

12 Lee, J. S., Frasher, W. G., and Fung, Y. C. B., "Two-Dimensional Finite Deformation Experiments on Dog's Arteries and Veins," AFOSR Sci. Rep. 67-1980, Defense Documentation Center, 1967.

13 Fung, Y. C. B., "Elasticity of Soft Tissues in Simple Elongation," *American Journal of Physiology*, Vol. 213, 1967, pp. 1532-1544.

14 Wiederhielm, C. A., "Distensibility Characteristics of Small Blood Vessels," *Federation Proceedings*, Vol. 24, 1965, pp. 1075-1084.

15 Oden, J. T., "Finite Plane Strain of Incompressible Elastic Solids by the Finite Element Method," *The Aeronautical Quarterly*, Vol. 19, Aug. 1968, pp. 254-264.

16 Oden, J. T., "Finite Element Formulation of Problems of Finite Deformation and Irreversible Thermodynamics of Nonlinear Continua—A Survey and Extension of Recent Developments," Proc. of U. S.-Japan Seminar on Matrix Methods of Structural Analysis and Design, Alabama University Press, 1970.

17 Oden, J. T., and Key, J. E., "Numerical Analysis of Finite Axisymmetric Deformations of Incompressible Elastic Solids of Revolution," to be published.

18 Bergel, D. H., "The Static Elastic Properties of the Arterial Wall," *Journal of Physiology*, Vol. 156, 1961, pp. 445-457.

19 Patel, D. J., Janicki, J. S., and Carew, T. E., "Static Anisotropic Properties of the Aorta in Living Dogs," *Circulation Research*, Vol. 25, 1969, pp. 765-775.

20 Opatowski, I., "Elastic Deformation of Arteries," Comm. on Math. Biol., Univ. of Chicago, Jan. 1965.

21 Opatowski, I., "Elastic Deformation of Arteries," *Journal of Applied Physiology*, Vol. 23, No. 5, 1967, pp. 772-778.

22 Fung, Y. C. B., "Theoretical Considerations of the Elasticity of Red Cells and Small Blood Vessels," *Federation Proceedings*, Vol. 25, No. 6, 1966, pp. 1761-1772.

23 Fung, Y. C. B.: "Microscopic Blood Vessels in the Mesentery," *Biomechanics*, Y. C. B. Fung (ed.), ASME publication, 1966.

24 Fung, Y. C. B., Zweifach, B. W., and Intaglietta, M., "Elastic Environment of the Capillary Bed," *Circulation Research*, Vol. 19, 1966, pp. 441-461.

## APPENDIX

### Basic Finite Element Theory

The basis for finite element analysis is the principle of minimum potential energy which states that, for a given state of displacements and strains, the equilibrium conditions are satisfied when the total potential energy associated with that state is a minimum. A plane structure to be analyzed is considered to be divided into an assemblage of finite elements (triangles) connected at nodes (corners). Using the notation of Oden [15], the total potential energy  $V$  for one finite element triangle with zero body forces is

$$V = \int_{v_0} W dv_0 - \int_s S_\alpha u_\alpha ds \quad \alpha = 1, 2 \quad (\text{A-1})$$

where  $v_0$  = volume of undeformed element and  $S_\alpha$  = components of surface traction per unit of deformed surface area,  $s$ .

Assuming that the displacement field  $u_\alpha$  may be approximated by a linear function in each triangular element yields

$$u_\alpha \doteq (k_N + C_{\beta N} x_\beta) u_{N\alpha} \quad N = 1, 2, 3; \alpha, \beta = 1, 2 \quad (\text{A-2})$$

where  $k_N$ ,  $C_{\beta N}$  = constants dependent on the undeformed area and initial nodal coordinates and  $u_{N\alpha}$  are the nodal displacements for each triangle.

In terms of nodal displacements the strains become constant in each finite element and are given by

$$2e_{\alpha\beta} = C_{\alpha N} u_{N\beta} + C_{\beta N} u_{N\alpha} + C_{\alpha N} C_{\beta M} u_{N\gamma} u_{M\gamma} \quad N, M = 1, 2, 3 \quad (\text{A-3})$$

$$\alpha, \beta, \gamma = 1, 2$$

$$2e_{\alpha\alpha} = 0 \quad (\text{A-4})$$

$$2e_{33} = \lambda^2 - 1 \quad (\text{A-5})$$

The total potential energy is

$$V = v_0 W - P_{N\alpha} u_{N\alpha} \quad (\text{A-6})$$

where  $P_{N\alpha}$  = generalized nodal forces (corresponding to  $u_{N\alpha}$ ) given by

$$P_{N\alpha} = \int_s S_\alpha (k_N + C_{\beta N} x_\beta) ds \quad (\text{A-7})$$

For incompressible materials  $V$  is to be minimized subject to the constraint

$$J - 1 = |(\delta_{ij} + 2e_{ij})| - 1 = 0 \quad i, j = 1, 2, 3 \quad (\text{A-8})$$

Multiplying the above constraint by a Lagrange multiplier  $v_0 h$ , and adding the result to  $V$  yields a functional  $V^*$

$$V^* = v_0 W - P_{N\alpha} u_{N\alpha} + v_0 h (J - 1) \quad (\text{A-9})$$

which will be a minimum when

$$\delta V^* = \frac{\partial V^*}{\partial u_{N\alpha}} \delta u_{N\alpha} = 0 \quad (\text{A-10})$$

Equations (A-6) and (A-10) imply that

$$P_{N\alpha} = v_0 \left( \frac{\partial W}{\partial u_{N\alpha}} + h \frac{\partial J}{\partial u_{N\alpha}} \right) \quad (\text{A-11})$$

The foregoing represents the equilibrium requirements for a single finite element. The incompressibility constraint (A-8) implies that the initial volume  $v_0$  and the final volume  $v$  of the finite element are equal, i.e.,

$$v - v_0 = la(u_{N\alpha}) - l_0 a_0 = 0 \quad (\text{A-12})$$

where  $a_0$  and  $a$  are the initial and final areas of the triangle, respectively. Now

$$\lambda = \frac{l}{l_0} \quad (\text{A-13})$$

and

$$2a_0 = (x_{11}x_{22} + x_{21}x_{32} + x_{31}x_{12}) - (x_{21}x_{12} + x_{31}x_{22} + x_{11}x_{32}) \quad (\text{A-14})$$

$$2a(u_{N\alpha}) = (y_{11}y_{22} + y_{21}y_{32} + y_{31}y_{12}) - (y_{21}y_{12} + y_{31}y_{22} + y_{11}y_{32}) \quad (\text{A-15})$$

where  $y_{N\alpha} = x_{N\alpha} + u_{N\alpha}$  and  $x_{N\alpha}$  = the initial nodal coordinates. Hence with the above (A-12) is of the form

$$a(u_{N\alpha}) - \frac{1}{\lambda} a_0 = 0 \quad (\text{A-16})$$

## DISCUSSION

### R. N. Vaishnav<sup>3</sup>

The authors of this paper have solved the problem of evaluating the distribution of tangential and radial stresses in an artery treated as a thick-walled pressurized circular cylinder constrained to a fixed length. The arterial wall has been considered to be composed of an incompressible, orthotropic, nonlinearly elastic material.

The authors show that the actual distribution of the tangential stress is far from being uniform and that even the Lamé solution (valid for isotropic, linearly elastic case) severely underestimates its maximum value. This conclusion is valid and stems from the facts that the wall material becomes increasingly stiffer with strain and the strain increases as one progresses from the outer to the inner surface. This important fact should be borne in mind when calculating stresses using a theory based on assuming the wall as thin. However, we may note that the cor-

<sup>3</sup> Professor of Mechanics, The Catholic University of America, Washington, D.C.

responding error in the constitutive constants is not large; the exponent  $k$  in equation (26) is unaffected and the coefficient  $A$  is affected only slightly as seen in Fig. 3 by comparing the solid curves with the dashed ones. This is so because of the integration in equation (20) through which the stresses affect the constitutive parameters.

Finally, attention should be drawn to the fact that, whereas equations (1) and (7) are correct starting points, equation (26) is valid only for a given longitudinal extension. A complete characterization of the wall material would involve postulation and experimental validation of a constitutive relation involving both the circumferential and longitudinal extension ratios as variables. We have developed such a constitutive relation

---

<sup>4</sup>Young, John T., "Determination of Constitutive Constants of Canine Aorta Under Large Deformations," Master's thesis, The Catholic University of America, Apr. 1970.

which was published as a Master's thesis<sup>4</sup> and will be detailed in a forthcoming publication.<sup>5</sup>

### Authors' Closure

The authors wish to thank Professor Vaishnav for emphasizing some of the salient but important findings of this paper. It should be noted that both the coefficient  $A$  and the exponent  $k$  in equation (26) were adjusted in determining the mechanical response for other aortas tested and for data reported in references [12 and 14]. The thick-walled pressure-radius response and tangential stress distribution show considerable deviations from the corresponding thin-walled tube results. These results will appear in a future publication.

---

<sup>5</sup>Patel, Dali J., and Vaishnav, Ramesh, N., "Rheology of Large Blood Vessels," in *Handbook of Cardiovascular Fluid Dynamics*, edited by Bergel, D. H., Academic Press, London, 1971.