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APPENDIX

Basic Finite Element Theory

The basis for finite element analysis is the principle of minimum potential energy which states that, for a given state of displacements and strains, the equilibrium conditions are satisfied when the total potential energy associated with that state is a minimum. A plane structure to be analyzed is considered to be divided into an assemblage of finite elements (triangles) connected at nodes (corners). Using the notation of Oden [15], the total potential energy V for *one* finite element triangle with zero body forces is

$$V = \int_{v_0} W dv_0 - \int_s S_\alpha u_\alpha ds \qquad \alpha = 1, 2 \qquad \text{(A-1)}$$

where v_0 = volume of undeformed element and S_α = components of surface traction per unit of deformed surface area, s.

Assuming that the displacement field u_α may be approximated by a linear function in each triangular element yields

$$u_\alpha \doteqdot (k_N + C_{\beta N} x_\beta) u_{N\alpha} \qquad N = 1, 2, 3; \alpha, \beta = 1, 2 \qquad \text{(A-2)}$$

where k_N, $C_{\beta N}$ = constants dependent on the undeformed area and initial nodal coordinates and $u_{N\alpha}$ are the nodal displacements for each triangle.

In terms of nodal displacements the strains become constant in each finite element and are given by

$$2e_{\alpha\beta} = C_{\alpha N} u_{N\beta} + C_{\beta N} u_{N\alpha} + C_{\alpha N} C_{\beta M} u_{N\gamma} u_{M\gamma} \qquad N, M = 1, 2, 3; \quad \alpha, \beta, \gamma = 1, 2 \qquad \text{(A-3)}$$

$$2e_{\alpha 3} = 0 \qquad \text{(A-4)}$$

$$2e_{33} = \lambda^2 - 1 \qquad \text{(A-5)}$$

The total potential energy is

$$V = v_0 W - P_{N\alpha} u_{N\alpha} \qquad \text{(A-6)}$$

where $P_{N\alpha}$ = generalized nodal forces (corresponding to $u_{N\alpha}$) given by

$$P_{N\alpha} = \int_s S_\alpha (k_N + C_{\beta N} x_\beta) ds \qquad \text{(A-7)}$$

For incompressible materials V is to be minimized subject to the constraint

$$J - 1 = |(\delta_{ij} + 2e_{ij})| - 1 = 0 \qquad i, j = 1, 2, 3 \qquad \text{(A-8)}$$

Multiplying the above constraint by a Lagrange multiplier $v_0 h$, and adding the result to V yields a functional V^*

$$V^* = v_0 W - P_{N\alpha} u_{N\alpha} + v_0 h (J - 1) \qquad \text{(A-9)}$$

which will be a minimum when

$$\delta V^* = \frac{\partial V^*}{\partial u_{N\alpha}} \delta u_{N\alpha} = 0 \qquad \text{(A-10)}$$

Equations (A-6) and (A-10) imply that

$$P_{N\alpha} = v_0 \left(\frac{\partial W}{\partial u_{N\alpha}} + h \frac{\partial J}{\partial u_{N\alpha}} \right) \qquad \text{(A-11)}$$

The foregoing represents the equilibrium requirements for a single finite element. The incompressibility constraint (A-8) implies that the initial volume v_0 and the final volume v of the finite element are equal, i.e.,

$$v - v_0 = la(u_{N\alpha}) - l_0 a_0 = 0 \qquad \text{(A-12)}$$

where a_0 and a are the initial and final areas of the triangle, respectively. Now

$$\lambda = \frac{l}{l_0} \qquad \text{(A-13)}$$

and

$$2a_0 = (x_{11}x_{22} + x_{21}x_{32} + x_{31}x_{12}) - (x_{21}x_{12} + x_{31}x_{22} + x_{11}x_{32}) \qquad \text{(A-14)}$$

$$2a(u_{N\alpha}) = (y_{11}y_{22} + y_{21}y_{32} + y_{31}y_{12}) - (y_{21}y_{12} + y_{31}y_{22} + y_{11}y_{32}) \qquad \text{(A-15)}$$

where $y_{N\alpha} = x_{N\alpha} + u_{N\alpha}$ and $x_{N\alpha}$ = the initial nodal coordinates. Hence with the above (A-12) is of the form

$$a(u_{N\alpha}) - \frac{1}{\lambda} a_0 = 0 \qquad \text{(A-16)}$$

DISCUSSION

R. N. Vaishnav[3]

The authors of this paper have solved the problem of evaluating the distribution of tangential and radial stresses in an artery treated as a thick-walled pressurized circular cylinder constrained to a fixed length. The arterial wall has been considered to be composed of an incompressible, orthotropic, nonlinearly elastic material.

The authors show that the actual distribution of the tangential stress is far from being uniform and that even the Lamé solution (valid for isotropic, linearly elastic case) severely underestimates its maximum value. This conclusion is valid and stems from the facts that the wall material becomes increasingly stiffer with strain and the strain increases as one progresses from the outer to the inner surface. This important fact should be borne in mind when calculating stresses using a theory based on assuming the wall as thin. However, we may note that the cor-

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responding error in the constitutive constants is not large; the exponent k in equation (26) is unaffected and the coefficient A is affected only slightly as seen in Fig. 3 by comparing the solid curves with the dashed ones. This is so because of the integration in equation (20) through which the stresses affect the constitutive parameters.

Finally, attention should be drawn to the fact that, whereas equations (1) and (7) are correct starting points, equation (26) is valid only for a given longitudinal extension. A complete characterization of the wall material would involve postulation and experimental validation of a constitutive relation involving both the circumferential and longitudinal extension ratios as variables. We have developed such a constitutive relation which was published as a Master's thesis[4] and will be detailed in a forthcoming publication.[5]

[4] Young, John T., "Determination of Constitutive Constants of Canine Aorta Under Large Deformations," Master's thesis, The Catholic University of America, Apr. 1970.

Authors' Closure

The authors wish to thank Professor Vaishnav for emphasizing some of the salient but important findings of this paper. It should be noted that both the coefficient A and the exponent k in equation (26) were adjusted in determining the mechanical response for other aortas tested and for data reported in references [12 and 14]. The thick-walled pressure-radius response and tangential stress distribution show considerable deviations from the corresponding thin-walled tube results. These results will appear in a future publication.

[5] Patel, Dali J., and Vaishnav, Ramesh, N., "Rheology of Large Blood Vessels," in *Handbook of Cardiovascular Fluid Dynamics,"* edited by Bergel, D. H., Academic Press, London, 1971.