On Quantum Electrodynamics without Subsidiary Conditions

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The formalism of quantum electrodynamics without subsidiary conditions was obtained in a relativistically covariant way, starting from Heisenberg’s representation. Transforming the formalism of quantum electrodynamics in Heisenberg’s representation to the one in the interaction representation, the equivalence of this formalism to the ordinary quantum electrodynamics introduced by Fermi was proved.

§ 1. Introduction

The divergence difficulties of the norm of the state vector accompanying Lorentz’s condition in quantum electrodynamics are well-known facts from the old time.1) The procedure avoiding these divergence difficulties was proposed by Gupta2) and Bleuler,3) and a rather radical interpretation of this procedure was tried by using the indefinite metric in Hilbert’s space introduced by Dirac.4) Furthermore, the formulation of quantum electrodynamics without subsidiary conditions was proposed by Valatin,5) whose method starts out from the interaction representation, assuming that the interaction Hamiltonian is composed of the transverse part and the longitudinal photon part leading to the Coulomb energy.

In their above methods there are some defects: Gupta and Bleuler have used the concept of the indefinite metric, which seems to me to be inadequate and Valatin has adopted the method mentioned above, which seems to lack the necessity with respect to the derivation of the formulation.

The method proposed here starts from a formulation of quantum electrodynamics expressed rigorously in Heisenberg’s representation, and after dividing the electromagnetic field potential into the transverse and the other parts, the integrals of these wave equations were exactly obtained by using Green’s functions. And furthermore, by employing the procedure derived by Glauber6) and Umezawa7), the formulation in Heisenberg’s representation is transformed into the gauge invariant one in the interaction representation and the equivalence of this formalism to the ordinary quantum electrodynamics is proved. Here the gauge difference in a wide sense produced from the other parts of the electromagnetic field potential \( A_\mu \) gives the ambiguities of Coulomb potential energy in the interaction Hamiltonian, which can be vanished. This circumstance is due to the reason that only three components of \( A_\mu \) can be determined, while the remaining one component of \( A_\mu \) is undetermined.

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§ 2. Quantum electrodynamics in Heisenberg’s representation

The quantized wave equations for the system of electrons and electromagnetic field interacting with each other without using Lorentz’s condition are expressed as follows:

\[
\left\{ \gamma_{\mu} (\partial_{\mu} - ie A_{\mu} (x)) + \kappa \right\} \phi (x) = 0,
\]

\[
(\partial_{\mu} \Box - \partial_{\nu} \partial_{\nu}) A_{\nu} (x) = - j_{\mu} (x),
\]

\[
j_{\mu} (x) = ie \phi (x) \gamma_{\mu} \phi (x),
\]

where the dynamical variables in Heisenberg’s representations are denoted by the bold letters and the ones in the interaction representation by the usual letters. \( A_{\mu} (x) \) and \( \phi (x) \) are the quantized electromagnetic potential and the electron wave functions respectively.

Here we divide \( A_{\mu} (x) \) into the following two parts:

\[
A_{\mu} (x) = \mathfrak{A}_{\mu} (x) + \mathfrak{B}_{\mu} (x),
\]

where \( \mathfrak{A}_{\mu} (x) \) indicates the transverse part of the electromagnetic potential and satisfies two conditions:

\[
\partial_{\mu} \mathfrak{A}_{\mu} (x) = 0, \quad n_{\mu} \mathfrak{A}_{\mu} (x) = 0,
\]

where \( n_{\mu} \) is a time-like unit vector \( n_{\mu} n_{\mu} = -1 \). Now we introduce the projection operator \( T_{\mu \nu} \), which produces the transverse part of \( A_{\mu} \):

\[
\mathfrak{A}_{\mu} = T_{\mu \nu} A_{\nu}.
\]

Here \( T_{\mu \nu} \) operator satisfies the following conditions:

\[
\partial_{\mu} T_{\nu \mu} = 0, \quad n_{\mu} T_{\mu \nu} = 0, \quad T_{\lambda \mu} T_{\mu \nu} = T_{\lambda \nu}.
\]

Such a projection operator satisfying the above condition (7) is expressed as follows:

\[
T_{\mu \nu} = \delta_{\mu \nu} + (n_{\mu} \Box - \partial_{\mu} \partial_{\nu}) \left\{ (\Box + \partial_{\nu})^{-1} + \alpha \delta (\Box + \partial_{\nu}) \right\} n_{\nu} - (n_{\nu} \partial + \partial_{\nu}) \left\{ (\Box + \partial_{\mu})^{-1} + \alpha \delta (\Box + \partial_{\mu}) \right\} \partial_{\mu},
\]

where \( \alpha \) is any numerical constant.

Then the quantized wave equations of \( \mathfrak{A}_{\mu} (x) \), which satisfy the conditions (5), are given in the following by operating \( T_{\lambda \mu} \) to the equation (2):

\[
\Box \mathfrak{A}_{\mu} (x) = - j_{\mu} (x) - (n_{\mu} \Box - \partial_{\mu}) \left\{ (\Box + \partial_{\nu})^{-1} + \alpha \delta (\Box + \partial_{\nu}) \right\} n_{\nu} j_{\lambda} (x),
\]

where \( \partial = n_{\mu} \partial_{\mu} \), and \( \Box^{-1} \), \( (\Box + \partial_{\nu})^{-1} \) expresses the symbolical operator, which means the multiplication by \( (\partial \Box + n_{\mu} \partial_{\mu})^{-1} \) in Fourier’s transformation of functions considered now respectively. \( (\Box + \partial_{\nu})^{-1} n_{\nu} j_{\lambda} (x) \) is the expression related to the Coulomb potential energy, i.e.,

\[
(\Box + \partial_{\nu})^{-1} n_{\nu} j_{\lambda} (x) = \int \delta (x - x') \partial \mathcal{O} (x - x') d\sigma', \quad \text{if} \quad n_{\mu} (x_{\mu} - x'_{\mu}) = 0,
\]

where \( \sigma \) is the space-like surface and \( \mathcal{O} (x - x') \) is defined by
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\[ \partial^2 \mathcal{D} (x-x') = D(x-x'), \]  
(11)

and \( \partial \mathcal{D} (x) \) is the following function \( ^{89} \)

\[
\begin{align*}
\partial \mathcal{D} (x) &= (1/4\pi) 1/ \sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2}, \quad \text{for } x_{\mu}^2 > 0, \\
&= 0, \quad \text{for } x_{\mu}^2 < 0, \\
&= (1/8\pi) (1/n_{\mu} x_{\mu}), \quad \text{for } x_{\mu}^2 = 0.
\end{align*}
\]  
(12)

From equations (2) and (9), we get the quantized wave equation of \( \mathcal{W}_\mu (x) \) :

\[
\Box \mathcal{W}_\mu (x) - \partial_{\mu} \partial_{\lambda} \mathcal{W}_\lambda (x) = \left( \square n_{\mu} - \partial_{\mu} \partial \right) \left\{ (\square + \partial^2)^{-1} + \alpha \partial \left( \square + \partial^2 \right) \right\} n_{\mu} f_\lambda (x). \]  
(13)

These expressions (9) and (13) are gauge invariant, where the former in a narrow sense \( (\Box A = 0) \) and the latter in a wide sense. Before solving these equations, we prove that the original \( A_\mu \) has only three determined solutions and the remaining one component of \( A_\mu \) can not be determined. If we substitute the Fourier's transformation of \( A''_\mu \) :

\[
A_\mu (x) = \int a_\mu (k) e^{ikx} (dk)
\]  
(14)

into (2), we get from (2) :

\[
k^2 a_\mu (k) - k_\mu k_\lambda a_\lambda (k) = F_\mu (k),
\]

where \( F_\mu (k) \) expresses the Fourier's transformation of the source function and satisfies the following condition

\[ k_\mu F_\mu (k) = 0. \]

The rank of the following matrix to give the coefficients of the simultaneous algebraic equations to determine \( a_\mu (k) \) is 3 :

\[
\begin{pmatrix}
  k_1^2 - k_2^2 & k_1 k_2 & k_1 k_3 & k_1 k_4 \\
  k_1 k_2 & k_2^2 - k_3^2 & k_2 k_3 & k_2 k_4 \\
  k_1 k_3 & k_2 k_3 & k_3^2 - k_4^2 & k_3 k_4 \\
  k_1 k_4 & k_2 k_4 & k_3 k_4 & k_4^2 - k_5^2
\end{pmatrix}
\]

Therefore only three components of \( a_\mu (k) \) can be determined, while the remaining one component of \( a_\mu (k) \) is undetermined.

The solutions of quantized wave equations (1), (9) and (13) are expressed in the following integral forms :

\[
\phi (x) = \psi (x, \sigma) - i e \int S^\sigma (x, x') \gamma_\mu A_\mu (x') \phi (x') dx',
\]  
(15)

\[
\mathcal{W}_\mu (x) = \mathcal{W}_\mu (x, \sigma) + \int j_\nu (x') d_{\nu \mu} (x) D^\sigma (x, x') dx' + \int (n_\nu \Box - \partial_\nu \partial) d_{\nu \mu} (x) D^\sigma (x, x') \left[ (\Box'' + \partial''^2)^{-1} + \alpha \partial (\Box'' + \partial''^2) \right] n_\lambda f_\lambda (x') dx'',
\]  
(16)
\[ \mathfrak{A}_\mu(x) = \mathfrak{U}_\mu(x, \sigma) + n_\mu \{ (\square + \partial^2)^{-1} + \alpha \delta (\square + \partial^2) \} n_\lambda j_\lambda(x), \]  
(17)

where

\[ S^\sigma(x, x'') = (\gamma^\sigma \partial_\mu - \kappa) d^\sigma(x, x''), \]  
(18)

\[ \mathfrak{A}^\sigma(x, x'') = 1/2 \cdot \{ \epsilon (\sigma, x'') - \epsilon (x-x'') \} \mathfrak{A}(x-x''), \]  
(19)

\[ D^\sigma(x, x'') = 1/2 \cdot \{ \epsilon (\sigma, x'') - \epsilon (x-x'') \} D(x-x''), \]  
(20)

\[ d_{\mu\nu} = \delta_{\mu\nu} - (\partial_\mu \partial_\nu - \partial_\mu \partial_\nu \partial_\nu - n_\mu \partial_\nu \partial_\nu^{-1} + n_\nu \partial_\mu \partial_\mu^{-1}), \]  
(21)

\[ \epsilon (x-x'') = \begin{cases} +1, & \text{for } x_0 > x_0'', \\ -1, & \text{for } x_0 < x_0''. \end{cases} \]

Here \( x \) does not necessarily lie on \( \sigma \). \( \mathfrak{U}_\mu(x, \sigma) \), \( \mathfrak{A}_\mu(x, \sigma) \) and \( \psi(x, \sigma) \) satisfy the following free equations and the conditions:

\[ (\gamma^\sigma \partial_\mu + \kappa) \psi(x, \sigma) = 0, \quad \square \mathfrak{U}_\mu(x, \sigma) = 0, \quad (\square \partial_\mu - \partial_\mu \partial_\nu) \mathfrak{A}_\nu(x, \sigma) = 0, \quad \partial_\mu \mathfrak{A}_\nu(x, \sigma) = 0, \quad n_\mu \mathfrak{U}_\mu(x, \sigma) = 0. \]  
(22)

(16) is reduced to the following form:

\[ \mathfrak{U}_\mu(x) = \mathfrak{U}_\mu(x, \sigma) + \int j_\nu(x') d_{\mu\nu} D^\sigma(x, x') dx'' \]  
\[ - (n_\mu \partial_\nu + \partial_\mu) \partial_\nu^{-1} [(\square + \partial^2)^{-1} + \alpha \delta (\square + \partial^2)] n_\lambda j_\lambda(x), \]  
(24)

by making use of the properties:

\[ n_\mu d_{\mu\nu}^1 = 0, \quad \partial_\mu d_{\mu\nu}^1 = -(n_\sigma \partial_\nu + \partial_\nu) \partial_\sigma^{-1} \partial_\nu. \]  
(25)

It is easily proved that the integral (24) satisfies the two conditions (5);

\[ \partial_\mu \mathfrak{U}_\mu(x) = \partial_\mu \mathfrak{U}_\mu(x, \sigma) - \int j_\nu(x') (n_\mu \partial_\nu + \partial_\nu) \partial_\nu^{-1} D^\sigma(x, x') dx'' - \partial_\nu^{-1} (n_\lambda j_\lambda(x)) \]  
\[ = \int j_\nu(x') (n_\mu \partial_\nu + \partial_\nu) \partial_\nu^{-1} D^\sigma(x-x'') dx'' - \partial_\nu^{-1} (n_\lambda j_\lambda(x)) \]  
\[ = (n_\mu \partial_\nu + \partial_\nu) \partial_\nu^{-1} j_\mu(x) - \partial_\nu^{-1} (n_\lambda j_\lambda(x)) = 0. \]

Since the fact that the solution (15) satisfies the wave equation (1) is well known, the proof is omitted. As the peculiar Green's function: \( d_{\mu\nu} D^\sigma(x, x'') \) is used in (16) or (24), we prove that (24) satisfies the equation (9):

\[ \square \mathfrak{U}_\mu(x) = - \int j_\nu(x') d_{\mu\nu} D^\sigma(x-x'') dx'' - (n_\mu \square + \partial_\mu \square) [ (\square + \partial^2)^{-1} + \alpha \delta (\square + \partial^2) ] \]  
\[ \times n_\lambda j_\lambda(x). \]
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\[ \begin{aligned}
\phi(x) &= -j_\mu(x) + \partial_\mu \partial^{-1}(n_\mu \mathcal{J}_\mu(x)) - (\Box n_\mu - \partial_\mu)(\Box + \partial^2)^{-1} a \partial (\Box + \partial^2) n_\mu \mathcal{J}_\mu(x) \\
&\quad - \partial_\mu \partial^{-1} (\Box + \partial^2) (\Box + \partial^2)^{-1} a \partial (\Box + \partial^2) n_\mu \mathcal{J}_\mu(x) \\
\mathcal{B}_\mu(x) &= -j_\mu(x) - (\Box n_\mu - \partial_\mu)(\Box + \partial^2)^{-1} a \partial (\Box + \partial^2) n_\mu \mathcal{J}_\mu(x).
\end{aligned} \]

As \( \mathcal{B}_\mu(x) \) has two components, \( \mathcal{B}_\mu(x) \) must have one determined component. Then the remaining undetermined component \( \mathcal{B}_\mu(x, \sigma) \) is related to the difference of gauge \( \partial_{\mu} A(x) \) in a wide sense, where \( A(x) \) is an arbitrary function, because the wave equation which \( \partial_{\mu} A \) satisfies becomes an identity relation:

\[ (\Box \partial_{\mu} - \partial_\mu \partial_{\sigma}) \partial_{\nu} A(x) = 0. \]

It is proved in the next section that this arbitrary term \( A(x) \) and the term related to \( \alpha \) do not contribute to the final result. Thus the final solutions in Heisenberg’s representation are given by (15), (24) and (17).

\[ \text{§ 3. Quantum electrodynamics in the interaction representation} \]

Here we transform the formulation obtained above into the one in the interaction representation by using Glauber’s procedure\(^{9}\) (quite similar to Umezawa’s method). If \( x \) lies on \( \sigma \), we get from (15), (24) and (17):

\[ \begin{aligned}
\phi(x)|_{x \sigma} &= \phi(x/\sigma), \\
\mathcal{B}_\mu(x)|_{x \sigma} &= \mathcal{B}_\mu(x/\sigma) - \alpha (n_\mu \partial + \partial_\mu) \partial^{-1} \partial (\Box + \partial^2) n_\mu \mathcal{J}_\mu(x/\sigma), \\
\mathcal{B}_\mu(x)|_{x \sigma} &= \partial_{\mu} A(x/\sigma) + n_\mu (\Box + \partial^2)^{-1} n_\mu \mathcal{J}_\mu(x/\sigma) + \alpha n_\mu \partial (\Box + \partial^2) n_\mu \mathcal{J}_\mu(x/\sigma),
\end{aligned} \]

by using the properties:

\[ \begin{aligned}
S^a(x, x')|_{x \sigma} &= 0, \\
D^a(x, x')|_{x \sigma} &= 0,
\end{aligned} \]

where \( x \sigma \) and \( x/\sigma \) mean that \( x \) lies on \( \sigma \). The interaction Hamiltonian \( H(x', n) \) can be determined by the following relations and commutation relations:

\[ \begin{aligned}
[\psi(x/\sigma), H(x', \sigma)] &= i \cdot \partial \psi(x/\sigma)/\partial \Omega(x'), \\
[\mathcal{B}_\mu(x/\sigma), H(x', \sigma)] &= i \partial \mathcal{B}_\mu(x/\sigma)/\partial \Omega(x'), \\
\{\overline{\phi}_a(x), \phi_b(x')\} &= i \delta_{ab}(x-x'), \\
[\mathcal{B}_\mu(x), \mathcal{B}_\sigma(x')] &= id_{\mu \sigma} D(x-x'), \\
\text{other commutators} &= 0.
\end{aligned} \]

By differentiating functionally on the equations (15) and (24), we get:

\[ \begin{aligned}
i \cdot \partial \psi(x'/\sigma)/\partial \Omega(x') &= -i S(x-x') \overline{\psi}_a(x'/\sigma) \{\overline{\mathcal{B}}_\mu(x'/\sigma) + 1/2 \cdot n_\mu (\Box' + \partial^2)^{-1} n_\mu \mathcal{J}_\mu(x'/\sigma) \\
&\quad - \alpha/2 \cdot \partial_\mu \partial^{-1} \partial (\Box' + \partial^2) (\Box' + \partial^2)^{-1} n_\mu \mathcal{J}_\mu(x'/\sigma) + \partial_\mu A(x'/\sigma)\} \\
&\quad - i \epsilon (1/2 \cdot n_\mu (\Box' + \partial^2)^{-1} n_\mu \mathcal{J}_\mu(x'/\sigma) - \alpha \partial_\mu \partial^{-1} \partial (\Box' + \partial^2) n_\mu \mathcal{J}_\mu(x'/\sigma)) \\
&\quad \times S(x-x') \overline{\psi}_a(x'/\sigma), \\
i \cdot \partial \mathcal{B}_\mu(x'/\sigma)/\partial \Omega(x') &= -ij_a(x'/\sigma) d_{\mu a} D(x-x'),
\end{aligned} \]
through the use of
\[
\frac{\partial \varepsilon}{\partial \Omega'(x')} = 2 \delta'(x'-x''), \quad \frac{\partial \varepsilon}{\partial \Omega(x')} = \delta'(x'-x''), \delta(x-x'),
\]
\[
\frac{\partial D}{\partial \Omega'(x')} = \delta'(x'-x'') D(x-x'),
\]
where we used the following integral instead of (12):
\[
\phi(x) = \phi(x) - \frac{ie}{2} \int S^2(x, x') \gamma_{\mu} \{ A_{\mu}(x'') \phi(x'') + \phi(x'') A_{\mu}(x''') \} dx''.
\]
(33)

One can show that (31), (32) are satisfied by the following Hamiltonian by making use of (30):
\[
H(x) = -j_{\mu}(x) \mathfrak{H}_{\mu}(x) - \frac{1}{2} n_{\mu} j_{\mu}(x) \int n_{\lambda} j_{\lambda}(x') \partial \Omega'(x-x') d\sigma' \\
+ \alpha / 2 \left\{ [\partial_{\mu} \partial^{-1} \partial (\Box + \partial^2) n_{\lambda} j_{\lambda}(x)] j_{\mu}(x) + j_{\mu}(x) [\partial_{\mu} \partial^{-1} \partial (\Box + \partial^2) n_{\lambda} j_{\lambda}(x)] \right\} \\
- j_{\mu}(x) \mathcal{D}_{\mu} A(x),
\]
(34)

where we put
\[
\phi(x, \sigma) = \phi(x), \quad \mathfrak{H}_{\mu}(x, \sigma) = \mathfrak{H}_{\mu}(x), \quad n_{\mu}(x_{\mu}-x'_{\mu}) = 0
\]
(35)

and \( \phi(x) \) and \( \mathfrak{H}(x) \) satisfy the following relations:
\[
(\gamma_{\mu} \partial_{\mu} + \kappa) \phi(x) = 0, \quad \Box \mathfrak{H}(x) = 0, \quad n_{\mu} \mathfrak{H}_{\mu}(x) = 0, \quad \partial_{\mu} \mathfrak{H}_{\mu}(x) = 0.
\]
(36)

The equation of motion for state vector \( \mathcal{F}[\sigma] \) is
\[
i \cdot \frac{\partial \mathcal{F}[\sigma]}{\partial \Omega(x)} = H(x) \mathcal{F}[\sigma],
\]
(37)

where the third and the 4th terms vanish by making use of the following unitary transformation of \( \mathcal{F}[\sigma] \):
\[
\mathcal{F}[\sigma] = \exp(iA) \mathcal{F}[\sigma], \quad A = \frac{\alpha}{2} \int \sigma \left\{ j_{\mu}(x') [\partial^{-1} \partial (\Box' + \partial^2) n_{\lambda} j_{\lambda}(x')] \\
+ [\partial^{-1} \partial (\Box' + \partial^2) n_{\lambda} j_{\lambda}(x')] j_{\mu}(x') \right\} d\sigma' + \int j_{\mu}(x') A(x') d\sigma,
\]
(38)

\[
\frac{i e^{-IA}}{\partial \Omega(x)} \frac{\partial \mathcal{F}[\sigma]}{\partial \Omega(x)} = - \frac{\partial A}{\partial \Omega(x)} \phi + \left\{ \frac{\partial \phi}{\partial \Omega(x)} \right\} \\
= - \frac{\alpha}{2} \left\{ j_{\mu}(x) [\partial_{\mu} \partial^{-1} \partial (\Box + \partial^2) n_{\lambda} j_{\lambda}(x)] + [\partial_{\mu} \partial^{-1} \partial (\Box + \partial^2) n_{\lambda} j_{\lambda}(x)] j_{\mu}(x) \right\} \phi \\
+j_{\mu}(x) \partial_{\mu} A(x) \phi + i \frac{\partial \phi}{\partial \Omega(x)},
\]

\[
\exp(-iA) H(x) \exp(iA) = H(x) - i[A, H(x)] + \cdots.
\]
\[ [A, H(x)] = \int \frac{j_\mu(x) \mathcal{A}_\mu(x) + \frac{1}{2} n_\mu j_\mu(x)}{\int n_\lambda j_\lambda(x) \vartheta' \mathcal{D} (x-x') d\sigma'} - j_\mu(x) \vartheta \mathcal{A}(x) \]
\[ + \frac{\alpha}{2} \{ j_\mu(x) (\vartheta^{-1} \mathcal{D} (\vartheta' + \vartheta^2) n_\lambda j_\lambda(x)) + (\vartheta^{-1} \mathcal{D} (\vartheta' + \vartheta^2) n_\lambda j_\lambda(x)) j_\mu(x) \}, \]
\[ - j_\mu(x) \mathcal{A}(x') \mathcal{D} d\sigma'. \]

Here the ambiguous terms relating to \( \alpha \) become zero and if \( \mathcal{A}(x) \) is a \( c \)-number, all terms of the right hand side containing \( \mathcal{A}(x) \) vanish, while if \( \mathcal{A}(x) \) is any \( q \)-number independent of \( \psi(x) \), quite ambiguous results are produced. However, as the equation of \( \mathcal{A}(x) \) does not exist and furthermore \( \mathcal{A}(x) \) has not any definite commutation relation, \( \mathcal{A}(x) \) cannot describe the real physical field. Then we must consider \( \mathcal{A}(x) \) as the \( c \)-number not corresponding to any physical field. Therefore the final equation of motion for state vector \( \Psi[\sigma] \) becomes as follows:

\[ i \cdot \partial \Psi[\sigma] / \partial \Omega(x) = \{- j_\mu(x) \mathcal{A}_\mu(x) - \frac{1}{2} \int n_\mu j_\mu(x) n_\lambda j_\lambda(x) \vartheta' \mathcal{D} (x-x') d\sigma'\} \Psi[\sigma]. \quad (39) \]

This formula is completely equivalent to the form of the ordinary quantum electrodynamics. If we adopt only 2 times of the second term, (31) becomes as the following:

\[ [\psi(x), H(x')] = - \epsilon \mathcal{D}(x-x') j_\mu \psi(x') \{ \mathcal{A}_\mu(x') + n_\mu \int n_\lambda j_\lambda(x') \vartheta'' \mathcal{D} (x''-x') d\sigma'' \}. \quad (40) \]

If we calculate the left hand side of (40) by using (34), we get the formula (31). By making use of the following relation,

\[ [\psi(x), j_\mu(x')] = \epsilon \mathcal{D}(x-x') \gamma_\mu \psi(x'), \]

(31) becomes as follows:

\[ [\psi(x), H(x')] = - \epsilon \mathcal{D}(x-x') \gamma_\mu \psi(x') \{ \mathcal{A}_\mu(x') + n_\mu \int n_\lambda j_\lambda(x') \vartheta'' \mathcal{D} (x''-x') d\sigma'' \}
- \frac{e}{2} S(x-x') \gamma_\lambda n_\lambda \psi(x') \int \vartheta'' \mathcal{D} (x''-x') S(x''-x') \gamma_\mu n_\mu d\sigma''. \]

The second term is related to the Coulomb static self energy of electrons. Then in order to eliminate such a term, we must adopt the symmetrical expression \( 1/2 \{ A_\mu(x'') \phi(x''') + \phi(x'') A_\mu(x''') \} \) instead of \( A_\mu(x') \phi(x'') \) in (12).

**§ 4. Conclusions**

We obtained the relativistically covariant formulation of quantum electrodynamics without subsidiary conditions starting from Heisenberg’s representation, by dividing the electromagnetic field potential into the transverse and other parts. The latter part has the connection to the Coulomb static potential, which has the ambiguities relating to the
difference of gauge in a wide sense. But these ambiguous parts can be vanished by the unitary transformation of the state vector in the interaction representation. By making use of the suitable Green's functions and putting the differential operator of the source function outside the integrals, the equations of motion in Heisenberg's representation are solved. And furthermore, the equivalence of this formalism to the one of the ordinary quantum electrodynamics introduced by Fermi was proved by transforming the formalism in Heisenberg's representation into the one in the interaction representation. But in order to avoid the term related to the Coulomb static self energy in the final interaction Hamiltonian, we must take the symmetrized interaction term in the expression of the interaction term between electrons and electromagnetic field potential.

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Mathematical appendix

Calculation of $D(x)$ and $\partial D(x)$ functions are performed as follows. As the integral of $D(x)$ function is expressed as follows:

$$D(x) = -\frac{i}{(2\pi)^3} \int (dk) \int_{-\infty}^{\infty} da \exp[iax_\mu + ik_\mu x_\mu] \epsilon(k), \quad (A.1)$$

$\partial D(x)$ is written in the following integral form:

$$\partial D(x) = \frac{1}{(2\pi)^3} \int (dk) \int_{-\infty}^{\infty} da \exp[iax_\mu + ik_\mu x_\mu] \epsilon(k). \quad (A.2)$$

The integral representations

$$\epsilon(k) = -\frac{\epsilon_\mu k_\mu}{|\epsilon_\mu k_\mu|} = \frac{i}{\pi} \int_{-\infty}^{\infty} \exp[i\epsilon_\mu k_\mu \tau] d\tau, \quad (A.3)$$

$$p \frac{1}{n_\mu k_\mu} = -\frac{i}{2} \int_{-\infty}^{\infty} \exp(in_\mu k_\mu a) \frac{d}{da} \frac{d}{da} \quad (A.4)$$

enable the integration over $k$ space to be effected:

$$\partial D(x) = \frac{-i}{(2\pi)^3} \int (dk) \int_{-\infty}^{\infty} \exp(ibn_\mu k_\mu db \int_{-\infty}^{\infty} \frac{dc}{c} \exp (ibc)$$

$$\times \int_{-\infty}^{\infty} da \exp[iak_\mu + i\epsilon_\mu k_\mu \tau] \int_{-\infty}^{\infty} d\tau \exp (ie_\mu k_\mu \tau) \exp (ibc).$$

If we put

$$k'_\mu = k_\mu + \frac{1}{2a} (x_\mu + bn_\mu + \epsilon_\mu \tau)$$
and use the following relation:
\[ \int (dk') \exp(iak'^2) = \frac{ia^2}{a|a|}, \]

we get
\[ \partial D(x) = \frac{1}{2(2\pi)^i} \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} dc \int_{-\infty}^{\infty} \frac{da}{c} \int_{-\infty}^{\infty} d\tau \exp \left[ -\frac{i}{4a} (x_\mu + bn_\mu + \epsilon_\mu \tau)^2 \right] \exp(ibc) \]
\[ = \frac{2}{(2\pi)^i} \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} dc \int_{-\infty}^{\infty} d\tau \exp(ibc) \int_0^{\infty} d\alpha \left\{ \exp \left[ -i\alpha (x_\mu + bn_\mu + \epsilon_\mu \tau)^2 \right] \right\} \exp \left[ i\alpha (x_\mu + bn_\mu + \epsilon_\mu \tau)^2 \right]. \]

If we perform the \( b' \) integration by using the relation
\[ \int_{-\infty}^{\infty} \exp(\pm iab'^2) = \sqrt{\frac{\pi}{\alpha}} \exp \left( \pm i \frac{\pi}{4} \right), \]
and putting
\[ b' = b - \left( n_\mu x_\mu + \tau n_\mu \epsilon_\mu - \frac{c}{2\alpha} \right), \]
we get
\[ \partial D(x) = 2 \sqrt{\frac{\pi}{(2\pi)^i}} \int_{-\infty}^{\infty} dc \int_{-\infty}^{\infty} d\tau \exp(ic(n_\mu x_\mu - \tau)) \times \]
\[ \times \int_0^{\infty} d\alpha \sqrt{\alpha} \left\{ \exp \left[ -i\alpha \left( x_\mu^2 + \left( n_\mu x_\mu \right)^2 + \frac{c^2}{4\alpha^2} \right) + i\frac{\pi}{4} \right] \right\} \]
\[ - \exp \left[ i\alpha \left( x_\mu^2 + \left( n_\mu x_\mu \right)^2 + \frac{c^2}{4\alpha^2} \right) - i\frac{\pi}{4} \right], \]

where we put \( \epsilon_\mu = n_\mu \), because \( \epsilon_\mu \) is any time-like vector. If we put \( \alpha = a^2 \) and use the next relation
\[ \int_0^{\infty} da \exp \pm i \left[ x^2 + \epsilon \frac{c^2}{4a^2} - \frac{\pi}{4} \right] = \sqrt{\frac{\pi}{2x}} \exp \left( \pm i\epsilon \right), \]
the following result is obtained:
\[ \partial D(x) = 4 \sqrt{\frac{\pi}{(2\pi)^i}} \int_{-\infty}^{\infty} dc \int_{-\infty}^{\infty} d\tau \exp(ic(n_\mu x_\mu - \tau)) \times \]
\[ \times \int_0^{\infty} da \left\{ \exp \left[ -i \left( x_\mu^2 + \left( n_\mu x_\mu \right)^2 \right) a^2 + \frac{c^2}{4a^2} \right] \right\} \cdot \exp \left( i\frac{\pi}{4} \right) \]
\[ - \exp \left[ i \left( x_\mu^2 + \left( n_\mu x_\mu \right)^2 \right) a^2 + \frac{c^2}{4a^2} \right] \exp \left( -i\frac{\pi}{4} \right) \]
\[
\frac{8i\pi}{2(2\pi)^4} \int_{-\infty}^{\infty} \frac{dc}{c} \sin \{ (n_{\mu} x_{\mu} - \tau) c \} \frac{1}{\sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2}} \exp \left[ -ic \sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2} \right] \\
- \exp \left[ ic \sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2} \right] 
\]
\[
= \frac{4\pi}{(2\pi)^3} \frac{1}{\sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2}} \int_{0}^{\infty} \frac{dc}{c} \cos (n_{\mu} x_{\mu} c) \sin (c \sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2}), 
\]
where the next relation was used:
\[
\int_{-\infty}^{\infty} \frac{d\tau}{\tau} \cos \tau c = 0, \quad \int_{0}^{\infty} \frac{d\tau}{\tau} \sin \tau c = \pi \quad (c > 0). 
\]
By making use of the following formula:
\[
\int_{0}^{\infty} \sin qx \cos px \frac{dx}{x} = \frac{\pi}{2} \quad \text{for} \quad q > p, \\
= 0 \quad \text{for} \quad q < p, \\
= \frac{\pi}{4} \quad \text{for} \quad q = p,
\]
we get the final result:
\[
\partial \mathcal{D} (x) = \frac{1}{4\pi} \frac{1}{\sqrt{x_{\mu}^2}} \quad \text{for} \quad \sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2} > n_{\mu} x_{\mu}, \text{ i.e. } x_{\mu}^2 > 0, \\
= 0, \quad \text{for} \quad \sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2} < n_{\mu} x_{\mu}, \text{ i.e. } x_{\mu}^2 < 0, \\
= \frac{1}{8\pi} \frac{1}{n_{\mu} x_{\mu}}, \quad \text{for} \quad \sqrt{x_{\mu}^2 + (n_{\mu} x_{\mu})^2} = n_{\mu} x_{\mu}, \text{ i.e. } x_{\mu}^2 = 0. 
\]
Here in the case of \( n_{\mu} x_{\mu} = 0, \) \( \partial \mathcal{D} (x) = \frac{1}{4\pi} \frac{1}{\sqrt{x_{\mu}^2}} \) expresses the covariant formulation of Coulomb potential.

Next we perform the explicit integration of \( \mathcal{D} (x) \) function similarly as before.
\[
\mathcal{D} (x) = \frac{i}{(2\pi)^4} \int \left( \frac{dk}{n_{\mu} k_{\mu}} \right)^2 \int_{-\infty}^{\infty} da \epsilon (k) \exp \left[ iak_{\mu}^2 + ik_{\mu} x_{\mu} \right] \\
= -\frac{1}{\pi (2\pi)^5} \int \left( \frac{dk}{n_{\mu} k_{\mu}} \right)^2 \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} dg \int_{-\infty}^{\infty} dc \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{da}{a} \\
\times \exp \left[ i(n_{\mu} k_{\mu} (b + c + \tau)) \right] \exp (b g + c h) \exp (iak_{\mu}^2 + ik_{\mu} x_{\mu}), 
\]
where we put \( \epsilon_{\mu} = n_{\mu}. \) Integrating over \( k_{\mu}, \) we get
\[
\mathcal{D} (x) = -\frac{i}{2(2\pi)^5} \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} dg \int_{-\infty}^{\infty} dc \int_{-\infty}^{\infty} dt \exp (b g + c h) \\
\times \int_{-\infty}^{\infty} \frac{da}{a} \exp \left[ -\frac{i}{4a} \{ x_{\mu} + (b + c + \tau) n_{\mu} \}^2 \right]
\]
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\[ \frac{2i}{(2\pi)^2} \int_0^{\infty} \int_0^{\infty} \frac{dg}{g} \frac{dc}{c} \frac{db}{b} \frac{d\tau}{\tau} \exp \left( i(g + cb) \right) \]

and integrating over \( b' \) after transforming \( b \), we obtain the following result:

\[ \mathcal{D}(x) = \frac{2 \sqrt{\pi} i}{(2\pi)^3} \int_0^{\infty} \int_0^{\infty} \frac{dg}{g} \frac{dc}{c} \frac{db}{b} \frac{d\tau}{\tau} \exp \left( i(gn_x + cb - (c + \tau) g) \right) \times \]

\[ \times \left\{ \exp \left[ i \left( \alpha (x_\mu^2 + (n_x x_\mu)^2) + \frac{g^2}{4\alpha} - \frac{\pi}{4} \right) \right] - \exp \left[ \alpha (x_\mu^2 + (n_x x_\mu)^2) + \frac{g^2}{4\alpha} - \frac{\pi}{4} \right] \right\} \]

\[ = \frac{4i}{(2\pi)^4} \frac{1}{\sqrt{x_\mu^2 + (n_x x_\mu)^2}} \left\{ \int_0^{\infty} \frac{dg}{g} \int_0^{\infty} \frac{dc}{c} \frac{db}{b} \frac{d\tau}{\tau} \sin (g \sqrt{x_\mu^2 + (n_x x_\mu)^2}) \sin (g (n_x x_\mu - c - \tau)) \times \exp (ich) \right\} \]

\[ = \frac{4\pi i}{(2\pi)^4} \frac{1}{\sqrt{x_\mu^2 + (n_x x_\mu)^2}} \left\{ \int_0^{\infty} \frac{dg}{g} \int_0^{\infty} \frac{dc}{c} \frac{db}{b} \exp (ich) \cos g (n_x x_\mu - c - \tau) \sin g \sqrt{x_\mu^2 + (n_x x_\mu)^2} \right\} \]

where we used the next formula:

\[ \int_0^{\infty} \frac{d\tau}{\tau} \sin g (n_x x_\mu - c - \tau) = -\pi \cos g (n_x x_\mu - c). \]

Making use of the following relation:

\[ \int_0^{\infty} \frac{dc}{c} \exp (ich) \cos g (n_x x_\mu - c) = \pi \{ \delta (b - g) \exp (ign_x) + \delta (b + g) \exp (-ign_x) \}, \]

we get

\[ \mathcal{D}(x) = \frac{1}{(2\pi)^2} \frac{1}{\sqrt{x_\mu^2 + (n_x x_\mu)^2}} \left\{ \int_0^{\infty} \frac{dg}{g} \sin g \sqrt{x_\mu^2 + (n_x x_\mu)^2} \sin gn_x x_\mu \right\} \]

\[ = -\frac{1}{4\pi} \frac{n_x x_\mu}{\sqrt{x_\mu^2 + (n_x x_\mu)^2}} \text{ for } \sqrt{x_\mu^2 + (n_x x_\mu)^2} \geq n_x x_\mu, \ x_\mu^2 > 0 \]

\[ = -\frac{1}{4\pi} \text{, for } \sqrt{x_\mu^2 + (n_x x_\mu)^2} \leq n_x x_\mu, \ x_\mu^2 < 0 \] \hspace{1cm} (A·6)

where we used the next formula:

\[ \int_0^{\infty} \frac{dg}{g} \sin gX \sin gN = \frac{1}{2} \pi X, \text{ if } N \geq X, \]

\[ = \frac{1}{2} \pi N, \text{ if } N \leq X. \]

If we operate \( \partial \) to \( \mathcal{D}(x) \) expressed by (A·6), it is easily proved that \( \partial \mathcal{D}(x) \) satisfies the conditions (A·5) and next relation holds

\[ (\partial_\mu + n_\mu \partial) \mathcal{D}(x) = 0 \]

in the case of \( n_\mu x_\mu = 0 \).
References

   S. T. Ma, Phys. Rev. 75 (1949), 535; 80 (1950), 729.
   J. Schwinger, Phys. Rev. 74 (1948), 1439; 75 (1949), 651.

6) Glauber, see Pauli’s Ausgewählte Kapitel aus der Feldquantisierung.
8) see Appendix.
9) loc. cit. 6).