Asymptotic solutions to Cagniard’s problem

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SUMMARY

When considering the seismic response in Cagniard’s problem, where a plane interface separates homogeneous, isotropic media, high-frequency asymptotic representations are known to break down at critical angles, where head waves and reflected waves interfere. Formulae have been derived to correct this, to be used in conjunction with more standard asymptotic expressions. We present formulae that are more generally applicable, as they account for the contribution of leaky waves, which can be asymptotically significant. The importance of leaky waves is shown to occur for strong contrasts in velocity across the interface. We therefore arrive at a series of approximations, based on a ray approach, that can be used to model a single interface or a system of homogeneous layers in an efficient manner.

Key words: ray theory, seismic modelling, seismic reflection.

INTRODUCTION

The seismic response of a single plane interface to a point source has been extensively studied, with analyses being carried out using both Fourier and Laplace transforms. The former usually result in high-frequency asymptotic approximations, while the latter can be used to find the exact time-domain response (see Aki & Richards 1980 for a summary of techniques). Standard asymptotic approximations have a rather limited applicability, as they are invalid for some angles of incidence (close to critical angles). Červený & Ravindra (1971) and Brekhovskikh (1980) derived similar high-frequency solutions that are valid near critical angles, for elastic and acoustic waves respectively. Brekhovskikh & Godin (1992) derived more general acoustic results that included the presence of leaky waves. Smirnova (1966) has also studied this problem. These formulae prove to be inaccurate for a large suite of geological models. Previous derivations have neglected the presence of leaky pole singularities close to branch points, and we give formulae for more generally valid asymptotic expansions. The latter combined with standard asymptotic formulae are less costly to compute than a numerical integration of the exact formula. They also generalize to a (homogeneous) multilayered structure, when one desires a ray-based solution. The final result can be viewed as an alternative to the Cagniard–de Hoop method for computing seismograms for the simple model considered. In addition, the spectral analysis demonstrated here can be used to find asymptotic simplifications for more complicated problems that lack an exact solution. For example, ray parameter integrals that approximate body wave amplitudes in heterogeneous media can be found using Maslov’s method, and the integrands contain many of the same singularities we consider in our approximations. When curved interfaces and inhomogeneity are modelled, one can also use the more elaborate results of Thomson (1990), who has arrived at corrections to ray theory near critical angles for acoustic waves.

There has, in the past, been some resistance to the notion of lower Riemann sheet contributions to seismograms for the simple plane layer case (Gupta 1970). However, Chapman (1972) utilized the exact nature of the Cagniard–de Hoop method to demonstrate the role of the leaky pole contribution to seismograms for Lamb’s Problem. We shall also review the effects of leaky waves, including their influence upon the reflected P wave, using an asymptotic frequency domain approach.

THEORY

Using the approach of Červený & Ravindra (1971) (we shall henceforth refer to this book as C&R), we investigate the reflected P wave when the incident wave is spherically symmetric and in the far field. The interface is a welded contact between two homogeneous, isotropic media. The source and receivers are located at distances h and z above the interface, where we are using cylindrical coordinates and the z-axis is positive in the top medium. The incident wave is harmonic and the displacement is given by

\[
\begin{bmatrix}
  u_{1}^{\text{inc}} \\
  u_{2}^{\text{inc}}
\end{bmatrix} = \frac{e^{-j\omega t - iR/2}}{R} \left( 1 - \frac{z_{2}}{ioR} \right) \begin{bmatrix}
  \sin \theta_{1}^{\text{inc}} \\
  - \cos \theta_{1}^{\text{inc}}
\end{bmatrix},
\]

where \( R = \sqrt{r^{2} + (z-h)^{2}} \) and we use \( z_{1,2} \) and \( \beta_{1,2} \) to represent the compressional and shear velocities in the top and bottom media, respectively. The exact form of the reflected P wave is the familiar product of vertical and horizontal eigenfunctions, satisfying the boundary conditions at the interface. We only describe the vertical component of the reflected P wave, as
a similar analysis can be carried out for the horizontal component and other $S$-wave terms. After some manipulation and expanding for large $|\omega np|$ (this far-offset approximation is reasonable since we shall be investigating near-critical angles—see C&R for details), we find

$$u_z^{\text{ref}}(r, z, \omega) \sim z_1 e^{-i\omega + i\pi / 4} \left( \frac{\omega}{2\pi r} \right)^{1/2} \int_{C_0} \rho^{1/2} R(p) e^{i\omega p} dp,$$

where $C_0$ is the appropriate contour in the complex ray parameter plane (Fig. 1), $\omega > 0$, and $\phi(p) = \text{sgn}(p_c + (h + z)v_1)$. Results for $\omega < 0$ can be found by taking the complex conjugate of positive frequency solutions. $R(p)$ is the reflection coefficient generalized to complex values, for which exact expressions are given in C&R. We point out that it contains the four radicals $\sqrt{p_1 - p_2}$, $\sqrt{p_1 - p_3}$, $\sqrt{p_1 - p_4}$, and $\sqrt{p_0 - p_1}$, so long as one is careful to avoid points where $\text{sgn}(p_c + (h + z)v_1) = 0$. (We relax this slightly to avoid the singularity at the saddle point at which $p = 1/\beta_1$.)

The initial contour, $C_0$, must lie on the top Riemann sheet $z_1 > 0$ so that the reflected wave is travelling away from the boundary. In addition, $C_0$ must lie in the quadrants for which $S_n(\{v_1\}) \geq 0$ to ensure boundedness of the integrand for $|z| \to \infty$. (We relax this slightly to avoid the singularity at the origin arising from our far-offset approximation.) This contour can be deformed to the equivalent steepest descents contour, $C_s$, so long as one is careful to avoid points where $R(p)$ is not analytic. This may include excursions into other quadrants and Riemann sheets, and will not violate the boundedness condition, as we shall see.

Dominant points in the ray parameter plane

The main observable seismic phases arise from an asymptotic analysis about dominant points in the integral in eq. (2). There is one saddle point at which $\phi(p_c) = 0$, $0 < p_c < 1/\beta_1$, producing a reflected wave. The branch points at $p = 1/\beta_2$ and $p = 1/\beta_3$ cause regular head waves, whereas the branch point at $p = 1/\beta_1$ gives rise to an inhomogeneous head wave.

Depending upon the material parameters, there may exist top sheet poles on the real axis past $p = 1/\beta_1$, and these represent Stoneley interface waves. Finally, there is always a variety of lower sheet (leaky) poles present (Gilbert & Laster 1962). In this paper we shall not consider inhomogeneous head waves or Stoneley waves.

The steepest descents contour is shown in Fig. 2 for a post-critical angle of incidence. Since the saddle point is now located on a branch cut, the contour must continue onto the lower sheet $(-+++)$. The contour must also loop around the branch point at $p = 1/\beta_2$, and this loop has one leg on the $(+++)\text{ sheet}$ and the other on the $(-+++)\text{ sheet}$. Also shown is a possible leaky pole location $(p_1\text{)}$ which forces a residue to be taken, if the contour is to be deformed continuously from $C_0$. A second and similar loop contour would be drawn for $p_0 > 1/\beta_2$. We assume that $p_0$, $p_L$ and $1/\beta_2$ are sufficiently ‘far apart’ so that each contribution may be considered independently. Later we shall make precise the meaning of ‘far apart’. The terms arising from the branch point and saddle point integrations have been well documented, and thus we only examine that due to the leaky pole,

$$u_z^{\text{leaky}}(r, z, \omega) \sim z_1 e^{-i\omega + i\pi / 4} \left( \frac{2\pi \omega}{\rho} \right)^{1/2} p_L^{1/2} R_L,$$

with

$$R_L = \text{Res}_{p=p_L \{R(p)\}}.$$

The condition for which the residue needs to be evaluated is derived from the fact that $S_n(\phi(p))$ is constant along a steepest descents contour. Hence, the residue is included if

$$S_n(\phi(1/\beta_2)) < S_n(\phi(p_L)) < S_n(\phi(p_0)),$$

with $S_n(\phi(p_0)) \geq 0$.

$$S_n(\phi(1/\beta_2)) = \tau$, the reflected wave traveltime, and $S_n(\phi(1/\beta_3)) = \tau$, the head wave traveltime, we see that the leaky wave arrives between the head wave and the reflected wave on the seismogram. Given that the residue was added when the pole was crossed by the steepest descents contour and remains in a descent region after the contour has passed, it

Figure 1. The contours of integration in the complex ray parameter plane for $\omega > 0$. $C_0$ and $C_s$ are the initial and steepest descents contours, respectively.

Figure 2. The steepest descents contour for a post-critical angle of reflection, which must loop about the branch point at $p = 1/\beta_2$. Sample leaky poles from the $(-+++)\text{ sheet}$ are shown, one of which requires a residue.
follows that \( \Re\{\phi_p(z)\} < 0 \). Thus, \( w^{\text{leaky}} \) is itself negligible compared to the leading asymptotic term, since it is exponentially small as \( \omega \to \infty \). However, as we shall see in the next section, the leaky pulse produces an asymptotically significant contribution to the seismogram when it is located sufficiently close to the branch point or saddle point. The two general characteristics of the leaky pulse mentioned above are identical in the work of Chapman (1972) and Gilbert & Laster (1962), who did not use a frequency-domain method to compute seismograms. They found the leaky pulse contribution to be a broad and low-amplitude pulse arriving between the reflected and head waves.

Since the pole lies in the first quadrant [on the sheet \((-+++)\)], it is clear that \( \Re\{\sqrt{1/z_1^2-p_L^2}\} < 0 \), making the real coefficient of \( z \) positive in the exponent. From a plane wave perspective, this leads to an apparent violation of the boundedness condition for the integrand for \( z \) large. However, for the spherical wave we see that \( z \to \infty \Rightarrow p_0 \to 0 \), removing the residue. One would expect an initial increase followed by a decrease in amplitude of the leaky pulse when moving away from the interface, as has been touched upon by Brekhovskikh (1980). The real coefficient of \( r \) in the exponent is negative, making the leaky pulse decay exponentially with epicentral distance. We therefore conclude that the leaky pole will have maximum influence upon the seismogram when the pole has just crossed by the steepest descents contour, which corresponds in the time domain to the leaky pulse interfering with (emerging from) the front of the reflected wave. Poles located to the left of the portion of the steepest descents contour looping the branch point will not contribute a residue. These poles’ greatest influence will be exerted upon the interfering head and reflected waves near (and just past) the critical angle.

**ASYMPTOTIC APPROXIMATIONS**

We identify several cases in which two or three dominant points are sufficiently close in the complex ray parameter plane that standard asymptotic formulae become invalid. In each case we provide an asymptotic expansion, accurate when the dominant points are within the boundary layer, the limits of which will be examined in the next section. While we follow the general methodology of Bleistein & Handelsman (1986), we do not produce uniform expansions for two reasons: first, the uniform expansion leads to rather complicated coefficients; second, it is preferable to use a standard asymptotic formula outside the boundary layer for computational efficiency, where the non-uniform approximations are no longer valid. We refer to Appendix A for some details of the derivation. Below we use the symbol \( p_b \) to refer to the branch point of interest.

**Case 1: a simple pole, saddle point and branch point in proximity**

\[
\tilde{\mu}_1 \approx \zeta_1 e^{-\omega t(\cdot - \gamma)} \left\{ \begin{array}{l}
\left. a_0 \left( \frac{\omega}{7} \right)^{1/2} e^{i \omega/4 - \omega^2/4} D_{-1}(y) + \frac{d_1}{\sqrt{r^2-p_b^2}} \right|_{r=p_b} \\
\quad + b_0 \frac{e^{i \omega/2} w_{-1/2}(z, w)}{(r p_b)^{1/2}} \right. \\
\quad + b_1 \frac{e^{i \omega/2} w_{1/2}(z, w)}{(r p_b)^{1/2}} \right. \\
\quad + b_2 \frac{e^{-i \omega/2} w_{3/2}(z, w)}{(r p_b)^{1/2}} \right. \\
\left. + D_{1/2}(z) \right. \right\},
\]

\( (6) \)

Above, \( D_p \) for \( \mu = 1/2, 3/2 \) are parabolic cylinder functions (see Magnus & Oberhettinger 1954), whereas \( \tilde{\mu} \) is not identifiable as a standard special function. It has the integral definition

\[
\tilde{\mu}(z, w) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{1-x^2} e^{-x^2/2 - it} dt, \quad (7)
\]

where the contour of integration passes above the singularity at \( t = w \) as well as the branch point at the origin, and \( \Re\{\sqrt{t}\} > 0 \) for \( t \in \Re^+ \). The variables in eq. (6) are

\[
\begin{align*}
\rho &= |\omega^{(p_0)}|^{1/2}, \\
\tau &= e^{i \omega/4 \sqrt{p_b}} p_b - p_b, \\
\eta &= e^{i \omega/4 \sqrt{p_b}} (p_b - p_L), \\
\zeta &= e^{i \omega/4 \sqrt{p_b}} (p_b - p_b) - (p_b - p_b),
\end{align*}
\]

The coefficients are determined by the factorization used to find the asymptotic series. In this case

\[
\sqrt{\rho}(p) = A(p) + B(p)\sqrt{p_b - p}, \quad (8)
\]

where both \( A(p) \) and \( B(p) \) do not contain the term \( \sqrt{p_b - p} \). The locally smooth functions, which were approximated by polynomials (see Appendix A), are \( A(p)(p - p_b) \) and \( B(p)(p - p_b) \), and the polynomials are exact at \( p = p_b, p = p_b \) and \( p = p_b \). The coefficients are

\[
\begin{align*}
\begin{align*}
\alpha_0 &= \text{Res}_{p=p_b} \{A(p)\}, \\
\beta_0 &= \text{Res}_{p=p_b} \{B(p)\}, \\
\alpha_1 &= A(p_b) - \frac{\alpha_0}{p_b - p_b}, \\
\beta_1 &= B(p_b) - \frac{\alpha_0}{p_b - p_b}, \\
\beta_2 &= B(p_b) - \frac{\alpha_0}{p_b - p_b}, \\
\gamma_1 &= \frac{\alpha_0}{p_b - p_b} (p_b - p_b) - (p_b - p_b), \quad (9)
\end{align*}
\]

and are continuous in the limit \( p_b \to p_b \). Eq. (6) expresses the interference between the reflected wave, a head wave and a leaky wave. It can be further augmented in a similar manner to incorporate a second distinct pole location; this situation arises often close to the smallest branch point, where two complex conjugate poles are nearby, without being separated by a branch cut. We note that \( B(p_b) = -\sqrt{21} g_{1,3,4,4,4, k = 3, 4, 4} \) and \( g_{1,3,4,4,4,4} \) are the well-known head wave amplitude coefficients (C&R).

**Case 2: a saddle point and branch point in proximity**

\[
\begin{align*}
\tilde{\mu}_2 \approx \zeta_1 e^{-\omega t(\cdot - \gamma)} \left\{ \begin{array}{l}
\left. a_0 \left( \frac{\omega}{7} \right)^{1/2} e^{i \omega/4 - \omega^2/4} D_{-1}(y) + \frac{d_1}{\sqrt{r^2-p_b^2}} \right|_{r=p_b} \\
\quad + b_0 \frac{e^{i \omega/2} w_{-1/2}(z, w)}{(r p_b)^{1/2}} \right. \\
\quad + b_1 \frac{e^{i \omega/2} w_{1/2}(z, w)}{(r p_b)^{1/2}} \right. \\
\quad + b_2 \frac{e^{-i \omega/2} w_{3/2}(z, w)}{(r p_b)^{1/2}} \right. \\
\left. + D_{1/2}(z) \right. \right\}
\end{align*}
\]

\( (10) \)

and all symbols are identical to Case 1. We approximate \( A \) and \( B \) directly with polynomials that are exact at \( p = p_b \) and \( p = p_b \).

Note that this is a variation of the result given by C&R, describing the interference between a reflected and a head wave. In their formula it is assumed that

\[
b_2 \approx \frac{dB}{dp} \bigg|_{p_b} \approx 0,
\]

leaving only the \( D_{1/2} \) term.
Case 3: a branch point and simple pole in proximity

This case describes the loop contour around the branch point for post-critical angles.

\[ a_{zd} \sim z_1 e^{\frac{1}{2} \frac{\phi'(p)}{p}} \left( 1 - \frac{1}{2} \frac{\phi''(p)}{p} + \frac{1}{6} \frac{\phi'''(p)}{p^2} + \cdots \right) \]

\[ \times \left\{ (b_2 - 2b_3 \sqrt{2} z_3^2 (1 - y_3 e^{\frac{1}{4} D \cdot (y_3)}) \right\} \]

+ saddle point contribution,

where several new variables need to be defined:

\[ p_3 = \phi'(p_0)^{1/2}, \quad w_3 = e^{\frac{1}{4} \phi(p_0 - p_L)} \]

\[ z_3 = e^{\frac{1}{4} \phi} \psi_3, \quad y_3 = \sqrt{-2w_3 z_3}. \]

Eq. (11) expresses the interference between a head wave and a leaky wave. The coefficients are the same as defined for Case 1, but there is no contribution from \( A \) since the contour loops the branch point (and \( A \) does not have a branch point at \( p = p_0 \)).

For this case a polynomial approximates \((p - p_0)B(p)\), and is exact at the points \( p = p_0, p_1 \) and \( p = p_1 \). Implicit in the derivation of the above equation was the assumption that \( |\arg y_3| < \frac{3\pi}{4} \), which serves to determine the sign of \( \sqrt{-2w_3 z_3} \). The saddle point contribution is not specified because it can take more than one form. Above, we have only specified that a pole is close to a branch point, and the saddle point may be described by a standard asymptotic formula or another one of the case formulae, depending upon its position in the complex ray parameter plane.

Case 4: a saddle point and simple pole in proximity

The interference between the reflected wave and leaky wave is given by

\[ a_{zd} \sim z_1 e^{-\frac{1}{2} \left( \text{Re} \left\{ \frac{1}{\sqrt{\phi} (p_R - p_0 - p_L) (p_R - p_0) \} \right\} \right)} \]

\[ + p_1^{1/2} R_L \left( \frac{\phi'(p)}{p} \right) e^{\frac{1}{4} \phi^2 + \frac{3\pi}{4} D \cdot (iy)} \]

+ branch point contribution.

Again we do not give the precise branch point contribution, as we have made no assumption about the location of poles relative to branch points in deriving eq. (12). The function we approximated by a polynomial was \( \sqrt{\phi} R(p)(p - p_L) \), and this was made exact at \( p = p_0, p = p_0 \).

More case formulae could be found, depending upon the medium under study. The distance between branch points is a function of \( \beta_2 / \beta_2 \) and \( x_1 / x_2 \), and it is conceivable that two branch points (as well as a saddle point and poles) could interfere. Under Case 1 we mentioned that the formula can be extended to accommodate more than one distinct (simple) pole. If we are to model rays in a multilayered structure, multiple reflections from an interface near critical angles would give rise to poles in the integrand of order greater than unity; the methods used to derive these formulae would still apply (Appendix A); however, the result would require new functions to be found, in addition to \( f \).

It is important to see that in using eq. (8) we have isolated the radial of interest, \( \sqrt{\phi} - p \), by making \( A(p) \) and \( b(p) \) functions of \( (p_0 - p) \). An individual asymptotic analysis is carried out on each term. This division of \( \sqrt{\phi} R(p) \) brings lower sheet poles to the top sheet in both \( A \) and \( B \), forcing residues to be taken separately as the steepest descents contour crosses them.

(Before bringing poles to the top sheet only a small subset required residues—see eq. 5). If a pole is situated such that no residue is warranted from the perspective of \( \sqrt{\phi} R(p) \), it should cancel upon summation of the asymptotic terms from \( A \) and \( B \). Cancellation errors can be catastrophic when carrying our the numerical computations, depending upon the location of the poles. For \( f \text{am} \{p_L \} < 0 \), the residues become exponentially large, and due to the finite precision of floating point number representation, cancellation is imperfect and the other terms in the formula are swamped. It is best to calculate both \( f \) and \( D_{-1} \) without any residues (so they are discontinuous functions) and as a final step in computation add in required residues according to eq. (5). The following identity is useful:

\[ D_{-1}(-y) = -D_{-1}(y) + \sqrt{2\pi} e^{\pi/4} \]

which we can use to find \( D_{-1} \) for a restricted range of argument, such that no residue is taken. See Magnus & Oberhettinger (1954) for a summary of the properties of parabolic cylinder functions.

**Boundary Layers**

The case formulae in eqs 6, 8 and 10 depend upon variables \( w, y \) and \( z \), of which only two are independent. These variables are, in turn, functions of the separation of the points \( p_0, p_0, p_1 \) in the complex ray parameter plane, as well as frequency and \( \phi(p_0) \), which is proportional to the radius of curvature of the wave front in the plane of incidence. For large values of the variables, the case formulae may be replaced with simpler expressions that match the standard asymptotic expansions. Thus, for a given geological model and signal spectrum, only a subset of the receivers and signal frequencies require the use of the case formulae, and we shall refer to these subsets as boundary layers.

For practical computation it is desirable to have a numerical estimate of the size of the boundary layers. As a rough guide we mark the boundary as the point at which the special function, of a given case formula, can be replaced by the leading term of its asymptotic expansion, such that the difference does not exceed some maximum percentage of the original function. This, of course, is only an estimate as it does not bound the error in the overall formula, which clearly depends upon other factors.

Starting with Case 2, we look at replacing \( D_{1/2}(iz) \) and \( D_{1/2}(z) \) for large \( |z| \) by their leading terms, with one exception: for \( \text{arg} z = \pi/4 \), we include the higher-order term representing the head wave. Fig. 3 shows that we expect a maximum error of about 10 per cent if we make the boundary \( |z| = 2 \). Case 4 depends upon \( e^{\pi/4} D_{-1}(y) \), which can be replaced by its leading term for \( |y| \geq 4 \), with an error of between 5 and 10 per cent, depending upon \( \text{arg} z \). The error boundary is plotted in Fig. 4, where we only need consider \( \text{Re} \{y\} > 0 \) due to eq. (13), and \( f \text{am} \{y\} > 0 \) due to the fact that \( e^{\pi/4} D_{-1}(-y) = e^{\pi/4} D_{-1}(y) \) (the bar indicates complex conjugate). Case 3 has the same dependence on \( D_{-1}(y) \), although it must be approximated by
the leading two terms to provide the proper head wave contribution. This means that the boundary for Case 4 would overestimate the size of this layer (see Fig. 4). We next make the assumption that the above boundaries in general reflect the limits of pole–saddle point, branch point–saddle point and branch point–pole influence. Using the above assumption (as well as $w_1 \approx w_2$), we defined the following tentative regions to represent the boundary layers:

$$C_1 = \{ |z| \leq 2 \} \cap \{ |y| \leq 4 \} \cup \{ |w| \leq 4 \},$$

$$C_2 = \{ |z| \leq 2 \} \cap \{ |y| > 4 \} \cup \{ |w| > 4 \},$$

$$C_3 = \{ e^{-iz/4} z > 2 \} \cap \{ \sqrt{2wz} \leq 4 \},$$

$$C_4 = \{ e^{-iz/4} z > 2 \} \cap \{ |y| \leq 4 \},$$

where the $C_k$ are the regions for the $k$th case formula. The domain of standard asymptotic expansions is the complement of the set $C_1 \cup C_2 \cup C_3 \cup C_4$. These boundary layers generally form regions surrounding the critically reflected ray in terms of depth and offset. We shall show some examples of this in the Numerical Results section.

**Approximation of the exponent**

Our case formulae have made use of the approximation (Appendix A)

$$\phi(p) \approx \phi(p_0) + \phi'(p_0)(p - p_0)^{3/2},$$

which is accurate only for the region surrounding the saddle point, when $\phi$ is analytic. The function $\phi$ contains a branch point at $p = 1/z_1$, and the saddle point approaches it for large offsets (that is, grazing angles of incidence). This limits the applicability of the formulae presented to cases where the branch points of interest [in $R(p)$] are not close to the branch point in $\phi(p)$. Thus, for the reflected $P$ wave presented, the formulae would break down for small contrasts in velocity.
across the interface \( \alpha_1/\alpha_2 \rightarrow 1 \), and also for \( \beta_2/\alpha_1 \rightarrow 1 \). To deal with these cases one must employ a more general transformation of the exponent (Bleistein & Handelsman 1986).

**LOCATION OF POLES**

It is useful to gain a qualitative idea of the movement of the leaky poles for various geological models. The location of leaky poles in the complex ray parameter plane is a function of the material properties of the two media, \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) and \( \rho_1/\rho_2 \). If we introduce a dimensionless variable \( \nu \), where \( \nu \) is one of the four wave velocities, then the poles vary only as functions of \( \alpha_1/\alpha_2, \sigma_1, \sigma_2 \) and \( \rho_1/\rho_2 \), where \( \sigma \) is Poisson’s ratio. The velocity \( \nu \) can be chosen to normalize the pole location relative to the branch point of interest. We therefore use the complex \( \rho_2 \) plane when mapping the leaky poles from the \((- +/- +)\) sheet. Pole locations are shown only for the first quadrant, keeping in mind that corresponding complex conjugate poles are to be found in the fourth quadrant. The latter poles’ existence follows from the fact that \( R(p) \) coincidentally obeys the ‘reflection principal’ on a given Riemann sheet, from the theory of complex variables. The poles have not been represented when they lie on the real axis, where they are generally isolated.

We have contoured pole locations by varying \( \nu = \alpha_1/\alpha_2 \) and \( \sigma_1 \), while holding \( \sigma_2 = 0.315 \) constant at a value typical for limestones (Fig. 5). The density ratio is not held constant, but varies according to \( \rho_1 = 0.315 \rho_2^{1/4} \) (Sheriff & Geldart 1983) with \( \rho_2 = 2.5 \text{ g cm}^{-3} \). In this way we can change \( \nu \) to model varying velocity contrasts and \( \sigma_1 \) to reflect different top media lithologies whilst keeping the densities at realistic values. Fig. 6

![Figure 5](https://example.com/figure5.png)

**Figure 5.** The location of poles on the \((- +/- +)\) sheet plotted as a function of \( \nu = \alpha_1/\alpha_2 \) and \( \sigma \), the latter being Poisson’s ratio in the upper medium. Other parameter values are described in the text. The contour interval for \( \nu \) is 0.1, unless otherwise labelled. The contour interval for \( \sigma \) is 0.05, unless otherwise labelled.

![Figure 6](https://example.com/figure6.png)

**Figure 6.** The location of poles on the \((- +/- +)\) sheet plotted as a function of \( \nu = \alpha_1/\alpha_2 \) and \( \sigma \), the latter being Poisson’s ratio in the lower medium. Other parameter values are described in the text. The contour intervals for \( \nu \) and \( \sigma \) are 0.1 and 0.05, respectively.
maps the pole locations as a function of $n$ and $\sigma_2$, whilst keeping $\sigma_1 = 0.25$ constant.

Both figures show that the poles move closer to the branch point $p = 1/\sigma_2$ for lower $n$. Therefore, the need for Cases 1, 3 and 4 arises from a combination of small $|\cos'(p_0)|$ and large velocity contrasts across the interface. This is in agreement with the observations of C&R, who found that their formulae (Case 2) were not applicable for strong contrasts in velocity, and recommended their use only for $0.75 \leq n \leq 0.95$. Changing Poisson’s ratio tends to rotate the poles about the branch points. Increasing $\sigma_1$ moves the poles to the left of the branch cut, whilst the opposite holds true for changes in $\sigma_2$. Thus, for low $\sigma_1$ (high $\sigma_2$) it is more likely that a residue will need to be evaluated, whereas for high $\sigma_1$ (low $\sigma_2$) the pole will probably avoid the steepest descents contour. However, as we shall see, poles that do not require a residue may still have a significant influence upon the reflected wave.

**NUMERICAL RESULTS**

We apply our results to a system of homogeneous layers, taking the parameters from Ogilvie & Purnell (1996). This model is particularly suitable as it consists of a salt bed beneath layers of sediment, providing both high- and low-velocity contrasts across interfaces. Table 1 summarizes the model. We assume the source presented in the Theory section, and only model primary $P$-wave reflections. The receivers are located in the first medium, and the free surface effects are ignored. We model amplitude versus offset curves, and compare the results from using the case formulae to a numerical integration. Synthetic seismograms would simply confirm the results of the harmonic amplitude curves, without offering much additional insight. This is due to the fact (noted in the Theory section) that a leaky wave at high frequencies merely affects the amplitudes of the reflected and head waves near critical offsets, without producing a visible pulse distinct from the geometrical arrivals.

The amplitudes of reflected waves using standard asymptotic formulae (that is, plane wave reflection coefficients) are easily evaluated and not shown; we merely note that they differ greatly from the numerically calculated solutions near critical angles. The numerical integration is carried out using an integrand similar to eq. (2), but for pressure, since the first medium is water. We integrate along a complex contour in the ray parameter plane, and do not include contours that represent the inhomogeneous head wave and regular interface wave contributions.

The contrast in velocities across the first interface is not great, which suggests that the leaky poles are not important for an asymptotic description of the reflected wave. For the most part, only Case 2 is needed according to the boundary layers described earlier, and the result is plotted in Fig. 7. The oscillatory nature of the curves arises from the interference between the reflected and head waves for post-critical angles, keeping in mind that the source is monochromatic. We see a slight jump in amplitude at the edge of the Case 2 boundary layer, and this is a result of using a non-uniform expansion. That is, as the saddle point moves away from the branch point, amplitude discrepancies increase, and reach a maximum at the edge of the boundary layer. This accumulation of error is then abruptly corrected as we pass back into using the standard asymptotic formulae. The contrast in velocities across the third interface is large, and one expects the leaky pole influence to be greater. Fig. 8 compares the results for this interface, and shows where each case formula has been used. The match is imperfect at offsets of 2.4–2.6 km. This is actually where we pass from Case 1 to Cases 3 and 4. A better match can be found by extending the boundary between Case 1 and Cases 3 and 4 to

<table>
<thead>
<tr>
<th>Material</th>
<th>$\alpha$ (m s$^{-1}$)</th>
<th>$\beta$ (m s$^{-1}$)</th>
<th>$\rho$ (g cm$^{-3}$)</th>
<th>Thickness (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>1500</td>
<td>0</td>
<td>1.01</td>
<td>1036</td>
</tr>
<tr>
<td>Sediment</td>
<td>2040</td>
<td>772</td>
<td>2.05</td>
<td>464</td>
</tr>
<tr>
<td>Sediment</td>
<td>2106</td>
<td>850</td>
<td>2.10</td>
<td>542</td>
</tr>
<tr>
<td>Salt</td>
<td>4481</td>
<td>2530</td>
<td>2.14</td>
<td>884</td>
</tr>
</tbody>
</table>

![Figure 7](https://academic.oup.com/gji/article-abstract/138/3/820/578872)  
**Figure 7.** Comparison of numerical and asymptotic results for the $P$-wave reflection from the bottom of layer 1. The range over which the case formulae are used is shaded. Where no labels (or shades) exist, standard leading-order asymptotic expressions are used, except for post-critical angles, where first-order head wave terms are included. The triangle marks the critical offset, corresponding to the angle at which the transmitted $P$ wave is critically refracted. These conventions are adhered to in Figs 7–11.
$|z| = 3$ for post-critical reflection. Poisson’s ratio for layers 3 and 4 is 0.40 and 0.266 respectively. This is high in layer 3, and as a consequence the leaky poles are situated to the left of the steepest descents contours, so no residues are evaluated in the asymptotic calculation of the wave amplitudes. However, we see they have a significant influence when we compare Figs 8 and 9, the latter using only Case 2 to find amplitudes. It is easy to see that within the Case 1 region, Case 2 provides completely inaccurate results, whilst in the region where Cases 3 and 4 apply, we see that standard asymptotic expansions are also not adequate. In Fig. 10 we look at the effect of omitting poles below the real axis, hypothesizing that they might have a marginal influence, especially after adjusting the original model (poles have moved more to the right in the complex ray parameter plane). We see that this has no effect near the second critical offset, whilst it invokes a significant change near the first critical offset. The former is no surprise, for the pole below the real axis on the $(-+++) \text{ sheet}$ is separated from the steepest descents contour by the branch cut originating at $p = 1/\alpha_2$, and is therefore distant in terms of a continuous Riemann surface. In contrast, the pole below the real axis on the $(-+++) \text{ sheet}$ is relatively close to the steepest descents contour (see Fig. 2). In order to provide an example where a residue needs to be evaluated, we change the model by increasing the value of Poisson’s ratio in the fourth layer to 0.35. Fig. 11 shows the numerical and asymptotic results, and indicates at what offset a residue is first needed from the $(-+++) \text{ sheet}$. We see a good agreement, although it decreases somewhat again in the transition from Cases 1 to 3 and 4. The transition seems to occur where the residue is first taken, although this is merely coincidental; one can show this for higher frequencies, where the boundaries (since they are frequency-dependent) shift inwards and the residue location remains the same.

Figure 8. Comparison of numerical and asymptotic results for the $P$-wave reflection from the bottom of layer 3. Note the second critical offset, corresponding to the angle at which the transmitted (and converted) $S$ wave is critically refracted.

Figure 9. Comparison of numerical and asymptotic results for the $P$-wave reflection from the base of layer 3. Here the asymptotic label implies the use of the Case 2 formula only, in conjunction with standard asymptotic expressions.

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Using this final choice of parameters it is useful to look at the spatial distribution of the boundary layers. For the 20 Hz reflected wave, the layers form a fairly depth-invariant pattern surrounding the critically reflected ray (Fig. 12). At the much higher frequency of 80 Hz, some dynamic behaviour becomes visible (Fig. 13). The overall width of the layers has decreased, and the Case 1 region passes to Case 2 near the water/sediment interface, indicating that the influence of the leaky wave eventually diminishes away from the interface (from which it arises) at high frequencies. This contrasts with the head wave influence, which continues right up to the surface (Case 2). The latter trait can be seen from the definition of the parameters $z$, $y$ and $w$; regardless of the size of $\sqrt{\omega p_0}$, there will always be some region for which $\sqrt{\omega p_0 (p_0 - p_L)} < 2$. This does not hold true for $y$ and $w$, since generally $p_0 \neq p_L$.

**CONCLUSIONS**

We have presented formulae that accurately describe seismic wave amplitudes for reflected waves in the vicinity of critical angles using a high-frequency asymptotic ray-based approach. They apply to media approximated by a system of plane, homogeneous, isotropic layers. These formulae improve upon previous results because they account for the asymptotically significant contribution of leaky waves that arises when modelling interfaces with strong velocity contrasts. A set of bounds for the application of each formula has been proposed, and these have proven adequate for the models studied thus far. Calculating the reflected wave by using these formulae in conjunction with a standard ray approach leads to a computationally effective way for finding the seismic wave response.
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Asymptotic solutions to Cagniard’s problem


APPENDIX A: DETAILS OF ASYMPTOTIC EXPANSIONS

We assume an integral of the form

\[ I = \int_{C_s} f(p) e^{i\phi(p)} dp, \]  \hspace{1cm} (A1)

where \( C_s \) is the steepest descents contour through the saddle point, \( p_0 \). Since \( f \) is frequency-independent and \( \phi \) contains a saddle point, this method is restricted to finding ray-based solutions. We truncate \( C_s \), keeping only the portion in the vicinity of the saddle point, \( C_s \). This introduces asymptotically negligible errors, and we can now approximate \( \phi(p) \) in a Taylor expansion (to second order) about the saddle, and we call this \( \psi(p) \). We label the region surrounding the saddle point for which this expansion is valid \( D \). Thus,

\[ I \sim \int_{C_s} f(p) e^{i\psi(p)} dp. \]  \hspace{1cm} (A2)

We point out that this differs from the method of Bleistein & Handelsman (1986), Chapter 9, who transform \( \phi(p) \) to a canonical exponent instead of approximating it; our result applies only when critical points in \( f(p) \) lie close to the saddle point. As the saddle point moves past the critical point a contour must necessarily loop the latter, and the above approximation of the exponent shall introduce errors. Therefore, it is important to use more than one asymptotic expansion depending upon the saddle point’s proximity to the critical point (that is, this is a non-uniform approximation).

We next identify a set of critical points of \( f(p) \), namely \( \{p_k\} \) \( k = 1, 2, \ldots, N \), which are assumed to be ‘close’ in the complex \( p \) plane. By this we mean that the sums of the first terms of the standard asymptotic expansions calculated for each point independently are inaccurate. The function \( f(p) \) can be written as a product of functions, one of which is smooth in the region \( D \). Specifically,

\[ f(p) = C(p; p_k) S(p), \]  \hspace{1cm} (A3)

where \( S(p) \) is the smooth function in \( D \). As an example of this, the reflection coefficient in the vicinity of the branch point \( 1/2z \) could be expressed as

\[ R(p) = A(p) + B(p) \sqrt{1/2z - p}, \]

where \( A \) and \( B \) are smooth in some region surrounding \( p = 1/2z \). The same can be done in the vicinity of a pole, or a...
combination of poles and branch points. The next step is to expand $S(p)$ using the well-known Lagrange interpolation formula, making the polynomial exact at the $N + 1$ points $\{p_k\}, \ k = 1, \ldots, N$, and the saddle $p_0$. Thus

$$S(p) = L_N(p) + e_N(p),$$

(A4)

where

$$L_N(p) = \sum_{k=0}^{N} l_k(p) S(p_k), \quad l_k = \prod_{j=0}^{N} \frac{p - p_j}{p_k - p_j}, \ j \neq k,$$

and $e_N$ is the remainder from an $N$th-order interpolating polynomial, having the form

$$e_N(p) = \eta_N(p) \prod_{j=0}^{N} (p - p_j).$$

The above formula requires that $p_k, \ k = 0, 1, \ldots, N$, all be distinct, which is not always the case (e.g. for $p_0 \rightarrow 1/2$). The coefficients of the powers of $p$ can be rearranged into forms which exist in the limit as critical points coalesce. This also puts the integrals in a form that is more easily recognizable in terms of special functions. If we write

$$\Phi_0 = \sum_{k=0}^{N} S(p_k) \int_{C_\epsilon} C(p; p_k) l_k(p) e^{i\phi(p)} \, dp,$$

(A5)

then an asymptotic sequence of functions, $\{\Phi_j\}$, may be found by repeating the above process, where the functions $\eta_N(p)$ are interpolated to find higher-order terms in the sequence. One would need to include the neglected part of $\phi(p)$ for the higher-order terms in the asymptotic sequence to have any relevance. Finally, it can be shown that for $|\omega\phi'/(\omega^2)}| \rightarrow \infty$,

$$I \sim \sum_{j=0}^{N} \Phi_j, \ \{\zeta_j\},$$

(A6)

where $\{\zeta_j\}$ is an appropriately chosen auxiliary asymptotic sequence. One can see that because $C(p; p_k)$ cannot be expanded in a series, $\Phi_0$ will consist of a sum of integrals with parameters depending upon the distance between the points $p_k, \ k = 0, \ldots, N$. These integrals may in some cases be related to known special functions; otherwise, they must be tabulated.