1} B^2/2$, respectively. The scale factors for other quantities and the mathematical formulation of the boundary conditions can be found in Paper I.

2.2 Numerical technique

The numerical technique has been described in Olson & Glatzmaier (1995, 1996). Here it is briefly summarized, including an important modification of the time step control. The velocity and magnetic field vectors are represented by poloidal and toroidal scalars, and eqs (1) and (4) are written in terms of these functions. The resulting four scalar equations, along with the temperature equation (3) are advanced simultaneously at each time step. The five scalar variables are expanded in spherical harmonics in $\theta$ and $\phi$ up to degree and order $l_{\text{max}}$, and in Chebyshev polynomials in $r$. At each time step the non-linear terms in (1)-(4) are evaluated on a grid in physical space. Aliasing is avoided by using a larger number of grid points than would be required for a given truncation of harmonics, $3l_{\text{max}}/2$ in the $\theta$-direction and $3l_{\text{max}}$ in the $\phi$-direction. This technique differs from that employed by Glatzmaier & Roberts (1995a,b, 1996) by an explicit treatment of the Coriolis force.

Because the terms describing advection, Lorentz and Coriolis forces in (1), (3) and (4) are treated explicitly, stability requirements set an upper limit for the time step $\Delta t$. The Coriolis term requires $\Delta t < E/4$, although in practice instabilities were encountered when $\Delta t$ exceeded $E/6$.

Advection requires that the Courant criterion,

$$\Delta t < (|u_r|/\Delta r; |u_\theta|/\Delta \theta)_{\text{max}}^{-1},$$

be satisfied, where $u_r$ and $u_\theta$ are the radial and horizontal velocity components, respectively, $\Delta r$ is the radial grid interval, $\Delta \theta = \theta_{\text{max}}/(l_{\text{max}} + 1) - 1/2$ and the maximum in eq.(5) is evaluated over the entire grid.

The explicit treatment of magnetic induction and Lorentz force requires an additional criterion for the time step, based on the magnetic field strength. Originally, this was accounted for by taking for $\alpha$ in (5) the sum of the absolute values of the fluid velocity, $u_r$, and the local Alfvén velocity.

$$u_A = (PmE)^{-1/2} B.$$  \hspace{1cm} (6)

Ignoring the Alfvén velocity can lead to the excitation of spurious Alfvén waves or torsional oscillations (Gubbins & Roberts 1987). The small values of $\Delta r$ near the inner boundary of the non-uniform grid in combination with the large maximum values of $B$, up to $\sim O(10)$ in this region imply that for many of our dynamos the time step is essentially limited by the Alfvén velocity, hence the field strength, and not by the fluid velocity or the constraint based on rotation. However, employing (6) in a Courant-type condition does not account for the viscous and ohmic damping that suppresses short-wavelength Alfvén-type oscillations. The dissipation has a stabilizing influence, especially in the regions with small grid spacing. Following Chandrasekhar (1961), damping can be accounted for by defining a complex Alfvén velocity whose imaginary part is, in our scaling, $(1 + Pm^{-1})k/2$, with $k$ the wavenumber. For the largest possible $k \sim 1/\Delta r$ on the grid, the imaginary component may easily exceed the real part of the Alfvén velocity. We account for this by employing a reduced Alfvén velocity, setting in (5)

$$|u_r| = c_F |u_{F,r}| + c_A u_{A,r}/\sqrt{u_{A,r}^2 + [(1 + Pm^{-1})/(2\Delta r)]^2}.$$ \hspace{1cm} (7)

Here $u_{F,r}$ is the radial component of the fluid velocity and $u_{A,r} = (PmE)^{-1/2} B_r$ is the radial Alfvén velocity. An equivalent expression is used for $|u_\theta|$. The use of eq. 7 is not based on a rigorous stability analysis, but various tests have indicated that this scheme remains stable for values of the factor $c_F$ as low as $0.6-0.8$. In the dynamo runs $c_F = 2.5$ and $c_A = 1$ have been set. In many cases the new scheme increases the time step by a factor of 2-4 compared to the step calculated with the unmodified Alfvén velocity.

2.3 Modelling strategy

A few of our model runs are started from magnetocoonvection simulations as described in Paper I. In most cases, however, we use the results of dynamos with different parameters as initial conditions. All calculations start with a strong dipole-dominated field with $B \sim O(1)$. Each dynamo is run for at least three magnetic dipole decay times. In our shell geometry this is $t_{\text{dipole}} = 0.236 Pm$, marginally longer than $t_{\text{dipole}} = 0.240 Pm$, appropriate for a full conducting sphere with the same outer radius. In many cases the initial transient behaviour takes less than one dipole time. For these, the time-averaged properties of the dynamo are calculated with data from the subsequent two dipole times. In some calculations the initial transient took
longer. For those we continued the run, up to 10 dipole times in a few cases. Examples of the time variation of the magnetic energy are shown in Fig. 1.

Model runs exhibiting field decay are usually continued until the magnetic energy has dropped by a factor of 10 or more before it is declared a failed dynamo. In most of these cases a logarithmic plot of magnetic energy versus time shows a linear trend, which provides an estimate of the exponential decay constant (e.g. case e in Fig. 1). We tested the stability of some dynamo solutions by multiplying the magnetic field by a factor of 1/20 or 1/50 and monitoring the subsequent evolution of magnetic energy. This often produces a period of exponential growth or decay (see Paper I for an example), which allows us to determine a growth constant, \( t_{\text{grow}} \).

For each successful dynamo we also calculate the corresponding solution for non-magnetic convection. These calculations are started from a conductive temperature profile with random noise superimposed. They reach a statistical equilibrium more rapidly than the dynamos, typically in a fraction of the viscous or thermal diffusion time.

At Ekman numbers \( E > 10^{-4} \) the solutions are full-sphere, without any assumed symmetry. For most of the \( E = 10^{-4} \) cases, which require higher resolution, we use two-fold symmetry in longitude. With this symmetry, only the even harmonic order coefficients \( m \) appear in the spherical harmonic expansion. In all but a few cases, the truncation parameter of the spherical harmonic expansion \( \ell_{\text{max}} \) is chosen to ensure that the spectral power distribution of the kinetic and magnetic energy drops by a factor of at least 100 from the spectral peak to the cut-off value, which ensures adequate resolution (see below). In the radial direction the grid points are located at \( r_j = r_1 + (1 + \cos(\pi j/N_r))/2 \) for \( j = 0, \ldots, N_r \). We employ \( N_r = 32 \) radial grid intervals in cases with \( E > 10^{-4} \), and \( N_r = 40 \) in most of the cases with \( E = 10^{-4} \), which ensures that the Ekman boundary layers are covered by three grid intervals.

### 3 RESULTS

Out of 74 trials we found 48 cases that are self-sustained dynamos. The control parameters, numerical parameters and time averages of various quantities for the dynamos are listed in Tables 1–4. Values in parentheses correspond to non-magnetic convection calculations with the same parameters. Table 5 gives the decay rates of the magnetic energy for failed dynamos. The magnetic Reynolds number \( R_m = u_m D/\eta \) in Table 5 is based on the characteristic velocity at a stage where the decaying magnetic field has an intensity of order one. The critical Rayleigh numbers used in Tables 1–5 are those for non-magnetic convection. They have been calculated by monitoring the growth or decay of weak thermal perturbations of the conductive state in a number of trial runs and their values are given in Table 6 with an estimated accuracy of 0.5 per cent. Almost all dynamos show chaotic time dependence. An exception is case 100/5 in Table 1 (the first number is the modified Rayleigh number and the second the magnetic Prandtl number) at Ekman number \( E = 10^{-3} \). This dynamo is quasi-stationary; that is, it is stationary apart from longitudinal drift. It exhibits equatorial symmetry and four-fold symmetry in longitude (Fig. 2a), although the initial condition did not have these symmetries. Case 125/4 in Table 1 shows periodic oscillations which break the equatorial symmetry. We first describe the dynamos with rigid boundaries in some detail and finally we compare these with stress-free boundary cases.

#### 3.1 Influence of resolution

Three cases are selected for systematic resolution tests. The quasi-stationary dynamo at \( E = 10^{-3} \) allows precise numerical comparisons at different resolutions, and chaotic dynamos at \( E = 10^{-4} \) (cases 418/2 and 750/2 in Table 3) allow comparison of time-averaged spectra and qualitative aspects of the magnetic field. In all cases we start from a well-resolved solution and then decrease the truncation parameter \( \ell_{\text{max}} \) and the number of radial grid intervals \( N_r \) in steps.

In the quasi-stationary dynamo with \( \ell_{\text{max}} = 24 \) and \( N_r = 20 \), global properties such as mean kinetic and mean magnetic energy and the Nusselt number agree with the best-resolved case to within 2 per cent. Increasing the resolution to \( \ell_{\text{max}} = 32 \) and \( N_r = 24 \) reduces the discrepancies to 0.2 per cent (Table 7). For \( \ell_{\text{max}} = 16, N_r = 16 \), the magnetic energy deviates appreciably, but the long-wavelength structure of the velocity and magnetic fields remains qualitatively unchanged. However, when the resolution is further decreased, the kinetic energy increases and the magnetic field dies away.

We find the same qualitative behaviour for the chaotic dynamos at \( E = 10^{-4} \). For the case with \( Ra = 418 \), stepwise reduction of the resolution to \( \ell_{\text{max}} = 32, N_r = 32 \) does not affect the time-averaged magnetic and kinetic energies, at least at the level of the chaotic fluctuations, about 10 per cent (Fig. 3). At lower resolution the magnetic field decays and the kinetic energy blows up. Fig. 4 shows time-averaged spectra of kinetic and magnetic energy obtained at the various resolutions. The distribution is very similar around the spectral maximum, even for comparatively coarse resolution. However, in the last 10 harmonics before the spectral cut-off, ‘tails’ of elevated energy appear, indicating that energy starts to accumulate near the end of the spectrum instead of cascading to shorter wavelengths where it can be dissipated by diffusive processes. When the resolution is too coarse this accumulation becomes catastrophic, leading to spurious increases in kinetic energy. When this happens the energy spectrum changes from red to blue (see \( \ell_{\text{max}} = 24 \) in Fig. 4). No obvious qualitative differences are seen in the flow and magnetic field structure for \( \ell_{\text{max}} \geq 42 \).
However, for $\ell_{\text{max}} = 32$, spurious short-wavelength features start to become prominent, especially in the velocity field, although the large-scale structure is not altered (not shown). At $Ra = 750$ the result is similar, but the mean kinetic and magnetic energies already differ by more than 10 per cent from those of well-resolved solutions when $\ell_{\text{max}}$ is lowered to 32.

These results suggest that the calculations are decently resolved when the spectral power of kinetic and magnetic energy drops by more than a factor of 100 from the spectral maximum to the cut-off wavelength. The general structure of the flow and magnetic field are still captured at even lower resolution. All the dynamo calculations reported here, with few exceptions, show a decay in the kinetic and magnetic energy spectra of more than a factor of 100, many of them by a factor of 1000 or more.

The influence of the assumed two-fold symmetry in longitude, which is made in the runs at $E = 10^{-4}$ in order to save computer time, is tested for a case with $Ra = 560$ and $Pm = 2$ by comparing it with a solution obtained in the full sphere. The time-averaged properties differ very little (Table 3) and the qualitative structure of the flow and magnetic field is very similar (not shown). The tilt of the dipole field at the outer surface, which must be zero when longitudinal symmetry is assumed, is only $2^\circ$ on average and never exceeds $5^\circ$ during the calculation. Although deviations from longitudinal symmetry are probably important for reversals of the dipole field, we conclude that, in our parameter regime, the restriction to even azimuthal modes has little effect on the qualitative behaviour of the dynamos.

### 3.2 Critical conditions and stability of dynamos

One aim of this study is to delineate in parameter space the conditions for the onset of dynamo action. A large number of runs have been performed in the region where the transition from stable to decaying magnetic solutions is anticipated. Fig. 5 shows the results of this search. The inferred boundary between stable and decaying solutions is shown for various Ekman numbers, as a function of the Rayleigh number normalized by its critical value and of the magnetic Prandtl number. A certain minimum value of the magnetic Reynolds number $Rm$ must be exceeded for self-sustained dynamos. Based on the rms velocity, we find the critical magnetic Reynolds number to be in the range $Rm_{\text{crit}} = 40–50$ (Tables 1–4). At low magnetic Prandtl number the velocity needs to be larger to reach $Rm_{\text{crit}}$, hence the Rayleigh number must be more strongly supercritical, as seen in Fig. 5.

An unexpected result is that the dynamos die out again when the Rayleigh number becomes too large for a given $Pm$. This

Table 1. Dynamos at $E = 10^{-3}$ with rigid boundaries.

<table>
<thead>
<tr>
<th>$Ra$ / $Pm$</th>
<th>100 / 2</th>
<th>125 / 4</th>
<th>155 / 4</th>
<th>190 / 4</th>
<th>240 / 4</th>
<th>300 / 4</th>
<th>800 / 4</th>
<th>190 / 3</th>
<th>240 / 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell_{\text{max}}$</td>
<td>42</td>
<td>42</td>
<td>42</td>
<td>53</td>
<td>53</td>
<td>53</td>
<td>64</td>
<td>42</td>
<td>42</td>
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<tr>
<td>$E_{\text{kin}}$</td>
<td>31</td>
<td>55.3</td>
<td>87.3</td>
<td>141</td>
<td>232</td>
<td>383</td>
<td>2472</td>
<td>166</td>
<td>247</td>
</tr>
<tr>
<td>$E_{\text{mag}}^\text{ax}$</td>
<td>9%</td>
<td>6%</td>
<td>8%</td>
<td>7%</td>
<td>7%</td>
<td>6%</td>
<td>5%</td>
<td>5%</td>
<td>6%</td>
</tr>
<tr>
<td>$\ell_{\text{peak}}$</td>
<td>4–5</td>
<td>5</td>
<td>5</td>
<td>4–5</td>
<td>4–5</td>
<td>3–7</td>
<td>4–9</td>
<td>4–5</td>
<td>5–7</td>
</tr>
<tr>
<td>$m_{\text{peak}}$</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3–4</td>
<td>3–4</td>
<td>3–4</td>
<td>3–4</td>
<td>3–4</td>
<td>3–4</td>
</tr>
<tr>
<td>$N_{\text{tt}}$</td>
<td>1.34</td>
<td>1.46</td>
<td>1.57</td>
<td>1.65</td>
<td>1.81</td>
<td>2.03</td>
<td>3.33</td>
<td>1.66</td>
<td>1.81</td>
</tr>
<tr>
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<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

* Quasi-stationary; † periodic with period 0.18; * non-dipolar dynamo.

Results for non-magnetic convection are given in parantheses. $E_{\text{kin}}^\text{ax}$: fraction of axisymmetric toroidal energy; $E_{\text{mag}}^\text{ax}$: peak range for $E_{\text{mag}}$ in kinetic energy spectrum; $m_{\text{peak}}$: peak range for $m$ in kinetic energy spectrum; $\ell_{\text{peak}}$: equatorial symmetry, + yes, − broken; $B$: rms field in spherical shell; $B_{\text{surf}}$: rms field at outer surface; $B_{\text{mag}}$: dipole field at outer surface.
Table 2. Dynamos at $E = 3 \times 10^{-4}$ with rigid boundaries.

<table>
<thead>
<tr>
<th>$Ra / Pm$</th>
<th>334/5</th>
<th>210/3</th>
<th>334/3</th>
<th>240/2</th>
<th>280/2</th>
<th>334/2</th>
<th>418/2</th>
<th>560/2</th>
<th>560/2</th>
<th>750/2</th>
<th>750/2</th>
<th>1050/1</th>
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<td>5.0</td>
<td>4.8</td>
<td>3.5</td>
<td>4.0</td>
<td>4.0</td>
<td>4.8</td>
<td>6.0</td>
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<td>8.0</td>
<td>8.0</td>
<td>11</td>
<td>11</td>
<td>8.0</td>
</tr>
<tr>
<td>$E_{kin}$</td>
<td>251</td>
<td>749</td>
<td>385</td>
<td>482</td>
<td>775</td>
<td>1458</td>
<td>2815</td>
<td>888</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m_{peak}$</td>
<td>5-6</td>
<td>5-6</td>
<td>6</td>
<td>5</td>
<td>4-6</td>
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</table>

For explanations see Table 1.

behaviour is seen at the third Ekman number values considered. The range of Rayleigh numbers for which dynamos exist broadens with increasing magnetic Prandtl number. Only at $E = 10^{-3}$ was it possible to study whether dynamo action restarts at a still higher Rayleigh number. This is indeed the case. Using the result of model 300/4 as the initial state, we find at $Ra = 800$ and $Pm = 4$ a self-sustained magnetic field dominated by higher-multipole components that persists over

Table 3. Dynamos at $E = 10^{-4}$ with rigid boundaries.

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<tr>
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<th>334/5</th>
<th>210/3</th>
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<th>334/2</th>
<th>418/2</th>
<th>560/2</th>
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<th>750/2</th>
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$N_i$: radial grid intervals; $m_i$: symmetry in $\phi$; for further explanations see Table 1.

© 1999 RAS, GJI 138, 393–409
Table 6. Dynamos with stress-free boundaries.

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For explanations see Tables 1 and 3; * quadrupolar dynamo, * non-symmetric dynamo.

Table 7. Resolution test at $E = 10^{-3}$, $Ra = 100$, $Pm = 5$.

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<th>$m_{\text{act}}$</th>
<th>$E_{\text{kin}}$</th>
<th>$E_{\text{mag}}$</th>
<th>$N_{\text{cut}}$</th>
<th>$E_{\text{cut}}^{\text{Kin}}/E_{\text{mag}}^{\text{Kin}}$</th>
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<td>$1.9 \times 10^{-4}$</td>
</tr>
<tr>
<td>24</td>
<td>20</td>
<td>1</td>
<td>30.94</td>
<td>644.8</td>
<td>1.3456</td>
<td>0.0023</td>
</tr>
<tr>
<td>16</td>
<td>16</td>
<td>1</td>
<td>32.08</td>
<td>699.5</td>
<td>1.355</td>
<td>0.0214</td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>1</td>
<td>137.8</td>
<td>0.0</td>
<td>1.67</td>
<td>1.0</td>
</tr>
</tbody>
</table>

$E_{\text{cut}}^{\text{Kin}}/E_{\text{mag}}^{\text{Kin}}$: ratio of kinetic energy near spectral cut-off to energy at spectral peak; see Table 1 for further explanations.

The shape of the boundaries in the dynamo domain diagram (Fig. 5) implies that a minimum magnetic Prandtl number $Pm_{\text{crit}}$ exists, below which self-sustained dynamo action does not occur, even at magnetic Reynolds numbers in excess of 100 (Table 5). The value of $Pm_{\text{crit}}$ depends strongly on the Ekman

Critical azimuthal mode $m$ in parentheses.

four dipole times (upper right corner in Fig. 5a). This type of intermittent dynamo action as a function of the Rayleigh number was also found by Morrison & Fearn (1999) using a so-called 2.5-D dynamo model. Because the magnetic energy spectrum drops only by a factor of 30 from the peak to the spectral cut-off, we have only moderate confidence in this solution. As our interest lies mainly in the dynamos with dipole-dominated fields that are found in the large white regions in Fig. 5, a more precise delineation of the boundary for the non-dipolar dynamos has not been attempted.
Figure 2. Radial magnetic field $B_r$ at the outer shell surface (left in each panel) and radial velocity $u_r$ at $r = 0.8r_o$ (right in each panel). Red colours indicate positive values and blue colours negative values. Model parameters $E$, $Ra$ and $Pm$ are (a) $10^{-3}$, 100, 5 (quasi-stationary dynamo); (b) $10^{-3}$, 800, 4 (non-dipolar dynamo); (c) $10^{-4}$, 240, 2; (d) $10^{-4}$, 1050, 1; (e) $10^{-4}$, 750, 2; (f) $10^{-4}$, 750, 0.5. Contour steps for $B_r$ are 0.25, 0.2, 0.02, 0.25, 0.4, 0.1 in (a)-(f), respectively, and for $u_r$ 1.25, 16, 2.5, 30, 25, 25.

Figure 3. Kinetic energy and magnetic energy versus time for a dynamo at $E = 10^{-4}$, $Ra = 418$, $Pm = 2$, decreasing stepwise the cut-off $l_{\text{max}}$ of the harmonic expansion and the number of radial grid intervals $N_r$.

Figure 4. Time-averaged spectra of kinetic and magnetic energy for the same dynamo as in Fig. 3 at various levels of resolution. The zig-zag pattern is due to the exclusion of odd harmonic orders $m$ because of the assumed two-fold symmetry in longitude. The best-resolved cases ($l_{\text{max}} = 64$) are highlighted by circles and crosses.

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number and decreases systematically from about 3 at $E = 10^{-3}$ to 0.5 at $E = 10^{-4}$. In order to ensure a sufficient value of $Rm$, the normalized Rayleigh number at the minimum of the dynamo stability curves in Figs 5(a)–(c) must shift to higher values as the critical magnetic Prandtl number becomes lower.

The rms values of the magnetic field $B$ are listed in Tables 1–4, and in Figs 5(a)–(d) they are indicated by different symbols. $B$ is always of order one and ranges between 0.37 and 3.8 in our calculations, corresponding to Elsasser numbers $\lambda = B^2$ from 0.14 to 14. There is some ambiguity in how to measure the relative importance of Lorentz and Coriolis forces. The usual definition of the Elsasser number assumes that the effective current density in the Lorentz force term scales as $j \sim \sigma u B$. Alternatively, one may take $j \sim \mu^{-1} V \times B \sim \mu^{-1} B/I$, where $L$ is the characteristic length scale of the magnetic field.

The second option leads to a modified Elsasser number $\lambda'$ that is related to $\lambda$ by $\lambda' = Rm^{-3/2} \lambda$. Here the mean harmonic degree in the magnetic energy spectrum, $\ell = \sum k \langle E_{mag}(k) \rangle / E_{mag}$, is used for $L^{-1}$. $\ell$ lies typically between 5 and 12 in our dynamos. $\lambda'$ is roughly an order of magnitude smaller than $\lambda$ and covers the range 0.02–0.71, with most cases falling between 0.1 and 0.5. This suggests that the Lorentz force is slightly weaker than the Coriolis force, but is of similar order even when using the revised estimate. Hence our dynamos fall into the so-called strong-field regime. At $E = 10^{-3}$ the field is comparatively strong, with $\lambda > 3$, already very close to the dynamo stability boundary. In contrast, the Elsasser number is less than one near the stability boundary at an Ekman number of $10^{-4}$ and rises gradually when moving into the interior of the stability field. $B$ increases with $Pm$ in a slightly weaker than linear way when the other parameters are held fixed.

Another influence of the Ekman number is illustrated by suddenly reducing the field strength of a stable dynamo by a large factor and monitoring the re-equilibration process. At $E = 10^{-4}$ the weakened field grows exponentially ($t_{grow} > 0$ in Table 3) in all tested cases, even close to the critical conditions for dynamo action. All cases tested at $E = 10^{-3}$ (Table 1) show exponential decay. At $E = 3 \times 10^{-4}$ the result is somewhat inconclusive. The magnetic energy fluctuates but on time average it stays at a given level for several dipole decay times without a clear upward or downward trend. Although this might indicate a second (‘weak field’) branch of dynamo solution, we think it more likely that these dynamos are very close to neutral growth ($t_{grow} = 0$). This interpretation is supported by the observation that the field strength stays near its reduced level, whether this was 1/20 or 1/50 of the original field. If these solutions represented a stable weak-field branch, the magnetic energy should equilibrate at one definite level instead. Our results indicate that the dynamos at $E = 10^{-4}$ are supercritical, in the sense that non-magnetic convection is unstable and a small seed magnetic field introduced into the convection will grow with time. The dynamos at $E = 10^{-3}$ are subcritical, in the sense that non-magnetic convection at the same parameter values constitutes a second, stable branch of solutions.

Because most of the dynamos are chaotic, the meaning of the term ‘stability’ is not wholly precise. In order to survive as a subcritical dynamo, the magnetic field must stay within the basin of attraction of the dynamo. For example, the intensity must remain above some threshold value. If a fluctuation happens to be too strong, the field may become too weak for a time and the dynamo may die accidentally. An example of this behaviour is the failed dynamo with $Ra = 300, E = 10^{-3}$ and $Pm = 3$. It has been initiated from the dynamo at $Ra = 240$ and maintains a fairly constant level of magnetic energy for about four dipole decay times. The magnetic field then drops suddenly, and eventually it decays nearly exponentially (case d in Fig. 1). This case is a counter-example to the rule of thumb that maintaining a magnetic field for three dipole times is proof of a stable self-sustained dynamo.

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Figure 5. Domain diagram for dynamos at different Ekman numbers and boundary conditions. No self-sustained dynamos are found in the shaded regions; small circles (•) indicate failed dynamos. The rms field intensity $B$ is in the range (+) 0.25–0.50, (△) 0.5–1.0, (○) 1.0–2.0 and (◇) 2.0–4.0. The case in the upper right corner in (a) is a non-dipolar dynamo. The boundary for this type is uncertain.
In another test, we tried to establish a dynamo with a quadrupole-dominated poloidal magnetic field (at \( E = 10^{-4} \), \( Ra = 418, \ Pm = 2 \)). The run was started as magnetoconvection, with a strong toroidal field imposed on the inner and outer boundaries as described in Paper I, but with a harmonic degree \( l = 1 \) and order \( m = 0 \) pattern. In the magnetoconvection stage a dominantly quadrupolar poloidal field is generated. However, within one dipole decay time after the imposed field has been turned off, the quadrupole field decays and is replaced by a dipolar field.

3.3 Flow patterns

As discussed in Paper I, the flow pattern is dominated by columnar convection outside the tangent cylinder. This pattern persists throughout most of the studied parameter range. Columnar convection is indicated by the N-S-stretched structures in map projections of the radial velocity \( u_r \) shown in Fig. 2 (right side in each panel). The ‘banana shape’ seen in Fig. 2(c), for example, is not due to a bending of the column itself. Instead, the columns form a pinwheel pattern and are sliced by the projection at different distances from the rotation axis at different latitudes. However, at higher Elsasser numbers the columnar flow structure becomes more broken, as can be seen when comparing Figs 2(e) and (f). They show otherwise similar cases at different magnetic Prandtl numbers \( Pm \) and hence different field strength; \( \Lambda = 7 \) in (e) compared to 0.6 in (f). The effect of the field on the column structure is also shown in Fig. 6, which compares the component of vorticity parallel to the rotation axis \( \omega_\theta \) in meridional slices through the sphere for three cases with \( E = 3 \times 10^{-4} \) and \( Ra = 334 \) but different \( Pm \). In non-magnetic convection \( (Pm = 0) \) and in the dynamo with moderate field strength \( (Pm = 4) \) the columns are well developed, whereas at large Elsasser number \( (\Lambda = 14 \) in Fig. 6c) the columnar structure breaks down. The case in Fig. 6(c) is the only one found with strong deviations from columnarity. All others with Elsasser numbers \( \Lambda < 10 \) are clearly columnar. This agrees with results from magnetoconvection calculations (Olson & Glatzmaier 1995), where columns were found to break up for \( \Lambda > 10 \).

At small and moderate supercritical Rayleigh numbers the solutions are equatorially symmetric (antisymmetric with respect to \( u_\theta, B_\theta \) and \( B_, \) as shown in Figs 2(a) and (c)). However, a large magnetic field tends to break the equatorial symmetry. This is summarized in Fig. 7 for Ekman number \( E = 10^{-4} \). Symmetry breaking coincides roughly with the occurrence of active convection in the two regions within the tangent cylinder, north and south of the inner core. As described in Paper I, these regions are stagnant at low supercritical Rayleigh number, but at higher Rayleigh number they start to convect, often with plumes near the polar axes.

The onset of tangent cylinder convection requires a higher supercritical Rayleigh number at low Ekman number, but is favoured by the presence of a magnetic field (Fig. 7). According to linear stability analysis for rotating Bénard convection (Chandrasekhar 1961) this can be expected when the magnetic field vector is parallel to the rotation and gravity vectors, which is approximately the case inside the tangent cylinder for a dipole-dominated magnetic field. When tangent cylinder convection occurs, breaking of equatorial symmetry is plausible because there can be little communication between the two regions north and south of the inner core.

The kinetic energy is almost always higher in non-magnetic convection than in the corresponding dynamos (Tables 1–3), indicating that the dynamo equilibration is assisted by a reduction of convective vigour, through the action of the Lorentz forces (Paper I). On the other hand, the Nusselt number \( Nu \), defined as the mean temperature gradient at the inner or outer boundary normalized by the respective conductive gradient, is in general slightly higher in the dynamo calculations. Also, the total work done by buoyancy forces, that is, the volume integral of \( RaE^{-1}u_r Tr \) \( r_o \), which is the source for kinetic energy and ultimately magnetic energy, is higher for the dynamos than for the corresponding non-magnetic models. For example, in the quasi-stationary case at \( E = 10^{-3} \) \( (100/5 \) in Table 1) the rate of energy generation is found to be 43 per cent larger for the dynamo than the non-magnetic reference run. Whilst in non-magnetic convection all the energy is dissipated by viscous friction, Joule dissipation accounts for 62 per cent of the total in this dynamo, and the contribution from viscous dissipation is reduced to about 1 per cent. This is partly due to the relatively high magnetic Prandtl number used in these calculations.

Figure 6. Component of vorticity parallel to the rotation axis \( \omega_\theta \) in slices in the \( r, \theta \)-plane, chosen to pass through a region with large positive \( \omega_\theta \), \( E = 10^{-4}, \ Ra = 334 \) and (a) \( Pm = 0 \), (b) \( Pm = 3 \), (c) \( Pm = 5 \). Full contours positive and dash-dotted negative values; contour interval 400. The Elsasser number \( \Lambda \) is (a) 0.0, (b) 2.6, (c) 14.2.

Figure 7. Symmetry breaking for rotating convection and dynamos with rigid boundaries at \( E = 10^{-4} \). + equatorially symmetric, \( \circ \) symmetry broken, \( \oplus \) symmetry broken and active convection inside the tangent cylinder.
of (a and c) harmonic degree $\ell$ (summing over all $m\leq \ell$ and averaging radially and in time) and as a function of harmonic order $m$ (summing over all $\ell$ with $m \leq \ell \leq \ell_{\text{max}}$). In Tables 1–4 the respective peak ranges of the spectra are listed, where the peak range is given by the minimum and maximum values of $\ell$ and $m$, respectively, for which the energy remains within 75 per cent of the absolute maximum (excluding $\ell = 1$ and $m = 0$ in Tables 1–3). At both $E = 10^{-3}$ and $E = 3 \times 10^{-4}$ there is little systematic difference between dynamos and non-magnetic convection. At $E = 10^{-4}$ a systematic shift of the spectral peak towards lower values of $\ell$ and $m$ in the dynamo becomes discernible. Fig. 8 further illustrates this effect, with two selected cases. The shift in the spectral distribution at Ekman number $10^{-4}$ is more pronounced in $\ell$, which could mean that the velocity changes more smoothly at the upper and lower edges of the convection columns. However, a slight shift in $m$ is also present at $E = 10^{-4}$, typically from peak values of around 8 to values of around 6, and the spectral power decays more rapidly for $m > 10$ in the dynamos, indicating that there are fewer and broader columns.

With rigid boundaries, we find that the axisymmetric toroidal component of the flow, the azimuthal wind, typically contributes only a small amount to the total kinetic energy, of the order of 10 per cent. Except at the highest Ekman number, its contribution is somewhat less in the dynamos than in corresponding non-magnetic convection (Tables 1–3). The axisymmetric poloidal modes, which describe a meridional circulation, contain very little energy. Their contribution increases with Rayleigh number, which can be explained by the onset of nearly axisymmetric convection inside the tangent cylinder. Even then, however, the contribution from meridional flow always remains below 1 per cent. There are two reasons for this. One is the smallness of the regions inside the tangent cylinder, which make up 14 per cent of the volume of the spherical shell. Another is the fact that for Rayleigh numbers up to 15 times critical, tangent cylinder convection remains weaker than the columnar flow outside the tangent cylinder. The exceptional case is the non-dipole dynamo 800/4 at $E = 10^{-3}$ shown in Fig. 2(b), where the flow has similar vigour inside and outside the tangent cylinder.

### 3.4 Magnetic field

In most cases the magnetic energy is larger than the kinetic energy, usually by a factor of 2–5. Overall the range is 0.5–25 (Tables 1–3). The fraction of poloidal magnetic energy ranges between 0.25 and 0.69; that is, toroidal and poloidal modes contribute similar amounts to the total magnetic energy. This agrees with our classification of the dynamo mechanism as being predominantly of the $\beta^2$-type. The $\alpha$-effect, which could potentially lead to a dominance of the toroidal field component, generally plays a secondary role in these dynamos.

Fig. 2 (left part of each panel) illustrates the influence of the control parameters on the field at the outer boundary. The radial field is dominated by the axial dipole component in all cases except 800/4 at $E = 10^{-3}$ (Fig. 2b). Panels (a) and (b) compare dynamos with low and high Rayleigh numbers at $E = 10^{-3}$ and panels (c) and (d) the same for an Ekman number of $10^{-4}$. Comparing (e) and (f) illustrates the effect of changing the magnetic Prandtl number at constant $E$ and $Ra$. As shown previously (Kageyama & Sato 1997; Christensen et al. 1998; Paper I), flux concentrations occur preferentially at about 60° latitude, that is, close to the intersection of the tangent cylinder with the outer boundary, and are correlated with downwellings. A pronounced flux minimum at the poles is found for dipole-dominated dynamos, especially in cases with tangent cylinder convection, where the upwellings along the rotation axis transport flux away from the poles (Figs 2d and e). In the equatorial region there are paired spots with reverse flux, which have been interpreted as the expulsion of the toroidal field (Bloxham 1986; Christensen et al. 1998; Paper I). The reversed flux spots are weak when the magnetic Reynolds number is comparatively low. In general, the non-dipolar field contains more small-scale structure at lower Ekman number and higher Rayleigh number, where the velocity field is dominated by shorter wavelengths. For fixed values of $E$ and $Ra$ the field becomes more small-scaled at higher $Pr$. In this situation the magnetic Reynolds number is large and diffusion smooths only the smallest structures of the magnetic field (Figs 2e and f).

At the outer surface the axial dipole contributes 40–80 per cent to the magnetic field energy (excluding the non-dipolar dynamo 800/4 at $E = 10^{-3}$). In the interior of the shell, where the field intensity is larger by a factor of 2.5–5, higher-order modes contribute more strongly to the poloidal field, but the dipole component still contains typically one-half of the poloidal magnetic energy (compare $E_{\text{pol}}^{\ell = 1}$ with $E_{\text{mag}}^{\ell = 1}$ in Tables 1–4). The relative contribution of non-dipole components to the field energy is strongly correlated with the
magnetic Reynolds number. For example, at $Ra=750$ and $E=10^{-4}$ the dipole contributes 73 per cent to the poloidal energy at a magnetic Reynolds number of 50 ($Pm=0.5$), but only 39 per cent at $Rm=164$ ($Pm=2$). At the outer surface the dipole contribution to the rms field intensity drops from 88 to 58 per cent (compare Figs 2f and e). The contribution of the dipole component becomes especially low, in both an absolute and a relative sense, for dynamos that lie close to the high-Rayleigh-number edge of the stability regions in Figs 5(a) and (b). This weakening of the lowest-order component of the field may cause the breakdown of the dynamos with increasing Rayleigh number. In the models at $E=10^{-3}$ and $3 \times 10^{-4}$, where no longitudinal symmetry was assumed, the dipole tilt is found to be only $1-4^\circ$, with a peak excursion of $10.5^\circ$ in case 300/4 at $E=10^{-3}$.

The non-dipolar field of the dynamo 800/4 at $E=10^{-3}$ is mostly created inside the inner core, a region near the inner core (Fig. 10d). The Nusselt number directed westwards near the inner core and eastwards near the Rayleigh number. In the models at $E=10^{-3}$ and $3 \times 10^{-4}$, where no longitudinal symmetry was assumed, the dipole tilt is found to be only $1-4^\circ$, with a peak excursion of $10.5^\circ$ in case 300/4 at $E=10^{-3}$.

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bundles near the equatorial plane (Figs 9a and d). The lack of a flux minimum on the surface at the poles and the weaker toroidal field bundles inside the tangent cylinder are due to the absence of tangent cylinder convection.

An experiment in which the field intensity has been reduced to 1/50 of its original value ($E \approx 10^{-4}$, case 418/2b in Table 4) leads to an unexpected result. Rather than decaying or recovering the original dynamo state, after some large fluctuations the field settles into a different non-dipolar configuration. On the surface the field is patchy, but compared to the non-dipolar dynamo in Fig. 2(b) the flux is less concentrated. Also, its intensity is greatest at low and mid-latitudes (not shown). A surprising asymmetry between the two hemispheres persists during the whole run, with the dynamo being active only in the northern hemisphere. The magnetic energy reaches approximately half the value of

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Figure 9. Comparison of dynamos with rigid boundaries (left column) and stress-free boundaries (right column) at $E \approx 3 \times 10^{-4}$, $Ra \approx 510$, $Pm \approx 2$. Red colours indicate positive values and blue colours negative values. (a) $B_r$ at the outer boundary, contour step 0.15; (b) $u_r$ at $r = 0.8r_o$, step 12.5; (c) temperature in equatorial plane, step 1/16; (d) zonally averaged properties: $B_{\phi}$ colour-coded in the frames to the left, step 0.25, with axisymmetric poloidal field lines superimposed, and $u_{\phi}$ colour-coded in the right frames with meridional streamlines superimposed.

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partitioned in favour of the toroidal components than in most of the dipolar dynamos, suggesting that \( \omega \)-effects play a larger (and more constructive) role than in the dipolar dynamos. In the only stress-free dynamo calculated at an Ekman number of \( 10^{-3} \) (case 240/4 in Table 4) the initial field with dipole parity decays and is eventually replaced by a comparatively weak field of quadrupolar symmetry that is dominated by equatorial flux patches of both polarities. In this case the azimuthal wind is hardly diminished compared to non-magnetic convection.

4 SUMMARY AND COMPARISON WITH THE GEODYNAMO

Well-resolved 3-D calculations without hyperdiffusion show that \( x^2 \)-dynamos exist in almost the entire parameter range under consideration. They produce dipole-dominated magnetic fields from quasi-columnar convection outside the inner core tangent cylinder. The initial condition used in the modelling implies a certain bias for this kind of dynamo, and other types may coexist, as was shown for stress-free boundaries. The columnar flow structure starts to break down at Elsasser numbers in excess of 10, in agreement with previous results for magnetoconvection at a higher Rayleigh number (Olson & Glatzmaier 1995). A minimum magnetic Reynolds number of approximately 50 must be exceeded for self-sustained dynamo action. This condition is reached at low Rayleigh number with a large magnetic Prandtl number, that is, comparatively low magnetic diffusivity, or at lower magnetic Prandtl number by a more strongly supercritical Rayleigh number. However, an unexpected result is that the dynamos die out again when, at fixed magnetic Prandtl number, the Rayleigh number becomes too large. In addition, there is a minimum magnetic Prandtl number \( \text{Pm}_{\text{crit}} \) below which self-sustained dynamo action does not occur, even when the magnetic Reynolds number exceeds 100. The critical magnetic Prandtl number is an increasing function of the Ekman number. From our data we obtain the following approximate relation:

\[
\text{Pm}_{\text{crit}} = 450E^{3/4}. \tag{8}
\]

One implication of this result is that, in order to simulate dynamos in the geophysically realistic regime \( \text{Pm}<1 \), it is necessary to use very low Ekman numbers. For example, if we use the molecular diffusivities in the core, then \( \text{Pm} \approx 10^{-6} \) and (8) requires that \( E \approx 3 \times 10^{-12} \). Such a model would be impossible in terms of computer resources. Alternatively, if we assume that turbulence leads to an effectively larger magnetic Prandtl number in the core, say \( Pm \approx 10^{-2} \), then (8) requires \( E \approx 6 \times 10^{-7} \), still too small for present-day calculations, but possibly attainable in the foreseeable future.

Other anticipated low-Ekman-number effects such as the increase of the wavelength of the flow in the presence of a strong magnetic field are just discernable at the lowest Ekman number we studied. The destabilizing effect of the field on thermal convection, predicted by linear theory for low Ekman numbers, was not observed. This effect would lead to more vigorous flow in the presence of the field and to the possibility of convection below the critical Rayleigh number. Because such effects occur only when viscous effects are very nearly negligible, it is desirable to push the limit of dynamo calculations towards smaller Ekman numbers.

Figure 10. Non-magnetic convection at \( E=10^{-4}, R_a = 560 \). Comparison of rigid boundaries (a and b) with stress-free boundaries (c and d). In (a) and (c) the zonal wind component (axisymmetric part of \( u_z \)) is shown with full lines indicating positive (eastward) values and dash-dotted lines negative values; contour step is 30 and greyscale highlights absolute values. (b) and (d) show isotherms in the equatorial plane, contour step 1/16.

Figure 11. Variation with time of total magnetic energy (full line), poloidal magnetic energy (dotted line) and axisymmetric toroidal kinetic energy (dashed line) for case 210/4 at \( E=10^{-4} \) with stress-free boundaries.

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Using stress-free boundaries, we find essentially the same type of dynamo operating as in the case of rigid boundaries. This shows that Ekman layer effects do not dominate the solutions with rigid boundaries. The differences may become larger for a more realistic treatment of the inner core, which in the Earth can rotate independently from the mantle. Angular momentum can be exchanged between inner and outer core through magnetic torques and in the case of a rigid boundary additionally by viscous torques, which can influence the flow structure near the inner core boundary. One interesting effect of a stress-free boundary condition is the dominance of strong zonal winds in non-magnetic convection. We anticipate that these winds will also occur with rigid boundaries when the Ekman number is lower. The wind is strongly suppressed by the magnetic field in our dipolar dynamos. However, the possibility of exciting such winds gives rise to a different type of dynamo. The two examples that we found have a non-dipolar field structure.

The difference in these models has implications for the geodynamo. Maps of the flow at the surface of the Earth’s core inferred from the secular variation (e.g. Bloxham 1988) do not indicate a dominance of zonal winds. This suggests that the geodynamo may operate in a regime similar to that found in our dipolar dynamos, where the wind is strongly reduced and usually directed westwards near the outer boundary at low latitudes (Fig. 9d).

Compared to the geomagnetic field, our dynamos with a dominant dipole component are too stable in time. We find no reversals or large excursions amongst them. They have too much equatorial symmetry and in most cases the axial dipole is more dominant than in the geomagnetic field. However, there is a large degree of correspondence in the non-dipole components of the field. This is most obvious for those of our dynamo models with a large magnetic Reynolds number, where the energy in the non-dipolar part of the field is comparatively large. In Fig. 12 we compare the geomagnetic field at the core–mantle boundary (Bloxham & Jackson 1992) with the field of two dynamo models. Comparing the geomagnetic field with the complete dynamo model fields in Figs 12(b) and (d) reveals a general similarity, but there are also pronounced differences. In particular, individual flux spots are stretched in the meridional direction in the model field, in contrast to their more rounded shape in the geomagnetic field. The number of individual flux patches at high latitudes is larger in the model than in the geomagnetic field, where only two are found in each hemisphere. However, this apparent discrepancy may be caused by the limited resolution used to represent the core field. Crustal magnetization screens the core field at or above the Earth’s surface for spherical harmonic degrees above approximately 13. Higher harmonics must be filtered out by strong damping when inverting for the core field. To simulate the limited resolution inherent in maps of the core field, we apply the following low-pass filter to the dynamo fields:

\[
F(\ell) = \begin{cases} 
1 & \text{for } \ell < \ell_0 - \Delta\ell, \\
(1 - \sin [\pi(\ell - \ell_0)/2\Delta\ell])/2 & \text{for } \ell_0 - \Delta\ell < \ell < \ell_0 + \Delta\ell, \\
0 & \text{for } \ell > \ell_0 + \Delta\ell.
\end{cases}
\]

Setting \( \ell_0 = 12 \) and \( \Delta\ell = 2.5 \) approximates the damping employed in core field inversions (Gubbins & Bloxham 1985). The filtered model fields (Figs 12c and e) resemble more closely the geomagnetic core field. The N–S stretching of field structures disappears and two dominant flux patches at high latitude remain in each hemisphere, with roughly matching longitudes, just as in the geomagnetic field. Although in other models or at different times the number of high-latitude patches in the filtered maps can be larger, finding just two of a given polarity is quite common. The high-latitude flux patches mark the positions where convection columns with a cyclonic sense of motion impinge on the core–mantle boundary (Gubbins & Bloxham 1987; Christensen et al. 1998). Comparing the filtered and unfiltered versions of the model fields suggests that the presence of two pairs of patches in the geomagnetic field does not necessarily imply that there are only two cyclonic vortices in the core. Instead, the patches may represent a large-scale modulation of smaller-scale variations in magnetic flux on the core–mantle boundary, associated with low Ekman number convection in narrow columns. According to this interpretation, the observed patches may correspond to locations where columnar convection is particularly intense, perhaps due to variations in heat flow over the core–mantle boundary. This interpretation offers an explanation for the difference in scale of magnetic flux patches predicted by convection models versus their observed scale. It also suggests that the core surface flow pattern obtained from low-pass-filtered projections of the geomagnetic field on the core–mantle boundary may also be strongly smoothed, that is, unrealistically large-scaled.

In Fig. 13 the time-averaged magnetic field spectrum at the surface of model 300/4 at \( E = 10^{-3} \) is compared to that of the geomagnetic field at the core–mantle boundary. When scaled to the Earth, the power in the model field is too large by a factor of 6. However, the overall pattern is similar, including the relative weakness of the quadrupole term and the weakly decreasing trend with increasing \( \ell \). The spectrum in the dynamo model of Glätzmäier & Roberts (1995b) has a similar form, although the power decreases more strongly with \( \ell \) than in the Earth’s field. This difference could be due to the use of hyperdiffusivity in their model. We obtained similar spectra in a few other models, in particular those in which the dipole contribution is not overwhelming. When the dipole dominates very strongly, the spectral power distribution often includes a secondary maximum in the range of harmonics where the power in the flow field reaches its peak, i.e. at \( \ell > 5 \), the precise location depending on the Ekman number. This becomes apparent in the weak high-frequency modulation of the strong dipole field in Fig. 2(f), for example.

It is not clear whether the geodynamo falls into the same generic class of dynamos that we describe here, that is, dynamos that create a dipolar field by a dominant \( \tau^2 \)-mechanism resulting from helical columnar flow outside the inner core tangent cylinder. The similarity in field morphology between these dynamo models and the geomagnetic field on the core–mantle boundary is suggestive enough to warrant further modelling efforts and comparisons. Based on our results, it appears that numerical studies at comparatively moderate values of the control parameters can contribute substantially to the general understanding of the geodynamo. Amongst the shortcomings of the models presented here, perhaps the biggest is the lack of reversals in the strongly dipolar...
dynamos. A possible cause is that convection inside the tangent cylinder is too weakly developed in this parameter range. Also, ingredients that have not been included in our models, such as inner core conductivity and rotation (Hollerbach & Jones 1995) and non-homogeneous boundary conditions (Glatzmaier et al. 1999) are perhaps essential for the understanding of time-dependent aspects of the geomagnetic field.

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Figure 13. Power spectrum of the magnetic field as a function of harmonic degree ℓ for the Earth’s field 1980 at the core–mantle boundary (circles) and at the surface of model 300/4 at ℰ=10^{-3} (crosses), scaled to the Earth assuming ℏ=1, Ω=7.27×10^{-5} and μ=11 000 (SI units).