On the Heisenberg's Non-linear Meson Equation

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In order to account for the multiple meson production in high energy nuclear events, the non-linear meson theory has been proposed by Heisenberg. In his theory, the Lagrangian is taken to be of the so-called Born type:

\[ L = \frac{1}{2} \left[ 1 + l^{-2} \left( \partial_{\mu} \phi \cdot \partial_{\nu} \phi + \kappa^2 \phi^2 \right) \right]^{1/2}, \quad (1) \]

in which \( \phi \) is the scalar field quantity, \( \kappa \) is the meson mass and, \( l^{-1} \) is a small constant. Accordingly, the field equation is given by

\[ \delta_{\mu\nu} + l^{-2} \left( (\partial_{\lambda} \phi \cdot \partial_{\lambda} \phi + \kappa^2 \phi^2) \delta_{\mu\nu} - \partial_{\mu} \phi \cdot \partial_{\nu} \phi \right) \partial_{\mu} \partial_{\nu} \phi \\
- \left[ 1 + l^{-2} \left( 2 (\partial_{\mu} \phi \cdot \partial_{\nu} \phi) + \kappa^2 \phi^2 \right) \right] \kappa^2 \phi = 0. \quad (2) \]

The purpose of this note is to investigate the solutions
of the above equation, on the basis of the characteristic theory of partial differential equation. 2) In what follows we restrict ourselves to the case of one dimensional motion and the mass \( \kappa \) is put equal to zero.

The equation (2) then becomes

\[
(-p+v^2)u_x + u(v_x - v_t) - (p+v^2)v_t = 0,
\]

(3-a)

\[
u_x + v_t = 0,
\]

(3-b)
in which \( u \) and \( v \) are given by the equations:

\[
u = \theta \phi / \partial t, \quad v = -\theta \phi / \partial x.
\]

(4-a), (4-b)

It can easily be seen that (3-a) is automatically satisfied by the relation:

\[
\nu = mu \pm \sqrt{1-m^2} l,
\]

(5)
in which \( m \) is an arbitrary constant.

Substituting (5) into (3-b), the original equation reduces to

\[
u_x \pm mu_x = 0.
\]

Therefore, we can obtain the solutions

\[
u = f(x \mp mu), \quad v = mu \pm \sqrt{1-m^2} l,
\]

(6)
corresponding to the initial conditions:

\[
u = f(x), \quad v = mu + \sqrt{1-m^2} l,
\]

at \( t = 0 \).

In the following, it can be shown that these solutions correspond to the simple wave in the hydrodynamics. (In fact, the equation (3-a, b) are quite analogous to the equations of the isentropic irrotational steady plane flow.)

Following the procedure similar to that employed in the hydrodynamics, we have the characteristic equations:

\[
C^+ : \quad \tau_x - \lambda x = 0,
\]

(7-a)

\[
C^- : \quad \tau_x - \lambda x = 0,
\]

(7-b)

\[
I^+ : \quad u_x - \lambda v = 0,
\]

(8-a)

\[
I^- : \quad u_x - \lambda v = 0,
\]

(8-b)

\[
\lambda^2 = \frac{uv \pm l(p+u^2-v^2)\pm}{v^2-p},
\]

which are equivalent to (3-a, b).

Instead of (8-a, b), we can also employ the following equations for the characteristics:

\[
\varepsilon (du^2 - dv^2) = (du - dv)^2,
\]

(9)
in which \( \varepsilon \) is given by \( \varepsilon = (p + u^2 - v^2)^{1/2} \).

It follows immediately that the general solution of (9) is given by the straight lines in the \( u-v \) plane:

\[
v = mu \pm \sqrt{1-m^2} l,
\]

(10)
in which \( m \) is an arbitrary constant.

The detailed calculations show that \( v = mu + \sqrt{1-m^2} l \) represents \( I^+ \), corresponding to \( \sqrt{1-m^2} l \), respectively; and \( v = mu - \sqrt{1-m^2} l \) represents \( I^- \), corresponding to \( \sqrt{1-m^2} l \) respectively. However, the \( C \) characteristics are not, in general, straight lines and their geometrical relation to the \( I \) characteristics can be given by the equation:

\[
\tau_x v - x u = 0,
\]

which indicates that the angle between \( I^- \) and \( u \)-axis and the angle between \( C^+ \) and \( x \)-axis are complementary to each other and the similar relation holds for \( I^+ \) and \( C^- \). It is quite obvious from these results that the simple wave solution of (7) and (8), can be given by (6). In non-linear theories, it seems to be rather peculiar that the simple wave propagates keeping its shape constantly with the development of time. In this respect, the meson field under consideration is analogous to the \( \kappa \) \( \text{Karman-Zien's gas in the hydrodynamics.} ** The author expresses his thanks to Prof. Y. Tanikawa for his kind interest and to Prof. Z. Koba for his discussions.

\* Our time coordinate \( x^0 \) is real, \( c = h = 1 \), \( g^{00} = -1 \), \( g^{11} = g^{22} = g^{33} = 1 \).

2) K. Husimi, Butsurigaku-Koenshii, (4) (1944), (in Japanese.)
3) R. Courant and K. O. Friedlichs, Supersonic flow and Shock waves, Chapt. II.
4) ibid. Chapt. IV. B.
5) ibid. Chapt. II.

** As was discussed by Imai on the basis of the hodograph method, the analogy between the Born's electromagnetic field and the \( \kappa \) \( \text{Karman-Zien's gas is by far closer in the static case.} \)