Connection of the Strong Coupling Theory with the Weak Coupling Theory in the Bound Meson Problem
--- The Symmetrical Scalar Theory ---

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§ I. Introduction

It is the purpose of this paper to find a unified stand-point from which the bound meson problem may be well investigated over all ranges of coupling constant. As is well known, the intermediate coupling theory is available in the intermediate region of coupling strength. But a result of that theory, e.g., the level of the ground state, is not yet represented in terms of analytical expressions of the coupling constant; and, in its current

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form, it seems hard to carry out the renormalization program. So it is desirable to fill gaps of such kinds, and one remarkable step has been made by Sawada\textsuperscript{2} to this direction, who has obtained an interesting result by means of a simple type of canonical transformation. We shall make an approach of similar kind and study a possibility to improve it.

We shall here treat the symmetrical scalar theory as a simple example of such problems that their solution is made complicated by virtue of spin variables. If we confine our attention to bound mesons, the Hamiltonian is given by\textsuperscript{3}

\[ H = H_0 + H' \]  

with \[ H_0 = \left( \mathbf{p}^2 + \mathbf{q}^2 \right) / 2 \] (the bound meson Hamiltonian),

and \[ H' = V(\tau \cdot \mathbf{q}) \] (the interaction Hamiltonian),

where \( \tau \) is the nucleonic isotopic spin, \( V \) is the effective coupling constant, and the oscillator's frequency is taken as unity; (as to the unit of energy, see reference \textsuperscript{1a}, especially p. 609); \( \mathbf{q} \) and \( \mathbf{p} \) denote position- and momentum-operator of the oscillator, respectively.

It is known that the level shift of the ground state is \(- (3/2) V^2\) in the weak limit, while it is \(- (1/2) V^2-1+O(1/V^2)\) in the strong limit.\textsuperscript{3} The numerical results are available at several points in the intermediate coupling region.\textsuperscript{4} Sawada's method gives results which are in good agreement with correct values in the region \( V < 1 \). But in the case \( V > 2 \), it is not successful; it predicts that the level shift in the strong limit is \(- (1/2) V^2 + 1\). Moreover, the sign of the isotopic spin-orbit coupling term is reversed when the coupling constant becomes larger than a certain value; for example, among the first excited states, the state \( I=1/2 \) is lower than the one \( I=3/2 \) in a strong coupling case. Nevertheless, it is instructive for us to investigate on what the conspicuous results of Sawada's method are based; hence we shall begin with re-examination of Sawada's result in next section.

By means of the analysis given in Sec. 2, we see how different the features of the system are in the strong and weak limits. These features are further analyzed by means of canonical transformations in Secs. 3 and 4. In Sec. 3, the transformation function

\[ U_R = \exp \{ i/2 \cdot \tan^{-1}[2V(\tau \cdot \mathbf{p})] \} \]  

will be introduced to treat the weak region of coupling constant. This transformation is of the type of half-arc tangent, which is often found useful in the treatment of spin variables. Actually the interpretation of the transformation (2) can be given by taking account of some rotation of the \( \tau \)-spin. Simultaneously we can verify that there exists some regional characteristics in the domain of \( \mathbf{p} \), which can be represented independently of \( V \) by taking \( 1/V \) as the unit of length of \( \mathbf{p} \). In Sec. 3, the transformation function

\[ U_T = \exp \{ iV(\tau \cdot \mathbf{p}) \} \]  

will be introduced to treat the most noticeable feature of the strong-coupling cases. The interpretation of the transformation (3) can be given in analogy to the Bloch-Nordsieck
transformation, if we take account of the fact that the angular modes of the system are apparently ineffective in the strong coupling case. But, through closer examination, this form of transformation function fails to reproduce finer details of the strong coupling theory.

In the course of analysis made in Secs. 2—4, qualitative aspects of our problems can be clarified. With these results in mind, the transformation function of general form

$$U = \exp\left\{i[\text{an odd function of } V(\tau \cdot p)]\right\} = \exp\left\{i\left(\frac{\tau \cdot p}{|p|}f(|p|)\right)\right\} \quad (4)$$

will be investigated in Sec. 5 to get some improvement to the Sawada’s results. In view of the unsatisfactory development of the operator calculus, one is forced to use a function of the form (4); but it will turn out that the improvement to be achieved is rather minor and our results are by no means quantitatively parallel to our qualitative prospects. In Sec. 6, it will be discussed what kind of calculation should become possible before we improve our results.

§ 2. On Sawada’s approximation

We shall re-examine Sawada’s method and try to analyze the background for the success of this method. In this method we use the transformation function

$$U_0 = \exp\{i\lambda(\tau \cdot p)\}, \quad (5)$$

where $\lambda$ is a variational parameter. Actually $\lambda$ can be determined by the condition

$$\langle \lambda - V \rangle + 2e^{-\lambda^2}[\lambda + (2\lambda^2 - 1)V] = 0. \quad (6)$$
We shall hereafter denote by $\lambda_8$ such $\lambda$ that satisfies this condition. This condition gives the lowest approximative level of the ground state, $E_0 + (3/2)$, where $3/2$ is the zero-point energy of meson oscillators and $E_0$ is defined by

$$E_0 = \langle U_8^{-1} H U_R \rangle_0 - 3/2 = (1 + (\lambda^2/2) - V\lambda) - (1 + 2V\lambda) e^{-V/2}. \quad (7)$$

($\langle \cdots \rangle_0$ means to take the vacuum expectation value with respect to meson oscillators.) Simultaneously, the matrix elements for the ground state to emit or absorb a single meson vanish by virtue of this condition. (Cf. Eq. (11) below.)

The relation between $\lambda_8$ and $V$ is illustrated in Fig. 1. The value of $E_0$ is plotted in Fig. 2 as a function of $V$. In the intermediate region of $V$, $\lambda_8$ and $E_0$ are not single-valued functions of $V$; it is clear on the physical grounds that such branch of $\lambda_8$ must be selected that gives the lowest $E_0$. Then a certain range of $\lambda_8$ is useless, since $\lambda_8$ jumps from the lower region into the upper one when $V$ becomes larger than a certain value*. This jump of parameter reminds us of a phase-transition of a condensing system, and it may be taken as an evidence that the nature of the problem is different in two regions. But we must be careful before we take this jump for something of real meaning from the physical view-point; we must examine thoroughly what is the physical meaning of Sawada's transformation in different regions of $\lambda_8$.

First we examine the situation in the weak-coupling region. It is instructive for us to compare the results for $\lambda = V$ with those obtained for $\lambda = \lambda_8$. In every respect it is better to use $\lambda_8$, when $\lambda_8$ is much smaller than $V$; this becomes clear if we analyze the transformed Hamiltonian in both cases.

The transformed Hamiltonian is given by

* This jump is shown by an arrow in Fig. 1.
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\[ U^{-1}HU = H_0 + H_{pot} + H_{sp-orb} + H'_{rad} + H'_{ang}, \]  
where

\[ H_{pot} = \frac{\lambda^2}{2} - \lambda V + \sin^2(\lambda |p|)/p^2 - V \cdot \sin(2\lambda |p|)/|p|, \]  
\[ H_{sp-orb} = \frac{V}{p} \cdot \sin(2\lambda |p|)/|p| - \sin^2(\lambda |p|)/p^2 \cdot (p \times q) \cdot \tau, \]  
\[ H'_{rad} = \frac{1}{2} \left( V - \lambda \right) \left[ (\tau \cdot p)/p^2 \cdot (p \cdot q) + \text{conj.} \right], \]  
\[ H'_{ang} = \frac{1}{3} V \cos(2\lambda |p|) - \frac{1}{4} \sin(2\lambda |p|)/|p| \left[ (\tau \cdot q) - \frac{(\tau \cdot p)}{p^2} (p \cdot q) \right] + \text{conj.} \]  

The ordering procedure of these terms can be performed according to the formula

\[ F(p) = \frac{1}{(2\pi)^3} \int d^3x d^3y \, e^{i(p \cdot x)} e^{-i(x \cdot y)} F(y) \]  
\[ = \sum_{m,n} \frac{1}{(2\pi)^3} \left( -1 \right)^m \frac{1}{m!} \frac{1}{n!} \left( \frac{1}{\sqrt{2}} \right)^{m+n} \int d^3x d^3y \, e^{-a/x} e^{-i(x \cdot y)} F(y) \, \left( \xi^* \cdot x \right)^m \left( \xi \cdot x \right)^n, \]  

here we introduce the creation- and annihilation-operators, \( \xi^* \) and \( \xi \), and put

\[ p = (\xi - \xi^*)/\sqrt{2} \quad i, \quad q = (\xi + \xi^*)/\sqrt{2}. \]

After some manipulations we have the ordered interaction terms,

\[ H'_{ang} + H'_{rad} = I_1 (\tau \cdot \xi^* + \xi \cdot \xi^*) + \ldots \]  
\[ + (\text{terms of other types, e.g. } I'_1 (\tau \cdot \xi^*) (\xi^* \cdot \xi), \text{ etc.)} \]  
\[ + \text{conj.}, \]  

where \( I \)'s are given by

\[ I_1 = \frac{\sqrt{2}}{3} \cdot \left( \frac{1}{2} (V - \lambda) + e^{-\lambda} \left[ (1 - 2\lambda^2) (V - \lambda) \right] \right), \]  
\[ I_1' = \frac{\sqrt{2}}{5} \cdot \left\{ (V - \lambda) + e^{-\lambda} \left[ \frac{3}{8} (V - \lambda) + \lambda^2 + 2V\lambda^2 (\lambda^2 - 3) \right] \right\}, \]  
\[ I_2 = \frac{\sqrt{2}}{15} \cdot \left\{ \frac{3}{8} (V - \lambda) - e^{-\lambda} \left[ \frac{3}{8} (V - \lambda) + \lambda^2 + 2V\lambda^2 (\lambda^2 - 3) \right] \right\}. \]

\[ I \)'s are plotted in Fig. 3 in both cases of \( \lambda = V \) and \( \lambda = \lambda_\beta \). The matrix element for the creation (or annihilation) of single meson vanishes when we put \( \lambda = \lambda_\beta \), as the result of the equation,
With respect to matrix elements for creation or annihilation of more than one mesons, the results with $\lambda = \lambda_s$ are much better than those with $\lambda = \lambda_s$ in the region $V \leq 1$.

If there were no involved effect caused by the $\tau$-spin, the problem must have been solved by means of the transformation function $U = \exp[iV(\tau \cdot p)]$ in analogy to the neutral scalar theory. As for the $\tau$-spin, it is only $\tau_3$ that is diagonal in the usual representation of $\tau$-matrices, but off-diagonal $\tau_1$ and $\tau_2$ appear in the Hamiltonian with non-vanishing coefficients as charged mesons are virtually emitted or absorbed. Then these off-diagonal terms change the eigenstates of $\tau$-matrices. This fact can be represented by saying that the $\tau$-spin is put into a precession when mesons are virtually emitted or absorbed. It is noticeable that $\lambda_s$ is smaller than $V$; the difference between $\lambda_s$ and $V$ must be responsible for the property of $\tau$-spin. The effect of a component in some direction of $\tau$-spin will be reduced smaller, if the precession is taken into account and the fluctuation is averaged out; and we may suppose that the generating function will be reduced from $V(\tau \cdot p)$ to $\lambda_s(\tau \cdot p)$.

If this conjecture is true, the striking success of Sawada’s method, as demonstrated above, shows that the precession of $\tau$-spin is primarily important in the cases of weaker coupling strength.

On the other hand, the situation is quite different in the region of stronger coupling strength. When $\lambda_s$ of the weak region is smoothly extrapolated beyond the jump-point, it gives poor results, (cf. Fig. 3). The best value for $\lambda_s$ is nearly equal to $V$ in the strong-coupling cases. If we put $\lambda = V$, all matrix elements for the emission or absorption of an odd number of mesons can be made exponentially small, as can be seen in Eqs. (10, a, b). And the level of the ground state can be made as large as $- (1/2) V^2$ only when we put $\lambda = V$. The transformation function will assume the form $\exp[iV(\tau \cdot p)]$, if the $\tau$-spin can be treated as if it were a c-number. Though we cannot directly prove here that the $\tau$-spin behaves in the strong-coupling limit in such a way, it is remarkable that the degrees of freedom of $\tau$-spin is apparently reduced to one in the strong-coupling theory. Thus we can expect results of correct order, at least to the first order of approximation, by putting $\lambda = V$. The effect of the transformation can be understood by considering a translation of $|q|$ by $V(\tau \cdot p)/|p|$, which is primarily important in the strong limit.

However, the level of the ground state is not correctly given at the next order of expansion into powers of $1/V^2$. We find, putting $\lambda = V$,

$$E_0 = \langle H_{\text{pot}} \rangle_0 = -V^2/2 + 1 - e^{-r^2}(1 + 2V^2) \sim -V^2/2 + 1$$  \hspace{1cm} (12)$$

in the strong limit. The correct value is known as $-(1/2)V^2 - 1$. This failure is due to the fact that in our formulation $H_{\text{pot}}$ gives rise to a peak of the effective potential in the neighbourhood of $|p| \simeq 1/V$; the shape of $H_{\text{pot}}$ is plotted in Fig. 5a, putting $\lambda = V$. In the outer region of $p$-space ($|p| \gg 1/V$), $H_{\text{pot}}$ is oscillating, except for the constant $-V^2/2$, and gives no contribution on the average, but the effect of the first and highest peak cannot be cancelled by virtue of the valleys around it, and it may give a contribution.
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§ 3. Elimination of the angular part of the interaction

We shall show in this section that we can eliminate the angular variables from the interaction Hamiltonian by means of a certain canonical transformation; simultaneously we can verify the conjecture made in the previous section concerning the qualitative difference between strong- and weak-coupling cases.

We begin with giving some general results of the transformation, assuming that the generating function of the transformation is an odd function of \( (\mathbf{r} \cdot \mathbf{p}) \). The validity of this assumption will be discussed later (Sects. 5 and 6). We put the transformation function in the form

\[
U = \exp \left\{ i f\left[ (\mathbf{r} \cdot \mathbf{p}) \right]\right\},
\]

and

\[
\cos \left[ f(\mathbf{r} \cdot \mathbf{p}) \right] = C, \quad \left(1 / \mathbf{p} \right) \sin \left[ f(\mathbf{r} \cdot \mathbf{p}) \right] = S,
\]

or

\[
U = C + i (\mathbf{r} \cdot \mathbf{p}) S.
\]

By the definition we have the identity

\[
C^2 + p^2 S^2 = 1.
\]

The variable \( q \) after the transformation is given by

\[
U^{-1} q U = q - \mathbf{r} C S + 2 (\mathbf{r} \cdot \mathbf{p}) \left[ \partial C / \partial (p^2) \cdot S - \partial S / \partial (p^2) \cdot C \right] + [\mathbf{p} \times \mathbf{r}] S^2.
\]

Consequently we have

\[
U^{-1} q U = q^2 + [2p^2 (\partial C / \partial (p^2) \cdot S - \partial S / \partial (p^2) \cdot C) - CS]^2 + 2S^2 - 2S^2 \left( \mathbf{p} \times \mathbf{q} \right) \cdot \mathbf{r}
\]

\[
- [CS(\mathbf{r} \cdot \mathbf{q}) + \text{conj.}] + 2 \left[ (\partial C / \partial (p^2) \cdot S - \partial S / \partial (p^2) \cdot C) (\mathbf{r} \cdot \mathbf{p}) (\mathbf{p} \cdot \mathbf{q}) + \text{conj.} \right].
\]

The results of transformation for \( (\mathbf{r} \cdot q) \) is given by

\[
U^{-1} (\mathbf{r} \cdot q) U = \left\{ (C^2 - \frac{1}{2}) (\mathbf{r} \cdot q) + \text{conj.} \right\} + \left[ S^2 (\mathbf{r} \cdot \mathbf{p}) (\mathbf{p} \cdot \mathbf{q}) + \text{conj.} \right]
\]

\[
+ 2CS(\mathbf{r} \cdot [\mathbf{p} \times \mathbf{q}]) + 2 (\partial C / \partial (p^2) \cdot S - \partial S / \partial (p^2) \cdot C) (\mathbf{r} \cdot \mathbf{p}) (\mathbf{p} \cdot \mathbf{q}) + \text{conj.}
\]

\[
= \frac{1}{2} [S^2 - df / dp] |p|
\]

where we use the identity (16).

By means of Eqs. (18) — (20) and (16), the transformed Hamiltonian is

\[
U^{-1} H U = H_0 + H_{\text{pot}} + H_{\text{sp-cov}} + H_{\text{rad}} + H_{\text{aug}}.
\]

\[H_{\text{pot}}\]

is given by

\[
H_{\text{pot}} = \frac{1}{2} (df / dp) |p|^2 - V(df / dp) |p| + S^2 - 2VCS,
\]
and can be regarded as the additional term to the potential of meson oscillators after the transformation; \( H_{\text{sp-orb}} \) is given by

\[
H_{\text{sp-orb}} = S(2VC-S) (\tau \cdot [p \times q]) = (\tau \cdot [p \times q]) S(2VC-S), \tag{23}
\]

(where it is remarked that this term is commutable with \( p^2 \)); and it gives rise to a separation between a state with parallel isotopic spin and charge vectors and a corresponding state with anti-parallel ones: \( H_{\text{pot}} \) and \( H_{\text{sp-orb}} \) as well as \( H_0 \) are even operators by which the occupation number of mesons is changed by an even number, if any change takes place. On the other hand, \( H'_{\text{rad}} \) and \( H'_{\text{ang}} \) are odd terms by which the occupation number of mesons is necessarily changed by an odd number. They are given by

\[
H'_{\text{rad}} = \frac{1}{2} \left( V - \frac{df}{d|p|} \right) \frac{(\tau \cdot p)}{p^2} (p \cdot q) + \text{conj.}, \tag{24}
\]

and

\[
H'_{\text{ang}} = V \left( C^2 - \frac{1}{2} \right) - \frac{1}{2} CS \left[ (\tau \cdot q) - \frac{(\tau \cdot p)}{p^2} (p \cdot q) \right] + \text{conj.} = \frac{1}{2} \left[ V \cos [2f] \frac{\sin [2f]}{2 |p|} \right] \left[ (\tau \cdot q) - \frac{(\tau \cdot p)}{p^2} (p \cdot q) \right] + \text{conj.} \tag{25}
\]

respectively. Here \( H'_{\text{rad}} \) is independent of the angular modes of meson oscillators, while \( H'_{\text{ang}} \) depends on them.

In general both types of interaction term, \( H'_{\text{rad}} \) and \( H'_{\text{ang}} \), appear after the transformation. Only when we put the transformation function in the form

\[
U_R = \exp \left( i/2 \cdot \tan^{-1}[2V(\tau \cdot p)] \right), \tag{29}
\]

the angle-dependent term \( H'_{\text{ang}} \) can identically be canceled; (see (25)). For the moment, we recall the fact that a transformation function of half-arctangent-type has often proved useful in a problem in which 1/2-spin variables appear; the treatment of a free Dirac particle with non-vanishing momentum or the reduction of \( P_\mu P_\nu \) coupling in the meson theory are the examples, and in every case the physical interpretation of results has been given by considering some rotation of spin vector. Here the same type of transformation is powerful in treating the precession of 3-spin. Accordingly, we name this type of transformation as the spin-rotation-type or rotation-type in short, and put the index \( 'R' \) to relevant quantities.

The results of the transformation generated by \( U_R \) are, according to Eqs. (22) — (24),

\[
H_{\text{pot}}^R = \frac{1}{4p^2} \left( 1 - \frac{1}{\sqrt{1 + 4V^2 p^2}} \right)^2 + \frac{V}{2} \left( \frac{1}{1 + 4V^2 p^2} - \frac{2V^2}{\sqrt{1 + 4V^2 p^2}} \right), \tag{29}
\]

\[
H_{\text{sp-orb}}^R = \frac{2V^2}{1 + \sqrt{1 + 4V^2 p^2}} \left( [p \times q] \cdot \tau \right), \tag{30}
\]

\[
H_{\text{rad}}^R = \frac{2V^2 p^2}{1 + 4V^2 p^2} \left( \frac{(\tau \cdot p)}{p^2} (p \cdot q) + \text{conj.} \right). \tag{31}
\]
They are plotted in Fig. 4 as functions of $|p|$ except for factors $(\tau \cdot p)(p \cdot q)/p^2$ or $(\tau \cdot [p \times q])$.

The $H_{\text{spin}}^R$ term is larger in the neighbourhood of the origin of $p$-space, but is damped in the region $|p| \gg 1/2V$ with the asymptotic form $V/|p|$. It should be remembered that $H_{\text{spin}}^R$ gives rise to a splitting of correct sign between states with common value of $[p \times q]^2$, namely the states with parallel charge and isotopic spin vectors are made lower.

On the other hand, the remaining interaction term $H_{\text{had}}^R$ vanishes at the origin and is small in the region $|p| < 1/2V$, but is equal to $V$ in the region far from the origin. ($|p| \approx 1/2V$).

The wave functions of low-lying states have an exponential factor $\exp(-p^2)$ and in the region $|p| > 1$ their values are very small. In a weak-coupling case, we may safely neglect $H_{\text{spin}}^R$, because $H_{\text{spin}}^R$ is small in region $|p| < 1$ since $1 \approx 1/2V$ and its matrix elements between low-lying states are very small. The isotopic spin-orbit coupling operator can readily be made diagonal in a given charge state, and the effective Hamiltonian $H^R_0 + H_{\text{spin}}^R + H_{\text{spin}}^R$ is free from any term which causes a change of meson occupation number by an odd number. Thus we can successfully eliminate the interaction Hamiltonian by virtue of the transformation $U_R$. In the strong-coupling case, on the contrary, we have $1 \gg 1/2V$, and the major part of wave function is perturbed by the interaction term of strength $V$; while $H_{\text{spin}}^R$ and $H_{\text{spin}}^R$ are of secondary importance, for they affect only a small portion of wave function. Since the interaction term is averaged over angle and is correlated only with the radial mode, we have verified that, in the strong-
coupling limit, it is important to treat the interaction of the radial mode, first disregarding angular modes.

\[(1/2)p^2 + H_{\text{pot}}\] is the effective potential of the meson oscillator after the transformation. \(H_{\text{pot}}\) has the value \(-(3/2)V^2\) at the origin but it is damped in the region \(|p| > 1/2V\) with the asymptotic form \(-V/|p|\). We can expect a level shift of about \(-(3/2)V^2\) for the ground state in a weak-coupling case. This is because we have \(1/2V \geq 1\) in such a case and the ground state wave function is extended only in the region \(|p| \lesssim 1\).

If the coupling constant \(V\) becomes larger, it is necessary to perform a second transformation to eliminate the interaction of radial mode, which will be considered in next section.

§ 4. Elimination of the radial mode in the interaction Hamiltonian

The radial mode can identically be eliminated from the interaction Hamiltonian, only when we put the transformation function in the form

\[U_T = \exp \{iV(\tau \cdot p)\}. \tag{3}\]

This transformation corresponds to some translation in the \(q\)-space, and accordingly, we put the index ' \(T\) ' to relevant quantities. The transformed Hamiltonian is given by

\[H_T = -\frac{1}{2}V^2 + \left\{ \frac{\sin^2(V|p|)}{|p|} - \frac{V^2|p|}{|p|} \right\}, \tag{32}\]

\[H_T = -\{\tau \cdot [p \times q]\} \left\{ \frac{\sin^2(V|p|)}{|p|} - \frac{V^2|p|}{|p|} \right\}, \tag{33}\]

\[H_{T_{\text{total}}} = \frac{1}{2} \left\{ 2V^2|p| - \frac{\sin(2V|p|)}{2|p|} \right\} \left\{ (\tau \cdot q) - \frac{(\tau \cdot p)(p \cdot q)}{|p|^2} \right\} + \text{conj.} \tag{34}\]

These functions are illustrated in Fig. 5. In this case the remaining interaction Hamiltonian depends on angular variables; hence the transformation \(U_T\) has a character complementary to \(U_R\) of the preceding section.

The effective coupling strength in \(H_{\text{total}}\) oscillates rapidly in the region \(|p| \approx 1/V\). In a strong-coupling case, \(H_{\text{total}}\) has small matrix elements between low-lying states; they are actually proportional to \(\exp (-V^2)\) (see Eqs. (10, a-c)). This is because the major part of the wave function lies in the region \(|p| > 1/V\), and are slowly varying. Consequently, we can consider the interaction Hamiltonian is approximately eliminated in strong-coupling cases by means of \(U_T\). The effective potential also exhibits a characteristic feature of the strong-coupling theory. \(H_{\text{pot}}\) oscillates rapidly about the value \(-V^2\) in the region \(|p| \approx 1/V\), and to the first approximation, the level of the ground state may be \(-V^2\).

As mentioned earlier, the transformation function \(U_T\) is derived by assuming that the system is perturbed only by the radial mode of meson oscillators with constant strength \(V\). Then we proceed in analogy to the Bloch-Nordsieck transformation. However, as is shown
in the preceding section, this assumption is justifiable only in the region \( |p| \lesssim 1/V \). In the inner region \( |p| \lesssim 1/V \), the components corresponding to angular modes can never be neglected.

The behavior of transformed Hamiltonian is in fact unsatisfactory in the inner region. The effective potential has a high peak near \( |p| = 2/V \), whose effect of wrong sign cannot be canceled by virtue of neighbour potential valleys; and it makes the level shift a little wrong, i.e., \( -(1/2)V^2 + 1 \), in contradiction to the correct value \( -(1/2)V^2 - 1 \). The shape of \( H_{\text{orb}}^p \) has some close relationship to that of \( H_{\text{orb}}^0 \) and \( H_{\text{orb}}^\pi \) has a deep valley near \( |p| = 2/V \). This valley makes the expectation value of \( H_{\text{orb}}^\pi \) negative in a strong coupling case, which is a wrong sign; and among the first low-lying excited states, for example, the state with \( I = 1/2 \) will be brought lower than the state with \( I = 3/2 \), if the coupling constant becomes larger than a certain limiting value (about 0.81).* We must make use of a generating function of such a form as discussed in the preceding section if we want to get rid of these defects in the inner region.

Summing up our analysis made so far, we must have a generating function of the form, which is similar to \( (1/2)\tan^{-1}(2V|p|) \) in the inner region \( (|p| < 1/V) \), but approaches \( V|p| \) asymptotically in the outer region \( (|p| \gg 1/V) \); the deviation from \( V|p| \) must be largest in the region \( |p| \sim 1/V \), since the situation is most involved there as is seen above.

* We have

\[
\langle 1|H_{\text{orb}}^\pi |1\rangle = -[I(I+1) - 2 - 3/4] [e^{-V^2(1+4V^2-4V^4)} - 1],
\]

where \( \langle 1|\cdots|1\rangle \) denotes the expectation value taken with respect to any one of the first excited states of meson oscillators. \( I \) is the quantum number of total isotopic spin. The second factor vanishes at \( V = 0.81 \) and is positive (negative) for \( V \) smaller (larger) than the critical value.
It is noticeable that \( V \) and \( |p| \) always appear in combination, so we can speak of regional character of \( p \)-space, independently of \( V \); if we use \( 1/V \) for the unit of length of \( p \); we put \( x=V|p| \) hereafter. The division of two regions at \( x=1 \) is somewhat arbitrary, and there may be some intermediate region. In fact, as will be discussed in § 6, our approach is to some extent powerless in the region \( x \sim 1 \). It requires a computational technique more powerful than is available at present to improve this point. But it seems tolerable for us to divide the \( p \)-space into two regions at \( x=1 \), as far as qualitative arguments are concerned. The curve of \( f(x) \) plotted vs. \( x \) is somewhat similar to that of Sawada's parameter \( \lambda_s \) plotted vs. \( V \), if the latter curve is made smooth in the intermediate region. This fact is more meaningful than a superficial similarity. The inner region \( x<1 \) gives features common in weak coupling cases to our system, while the outer region \( x>1 \) is essential for the features of strong coupling cases. As the wave functions extends approximately up to \( x=V \), the magnitude of the coupling constant determines the character of regions which are covered by wave functions. On the basis of this fact we must expect some close relationship existing between the regional character of \( p \)-space, which is represented by \( f(x) \), and the character of coupling strength, which is represented by \( \lambda_s \); where \( \lambda_s \) is the effective coupling constant in which some effects of higher order perturbation theory are taken into account in a kind of averaged form.

Incidentally, it may be natural for us to define the strong-coupling case as the one in which the weak-region branch of \( \lambda_s \) gives poorer results than \( \lambda=V \) does, or in other words, a case in which the precession of \( \tau \)-spin gives only a secondary effect. Then we have \( V \sim 3.5 \) for the lower limit of the strong coupling region. Thus we have the following scheme in the symmetrical scalar problem

\[
\begin{align*}
V &< 1/2 \quad \text{the weak coupling region,} \\
1/2 &< V < 3.5 \quad \text{the intermediate coupling region,} \\
3.5 &< V \quad \text{the strong coupling region.}
\end{align*}
\]

\( \text{§ 5. Variational calculation} \)

We have thus far exclusively used a generating function which is an odd function of the argument \( \tau \cdot p \). We are forced to consider this type of generating function, since otherwise we cannot perform ordering of operators in the transformed Hamiltonian by known techniques of operator calculus. This is concluded in the following way: An exponential function of operators can be ordered, at present, if and only if the argument is a linear or bilinear form of operators. Thus we can perform ordering of the transformed Hamiltonian, applying Fourier transformation to the terms to be ordered (cf. Eq. (9)), if the argument of the transformation function is a linear or bilinear form of operators. But no bilinear form is useful, since we want to eliminate the interaction term. Hence we should consider a generating function, which is an odd function of the argument

\[
\lambda(\tau \cdot p) + \mu(\tau \cdot q),
\]

(36)
where $\lambda$ and $\mu$ are arbitrary numerical constants. If we perform a transformation by
\[
R = \exp\left[-i(tan^{-1}\frac{\mu}{\lambda}) (p^2 - q^2)/2\right],
\]
the total Hamiltonian in the new representation is put into the form
\[
R^{-1}HR = H_0 + \frac{\lambda V}{\sqrt{\lambda^2 + \mu^2}}(\tau \cdot q) + \frac{\mu V}{\sqrt{\lambda^2 + \mu^2}}(\tau \cdot p),
\]
and at the same time the generating function is put into the form
\[
R^{-1}UR = \exp\left[i\left(\sqrt{\lambda^2 + \mu^2}(\tau \cdot p)\right)\right].
\]
Then it is evident that we must put $\mu = 0$ in the new representation, because the last term on the right hand side of Eq. (38) remains unchanged by the transformation generated by (39). Consequently, we are led to consider a transformation function of the form used in this paper.

Let us now try to minimize effects of interaction terms of any type by virtue of a suitable transformation. Some remark was given in preceding section about the shape of generating function. But it is impossible to invent a transformation function which can identically eliminate both radial and angular modes from the interaction Hamiltonian, as far as the generating function is restricted to the type considered here. Then we must require that our transformation should satisfy the condition, ‘The radial and angular interaction terms should give no remarkable effect to the lowest state.’. This condition may be represented by the following two equations
\[
\int_0^\infty \text{(effective coupling strength in } H_{\text{rad}}') \exp\left(-|p|^2 \right) \cdot p^3 d|p| \to \text{min.}
\]
\[
\int_0^\infty \text{(effective coupling strength in } H_{\text{ang}}') \exp\left(-|p|^2 \right) \cdot p^3 d|p| \to \text{min.}
\]
If these minimum values are actually very small, we may safely neglect effects of $H_{\text{rad}}'$ and $H_{\text{ang}}'$-terms in the transformed Hamiltonian, at least when we are concerned with low-lying states. Then we have only to use the effective Hamiltonian
\[
H_{ee} = H_0 + H_{\text{pot}} + H_{\text{sp-orb}}.
\]
In each charge state, the spin-orbit coupling operator in $H_{\text{sp-orb}}$ can be made diagonal, and we have an effective Hamiltonian representing oscillators in $p$-space with modified potential for each state.

At this point, some closer examination of $H_{\text{pot}}$ may be useful. It consists of two parts, as is shown in Eq. (22)
\[
H_{\text{pot}} = H_{\text{pot}}^1 + H_{\text{pot}}^2
\]
with
\[
H_{\text{pot}}^1 = \left[\frac{1}{2} (df/d|p|)^2 - V(df/d|p|)\right], \quad \text{and} \quad H_{\text{pot}}^2 = S^2 - 2VCS.
\]
Only $H_{\text{pot}}$ gives remarkable effect in the outer region of $p$-space, and its physical meaning is understandable, if we take $df/d|p|$ for the effective coupling strength of our system and adopt an analogy to the Bloch-Nordsieck transformation. On the other hand, we may regard the effect of rotation of $\tau$-spin as the cause of $H_{\text{pot}}$. There is another support to a conjecture that $H_{\text{pot}}$ may be closely related to the precession of $\tau$-spin; the effective strength of $H_{\text{sp-orbl}}$, which represents characteristically a result of $\tau$-spin's precession, is of the just same form but of the reverse sign as this term. We may naturally expect that if the terms $S^2-2VCS$ are correctly given, it will then be possible in the cases of strong coupling to get a level shift which is not far from the correct value $-(1/2)V^2-1$, and to obtain simultaneously correct spin-orbit splitting of levels.

From a practical point of view, we will here introduce a variational approach, and assume a rather simple functional form for the generating function, in which some parameters are varied as to satisfy above conditions. The form here adopted is

$$f(x) = \frac{1}{2} \tan^{-1} \left[ \frac{2x}{1 + \lambda x^2} \right] + \frac{x^3}{\mu + x^2},$$

or

$$f(x) = x - \frac{3x^3}{4 + \lambda x^2 + \mu x^4},$$

with two parameters to be varied. This form is in accordance with the general remark given in the preceding section, as it approaches to $f(x) = x$ as $x \to \infty$, and to $f(x) = (1/2)\tan^{-1}(2x)$ as $x \to 0$. The numerical results are given in Table 1. Calculations are made at $V = 0.2, 1, 5$, which are representing the weak, intermediate and strong-coupling region respectively.

<table>
<thead>
<tr>
<th>$V$</th>
<th>0.2</th>
<th>1.0</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle H_{\text{pot}} \rangle_0$</td>
<td>$-1.41 \times 0.04$</td>
<td>$-0.935 \times 1$</td>
<td>$-0.478 \times 25$</td>
</tr>
<tr>
<td>$\langle H_{\text{sp-orbl}} \rangle_1$</td>
<td>0.93 $\times 0.04$</td>
<td>0.315 $\times 1$</td>
<td>$-0.373 \times 0$</td>
</tr>
<tr>
<td>$\langle H'_{\text{rot}} \rangle_0$ (Eq. (40))</td>
<td>0.14 $\times 0.2$</td>
<td>0.18 $\times 0.2$</td>
<td>0.390 $\times 1$</td>
</tr>
<tr>
<td>$\langle H'_{\text{rot}} \rangle_0$ (Eq. (41))</td>
<td>$-0.05 \times 0.2$</td>
<td>$-0.01 \times 0.2$</td>
<td>$-0.345 \times 1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>parameter</th>
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<th>$\mu$</th>
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<tr>
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<td>7.2</td>
</tr>
<tr>
<td></td>
<td>7.2</td>
<td>4</td>
</tr>
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<td></td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.5</td>
</tr>
</tbody>
</table>

It is true that some improvement to Sawada’s results is obtained, but it is insufficient in the strong coupling cases, and we are far from reproducing finer details of the strong coupling theory.
We have here computed expectation values of various terms not using the eigenfunctions of the effective Hamiltonian but using the eigenfunctions of the unperturbed harmonic oscillators, in the hope that the former may well be approximated by the latter. Some further improvement on our results is possible, if we make use of the more exact wave functions for each state, which are eigenfunctions of the effective Hamiltonian, \( H_0 + H_{\text{pot}} + H_{\text{sp-orb}} \). However, it seems improbable that we can expect much in this way, because there is some serious difficulty in carrying out our approach* in not-weak-coupling cases perfectly in a line with the qualitative prospect thus far obtained. This difficulty arises from the limited form of our generating function. And the region \( x \sim 1 \) can never be well treated.

§ 6. Discussions

We cannot quantitatively obtain a result, which parallels the qualitative prospect thus far discussed, so long as the generating function is restricted to some odd functions of \( \tau \cdot p \) alone. The general form of the generating function \( G \) can be obtained from the solution of the operator equation

\[
\]

where \( B_n \)'s are Bernoulli numbers; this equation is obtainable by means of the method investigated by one of the authors (S.T.) ; the \( n \)-th coefficient is that of \( x^n \) in the expansion of \( x \cot x \). When the generating function \( G \) is expanded into powers of \( V \) and is put in the form

\[
G = VG^{(3)} + V^2 G^{(5)} + \cdots
\]

the first term should be the solution of the equation

\[
-i[H_0, G^{(3)}] = (\tau \cdot q).
\]

This equation can readily be solved and the result is

\[
G^{(3)} = (\tau \cdot p).
\]

Then \( G^{(5)} \) should be the solution of the equation

\[
i[H_0, G^{(5)}] = (1/3) [[(\tau \cdot q), G^{(3)}]G^{(2)}]
\]

\[
= (1/3) \{(p^2 + q^2) (\tau \cdot q) + \text{conj.}\}
\]

\[
+ (1/3) \{(p^2 - q^2) (\tau \cdot q) - [(p \cdot q) + (q \cdot p)](\tau \cdot p) + \text{conj.}\}.
\]

* Alone with the condition that \( H'_{\text{sp-orb}} \) should give a minimum effect or that \( H'_{\text{sp-orb}} \) should give a maximum effect, we are led to the answer that \( f(x) = (1/2)\tan^{-1}(2x) \); while alone with the condition that the \( H'_{\text{rad}} \) should give a minimum effect, we are led to the answer that \( f(x) = x \). Actually we are forced to determine the best value of parameters by the condition that the approximative level of ground state (computed with the unperturbed wave function) should become as low as possible.
If only the first member of the right hand side is retained we obtain a part of \( G^{(3)} \), which is given by
\[
G^{(3)\prime} = -\frac{1}{3} \left\{ (p^2 + q^2) (\tau \cdot p) + \text{conj.} \right\}.
\]
If we further make an approximation to replace \( q^2 \) by \( p^2 \), considering that the same expectation value is obtainable when applied to the ground state, then we have
\[
G^{(3)\prime} \sim -\left( \frac{4}{3} \right) (\tau \cdot p)^3
\]
which gives the second term of the expansion of \( (1/2) \tan^{-1}(2V(\tau \cdot p)) \). However neglect of the second member on the right hand side of Eq. (47) cannot be permissible. There is no more justification, if any, than that its inclusion causes so much complication of the generating function \( G \) that we can hardly give the ordered form of the transformed Hamiltonian explicitly. In fact the exact solution to Eq. (47) is given by
\[
G^{(3)} = -\left( \frac{2}{3} \right) (q^2 (\tau \cdot p) + \text{conj.}) + \left( \frac{1}{3} \right) \left\{ [(p \cdot q) + (q \cdot p)] (\tau \cdot q) + \text{conj.} \right\}.
\]
Particularly, omission of such a term \( G^{(3)\prime\prime} \),
\[
G^{(3)\prime\prime} \sim [(p \cdot q) + (q \cdot p)] (\tau \cdot q)
\]
causes a serious effect on the transformed Hamiltonian. Since \( i[H_0, G^{(3)\prime\prime}] \) is just the term of the type of \( H'_{\text{rad}} \), inclusion of \( G^{(3)\prime\prime} \) or a term of similar nature yields an effect that, to some extent, \( H'_{\text{rad}} \) is reduced smaller, while \( H'_{\text{ang}} \) remains nearly unchanged. Thus we can expect that the results of \( \omega \tau \tau \tau \) transformation will be characteristically affected, if we include a function of the form \( G^{(3)\prime\prime} \) into the generating function, though we will have some difficulty in performing the ordering procedure of the transformed Hamiltonian.

In analogy to the treatment of rotation of \( \tau \)-spin in more usual cases, we must have two types of \( \tau \)-operator in the argument of the generating function of our transformation. This means that we must include both \( (\tau \cdot p) \) and \( (\tau \cdot q) \) into the generating function. But as was discussed in the previous section, the linear combination
\[
\lambda (\tau \cdot p) + \mu (\tau \cdot q)
\]
is useless; and, as suggested above, the part containing \( (\tau \cdot q) \) must appear in the form
\[
(\tau \cdot q) \left\{ [(p \cdot q) + (q \cdot p)] + \cdots \right\}.
\]
The transformation function which is given in terms of \( (\tau \cdot q) (p \cdot q) \) or similar forms may be effective in the intermediate region of \( p \)-space. The properly inner \((|p| < 1/V)\) or outer \((|p| > 1/V)\) regions of \( p \)-space can be manipulated by generating functions of the type which are similar to \( f(x) = (1/2) \tan^{-1}(2x) \) or \( f(x) = x \). Indeed, it is in the region \( 1/V \lesssim |p| \lesssim 3/V \) that we are annoyed by a high peak of \( H_{\text{pot}} \). If a transformation function given in terms of \( (\tau \cdot q) (p \cdot q) \) or similar forms were successfully applied, \( H'_{\text{rad}} \) in the intermediate region of \( p \)-space could be reduced smaller and then the transformations of rotation-type, which has been discussed in Sec. 3, would have a wider range of applicability in \( p \)-space, so that \( H_{\text{pot}} \) after such transformations would become far less oscillating and assume a well-behaved form which we expect on the basis of qualitative analysis made.
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in this paper. If the intermediate region of $p$-space could successfully be dealt, then $H_{sp-orb}$ would correctly be given and the results of the strong-coupling theory might well be reproduced in its finer detail.

In conclusion, we have made analysis of the bound-meson problem of the symmetrical scalar theory, and pointed out the regional character in $p$-space. When the knowledge about these regional character are combined with the fact that the wave functions of low-lying states are not extended beyond a certain limit, we can predict how the character of our problem is changed as the coupling constant is varied from the weak-coupling region into the strong-coupling region. Because of our limited possibility in handling a complicated form of operators, the success achieved is only partial as regards to its quantitative aspects. However, the direction has been discussed in which the future improvement might be explored.

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References

4) D. Ito, Y. Miyamoto and Y. Watanabe, Prog. Theor. Phys. 13 (1955), 594; Y. Watanabe