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PARAMETER UNCERTAINTY PROPAGATION ANALYSIS FOR URBAN RAINFALL RUNOFF MODELLING

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ABSTRACT

The paper proposes a strategy for model uncertainty propagation analysis. As an example, parameter uncertainty propagation analysis in the runoff block of the HYSTEM-EXTRAN model is carried out. The model is a modification of the well-known SWMM (Storm Water Management Model). Uncertainty propagation methods such as first-order analysis, sensitivity analysis, statistical linearization and Monte-Carlo analysis are discussed and applied. A pathway of parameter uncertainty propagation analysis is given based on validity, simplicity, and computational requirements. The pathway starts with sensitivity analysis which may help to reduce the dimensions of a multidimensional model by discarding insensitive parameters. This is to obtain a mathematically tractable uncertainty propagation problem for a complicated model. Then, the nonlinearity of the model must be quantified to check the validity of first-order analysis. If first-order analysis is not valid, and if components of model output uncertainty need to be known, the application of statistical linearization is the only analytical alternative. Monte-carlo analysis can always be applied and taken as a reference as long as the components of the model output uncertainty are not of interest. The parameter sensitivity is characterized by its sensitivity coefficient which is defined as the ratio of the coefficient of variance of a model output to the coefficient of variance of the model parameter itself. A non-linear rainfall runoff model usually results in a variable parameter sensitivity. Hence, recommendations about parameter sensitivity cannot be generalized for a given rainfall-runoff model, but depend on the type and the range of the model output variable. It is shown that the type of probability density function describing the parameter uncertainty with known mean and variance has only a small effect on the results of the model output uncertainty.

KEYWORDS

Model uncertainty analysis; model output uncertainty; uncertainty propagation; sensitivity coefficient; sensitivity analysis; first-order analysis; statistical linearization; Monte-Carlo analysis; probability density function; coefficient of variance.

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INTRODUCTION

A model is the mathematical description of a natural (physical, chemical, biological, etc.) process. The mathematical description can consist of equations, graphics, tables, and logical expressions. Because a model is a simplification of reality, it cannot describe all the relevant variables of the process precisely. Hence, model output uncertainty (the deviation between the model output and the process that the model should describe) is inevitable. Whether a model can be trusted or not depends on the acceptability of the model output uncertainty.

Model output uncertainty results from many sources of uncertainty such as input data uncertainty, parameter uncertainty, model structure uncertainty, and uncertainty due to undetected numerical problems. Uncertainty propagation analysis predicts model output uncertainty as a result of these sources if their a priori levels of uncertainty are given. Model uncertainty analysis involves identification and quantification of the sources of uncertainty as well as the propagation of the uncertainty through the model (Fig. 1).

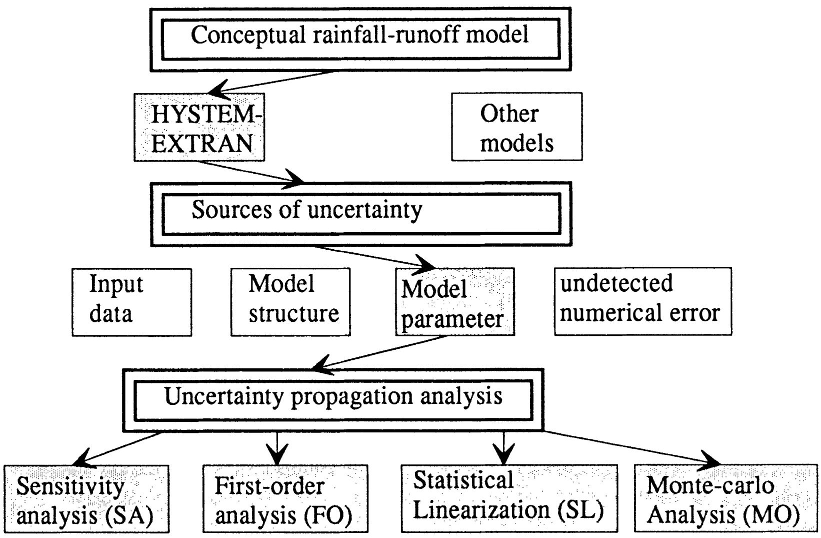

Fig. 1. Scheme of model uncertainty analysis. Hatched items have been accessed.

Applications of uncertainty propagation methods such as sensitivity analysis, first-order analysis and Monte-Carlo analysis can be found in the works of Burges et al. (1975), Mclaughlin (1983), Brown (1987, 1991) with respect to water quality models. Kitanidis et al. (198●) carried out a comprehensive analysis of uncertainty of the National Weather Service water quality model. Garen et al. (1981) investigated the effects of parameter uncertainty propagation through a twelve-parameter simplified version of Stanford Watershed Model. Kuczera (1988) investigated the validity of first-order analysis. In this paper a systematic uncertainty propagation pathway is described which can be applied for different cases and different tasks. The proposed pathway is illustrated with a parameter uncertainty analysis for a conceptual rainfall-runoff model.

A model requires parameters, each of them either having a well defined physical meaning (e. g. acceleration of gravity g), or referring to some kind of abstraction of the process (e. g. linear storage coefficient k). In the latter case the exact value of the parameter is unknown, and very often it cannot directly be measured. Hence the parameter might be described as a random variable. A probability density function can e used

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as a descriptive statistic to measure its uncertainty (i. e. probability of deviation from expected value). Uncertainty propagation methods then predict the model output uncertainty due to a given parameter uncertainty. Parameter uncertainty propagation analysis can also indicate which parameter should receive most of the effort in the estimation process, and give an indication on the robustness of the model at hand.

UNCERTAINTY PROPAGATION METHODS

If a rainfall-runoff model output Y (random variable) is a function of *an n-dimensional independent random vector* $\bar{X}$, $f(\bar{X})$, the essence of uncertainty propagation analysis is to explore the statistical properties of Y based on the statistical properties of $\bar{X}$. Generally, the probability density function of Y cannot be derived analytically from a-priori assumptions of the statistical properties of $\bar{X}$. Fortunately, it is often acceptable for engineering purposes to know the first and second moments of Y (i. e., the mean E[•] and the variance Var[•]). If the transfer function f(•) is linear, Benjamin and Cornell (1970) showed as follow:

$$Y = f(\bar{X}) = f(X_1, X_2, \cdots, X_n) = \sum_{i=1}^{n} \alpha_i X_i \quad (1)$$

$$E[Y] = \sum_{i=1}^{n} \alpha_i E[X_i] \quad (2)$$

$$Var[Y] = \sum_{i=1} \alpha_i^2 Var[X_i] \quad (3)$$

As in many cases f(•) is a nonlinear function, the common procedure to deal with the nonlinear problem is either to apply a linearization technique (first-order analysis or statistical linearization) or to apply Monte-Carlo analysis.

First Order Analysis

First-order analysis is derived from Taylor's linear approximation around the mean $E[\bar{X}]$ in which nonlinear components are truncated.

$$Y = f(\bar{X}) \cong f(E[\bar{X}]) + \frac{\partial f(\bar{X})}{\partial \bar{X}} (\bar{X} - E[\bar{X}]) \quad (4)$$

$$E[Y] \cong f(E[\bar{X}]) \quad (5)$$

$$Var[Y] \cong \sum_{i=1}^{n} \left(\frac{\partial f(\bar{X})}{\partial \bar{X}} \right)^2 Var[X_i] \quad (6)$$

One of the advantages of first-order analysis is that it allows the partitioning of the model output uncertainty into its sources which are the various contributions of the components of $\bar{X}$. Obviously, this is important information for model users. Another important attraction of first-order analysis is its simplicity. From equation 4, it can be seen that first-order analysis requires the existence of the partial derivative vector $\partial f(\bar{X})/\partial \bar{X}$ at the mean $E[\bar{X}]$. The validity of first-order analysis depends on the nonlinearity of f(•) in the region around $E[\bar{X}]$. Cornell (1972) suggested that first-order analysis is applicable to moderately nonlinear systems provided that the coefficient of variance of the parameter does not exceed about 0.2. Kuczera (1988) utilized Beale's nonlinearity measure to examine the validity of first-order analysis. Rainfall-runoff models often involve severe nonlinearity, such as saturation, threshold values, etc., where partial derivatives do not exist. It is impossible to handle such problems with first-order analysis. An alternative is statistical linearization (Gelb, 1978).

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Statistical Linearization

Suppose the linear approximation around $E[\bar{X}]$ for a nonlinear function f(•) of a vector random variable $\bar{X}$ is

$$Y = f(\bar{X}) \cong \beta + \bar{N}\,\bar{X} \qquad (7)$$

The parameters β and $\bar{N}$ are a constant and a vector, respectively, and have to be determined. Defining the error vector

$$\bar{e} = f(\bar{X}) - (\beta - \bar{N}\,\bar{X}) \qquad (8)$$

β and $\bar{N}$ shall be chosen so that

$$J = E[\bar{e}^{\,T} A\, \bar{e}] \qquad (9)$$

is minimized for some semi-definite matrix A. We have

$$\beta = E[f(\bar{X})] - \bar{N}\,E[\bar{X}] \qquad (10)$$

$$\bar{N} = \left(E[f(\bar{X})\,\bar{X}^{T}] - E[f(\bar{X})]\,E[\bar{X}]^{T}\right) P^{-1} \qquad (11)$$

where P is the co-variance matrix for $\bar{X}$. Therefore, statistical linearization of f(•) can be expressed as

$$f(\bar{X}) \cong E[f(\bar{X})] + \bar{N}\,(\bar{X} - E[\bar{X}]) \qquad (12)$$

As depicted in Fig. 2, statistical linearization is an optimum linear approximation in the sense of the minimum mean square error.

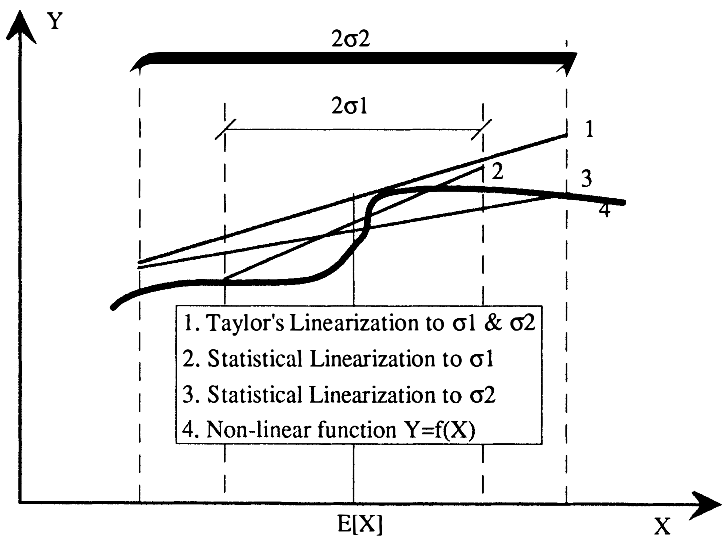

Fig. 2 Nonlinear single-degree freedom system (i. e. one independent variable X) and its linearization. E[X] is the mean value of random variable X; σ_1 and σ_2 are standard deviations of X ($\sigma_2 > \sigma_1$).

Therefore, statistical linearization can account for the variance of a random variable. Furthermore, it does not require the existence of partial derivatives. It also allows partitioning of the output uncertainty into its

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sources. On the other hand, statistical linearization demands the n-dimensional probability density function of $\overline{X}$ and much more computational effort, since numerical integration must be carried out in most cases. It is demonstrated below that the a-priori assumption on the probability density function is not too restrictive. Thus, computing time considerations become the main concern in the application of this method. An alternative to the linearization techniques above is the statistical tool of Monte-Carlo analysis.

Monte-Carlo Analysis

Roberts et al. (1990) pointed out that the theoretical foundation of Monte-Carlo analysis is associated with the fact that stochastic dynamic equations governing a system can be interpreted as an infinite set of deterministic equations. For each element of this set, the system input is a sample function of the input process, and the system output is the corresponding sample function of the response process. Because Monte-Carlo analysis is not restricted to nonlinear and discontinuous systems, it is a preferred method for dealing with any uncertainty propagation problem. Because of its general applicability and flexibility, it can be used as a reference method with which results from other methods are compared. In order to compute the statistics of both model inputs and outputs, a great number of iterations are needed. A generally valid number of iterations can hardly be given. Hence, the feasibility of Monte-Carlo analysis mainly depends on how heavy the computational burden becomes. Furthermore, Monte-Carlo analysis requires knowledge of the joint probability density function of the input variables (as with statistical linearization). With Monte-Carlo analysis a lumped response uncertainty is obtained, i. e. the individual contributions of each parameter remain unknown.

Sensitivity Analysis

Sensitivity analysis is used to examine the sensitivity of the model output with respect to perturbations of a specific parameter. Brown (1987) documented that sensitivity analysis is a "one-variable-at-a-time" approach. It gives an indication of where most effort should be concentrated in the parameter estimation procedure. Apparently, the emphasis in parameter estimation should depend on the questions the user is attempting to answer with the model. In other words, the sensitivity of a parameter can differ from task to task. The information obtained from sensitivity analysis is particularly useful if the analyst wants to reduce the dimensions of a multidimensional model. This might be necessary in order to obtain a mathematically tractable uncertainty propagation problem for a complicated model.

PATHWAY FOR UNCERTAINTY PROPAGATION

Based on the discussion above, a pathway for parameter uncertainty propagation analysis is recommended as shown in Fig. 3.

APPLICATION EXAMPLE

As an application example, the runoff block of the HYSTEM-EXTRAN model (Fuchs & Verworn, 1988) is applied to the Halden subcatchment of Fehraltorf near Zurich in Switzerland. The demonstration is restricted to the model uncertainty due to parameter uncertainty. Only the surface runoff computed for impervious areas is investigated. In order to quantify parameter uncertainty, questionnaires regarding parameter mean and range have been sent to international experts. Based on some responses (showing a surprising diversity), statistical descriptors of the parameters are assumed to be normally distributed (Normal PDF) with means and coefficients of variance (CV) summarized in Table 1. ANTUNA stands for percentage of the area which contributes to runoff *at the beginning* of the storm event. ANTUNE is the percentage of the area which contributes to runoff *at the end* of the storm event, VBENU is the wetting loss and VMULDU is the depression storage.

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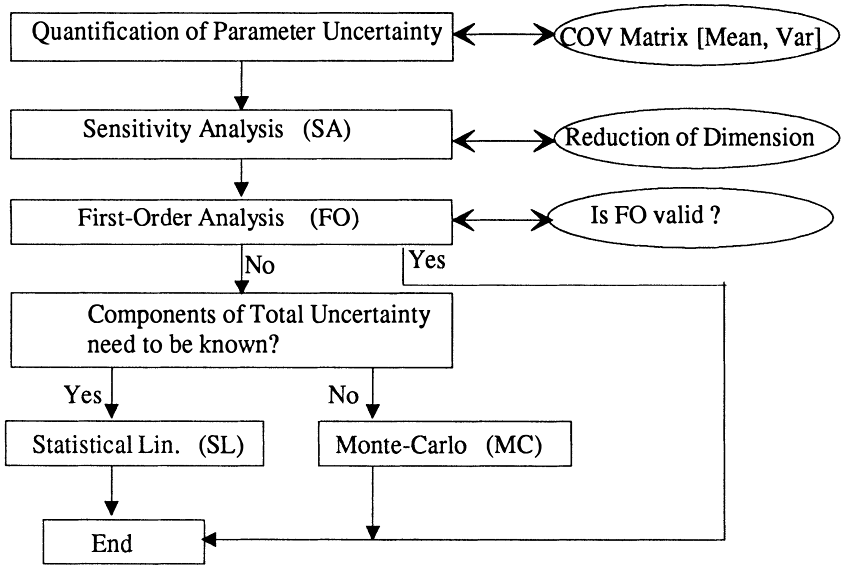

Fig. 3 Parameter uncertainty propagation analysis pathway

TABLE 1 Parameter mean values and coefficients of variation (CV)

Normal PDF	ANTUNA	ANTUNE	VBENU (mm)	VMULDU (mm)
Mean	25%	85%	0.7	1.8
CV	0.4	0.088	0.44	0.5

Sensitivity analysis is carried out from using the event of 28. 06. 1990. The ratio of the coefficient of variance of the model output to the coefficient of variance of a parameter itself is defined here as the sensitivity coefficient. Three outputs (runoff volume, peak flow, and time to peak) are investigated. From Fig. 4, it can be concluded that only the parameter ANTUNE (i. e. percentage of the area which contributes to runoff *at the end* of the storm event) is very sensitive to runoff volume and peak flow.

To evaluate the propagation of parameter uncertainty, the runoff uncertainty of eight historical events was calculated using statistical linearization, Monte-Carlo analysis, and first-order analysis. From Fig. 5 it can be concluded that, firstly, runoff uncertainty resulting from loss parameter uncertainty decreases with increasing rainfall depth and, secondly, the three methods statistical linearization, first-order analysis, and Monte-Carlo analysis show relatively good agreement. Apparently, the smaller the parameter uncertainty (CV), the better agreement of the methods. Needless to say, that first-order analysis would be recommended here because of its simplicity. Fig. 6 shows that except for events with small rain depths runoff uncertainty is practically only dominated by the parameter ANTUNE. (The same conclusion has been drawn from sensitivity analysis). Also we see that the sensitivity of the parameter depends on the type and the range of the output variable.

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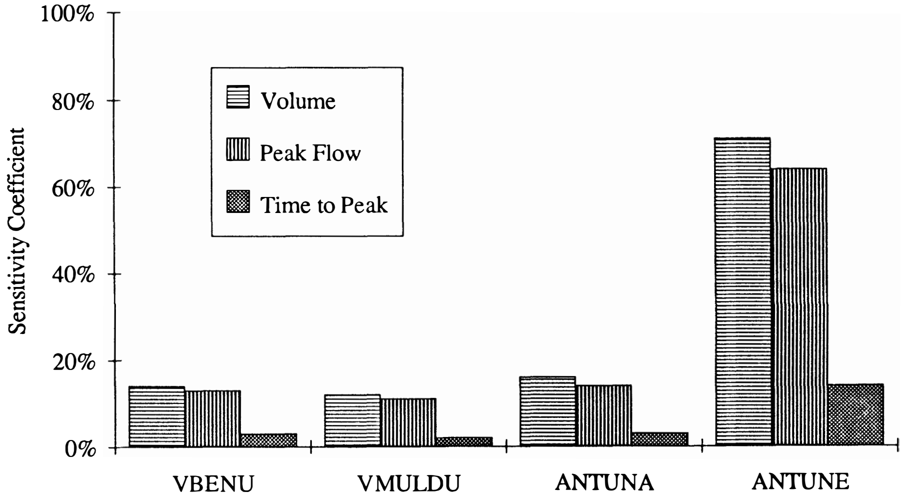

Fig. 4. Sensitivity coefficient (SC) of the parameters VBENU, VMULDU, ANTUNA, ANTUNE, and ALPHU with respect to runoff volume, peak flow and time to peak.

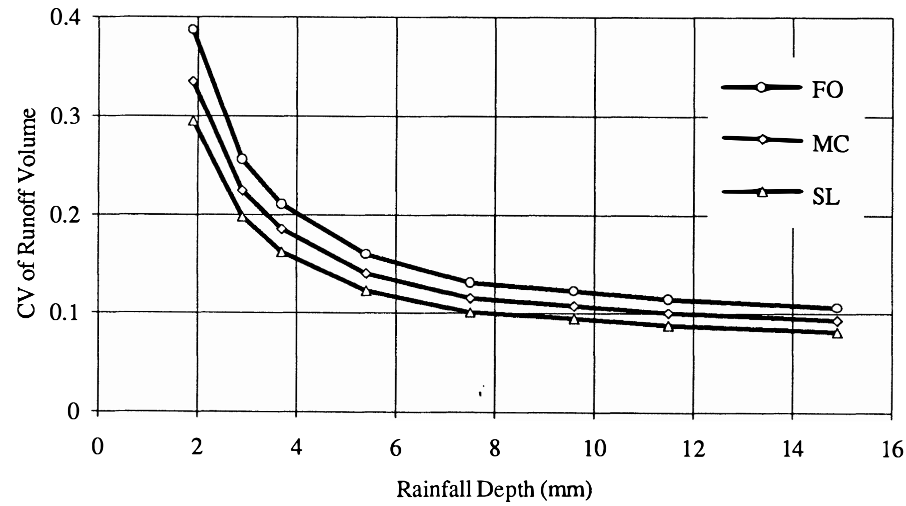

Fig. 5. Coefficient of variance (CV) of runoff volume resulted from the uncertainty of loss parameters versus rainfall depth (mm). Comparison of first-order analysis (FO), statistical linearization (SL), and Monte-Carlo analysis (MC) parameter uncertainty propagation analysis methods.

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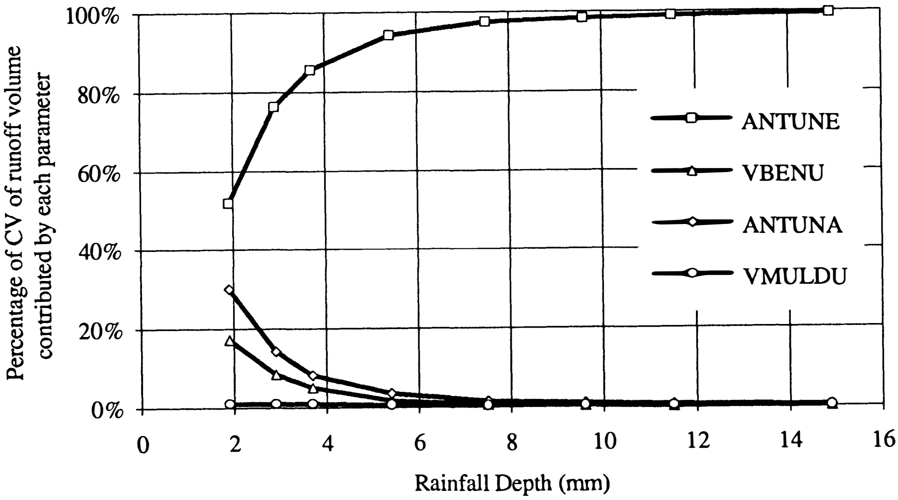

Fig. 6. Coefficient of variance (CV) of runoff volume partitioned into the contributions of each parameter (in %) versus rainfall depth (mm).

QUANTIFICATION OF PARAMETER UNCERTAINTY

In practice, it is difficult to specify the probability density function of a parameter which is necessary for the application of Monte-Carlo analysis and statistical linearization. However, estimating the mean value and the variance of a parameter seems to be more practical. Mean value and variance are two most important properties of a random variable, describing the most likely value and its scatter. Engineers are familiar with such descriptive statistics in that they are used to estimate a range within which a variable lies. Additionally, a shape of the probability density function can be assigned based on experience. Basically, using a non-uniform probability density function indicates that the true value of the parameter is more likely to lie in certain portions of the possible range than in other portions. As mean value and variance contain the most important statistical information, it can be expected that the actual shape of the probability density function is of minor importance as long as the mean and variance are estimated correctly. The results of this study support this hypothesis. Six probability density functions (i. e. Uniform, Normal, Log-Normal, Triangular, Gamma, Gumbel) with equal mean and variance, respectively, were assumed for the two most sensitive parameters ANTUNE and ALPHU. By simulating the event of 28. 06. 1990, Fig. 7 and Fig. 8 highlight that the computed uncertainty propagation does not necessarily strongly depend on the type of probability density function of the parameter.

FURTHER RESEARCH

Modelling (or simulation) is increasingly applied in almost every engineering field. It is also widely acknowledged that the model uncertainty is ubiquitous. It is the authors' view that any model application has to be accompanied with model uncertainty analysis because it is difficult to evaluate model validity and improve model performance without knowing model uncertainty. After further research in this field, it is expected that a general procedure could be proposed that allows to carry out model uncertainty analysis for

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urban rainfall-runoff modelling. Its components would be: the identification of sources of uncertainty, the quantification of identified sources of uncertainty as well as the propagation of uncertainty. As a result, it should be possible to show how to use the model uncertainty analysis as the criteria for choosing the most procedure for model uncertainty propagation are:

- uncertainty propagation of model inputs;
- uncertainty propagation of non-independent model inputs and parameters;
- criteria to measure the nonlinearity of the model.

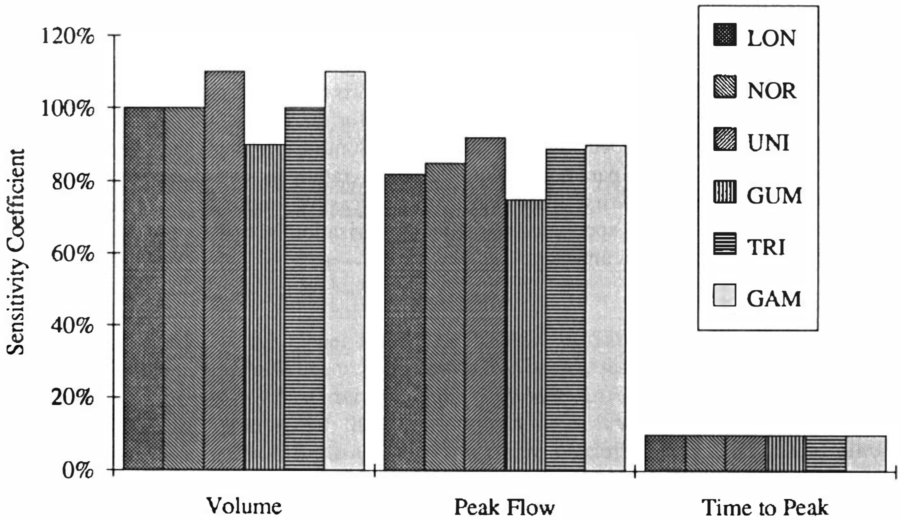

Fig. 7. Sensitivity coefficient (SC) of the parameter ANTUNE with respect to volume, peak flow and time to peak versus six probability density functions with the same mean and coefficient of variance (CV) as ANTUNE.

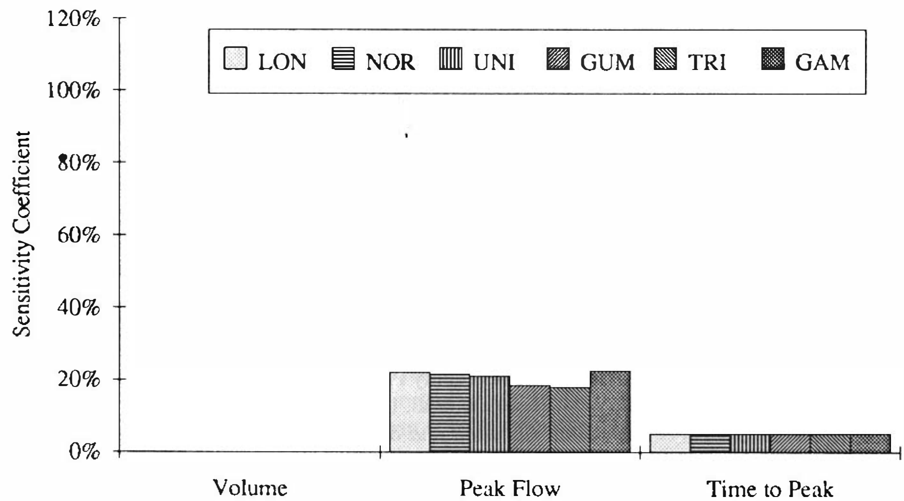

Fig. 8. Sensitivity coefficient (SC) of the parameter ALPHU with respect to volume, peak flow and time to peak versus six probability density functions with the same mean and coefficient of variance (CV) as ALPHU.

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CONCLUSIONS

In the runoff block of the HYSTEM-EXTRAN model the parameters "wetting loss of impervious areas" VBENU, "depression storage of impervious areas" VMULDU, and "percentage of the impervious area which contributes to runoff *at the beginning* of the storm event", ANTUNA are insensitive. The sensitivity coefficients for peak flow, time to peak, and runoff volume are around 10%. Only the parameter "percentage of the impervious area which contributes to runoff *at the end* of the storm event" ANTUNE is sensitive. Here, the sensitivity coefficient to runoff volume and peak flow is about 80%, but with respect to time to peak there is little sensitivity. First-order analysis is the recommended method for computing the uncertainty propagation for the surface runoff component of the HYSTEM-EXTRAN model because this method appears to be both valid and simple. The type of the probability density function that characterizes the parameter uncertainty has little effect on the results of the uncertainty propagation analysis, provided the mean values and variances of the parameters are known. This conclusion supports the application of Monte-Carlo analysis and statistical linearization. Monte-Carlo analysis is applicable for the estimation of response statistics of any nonlinear and/or discontinuous model. A disadvantage of Monte-Carlo analysis is that it cannot show the uncertainty contributions of each parameter. Numerical integration in statistical linearization requires substantial computational effort, but if first-order analysis is not feasible, and if the uncertainty contribution of each parameter is required to be specified, statistical linearization seems to be the only applicable alternative for uncertainty propagation analysis.

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