DISCUSSION

On: "Analysis of the Rayleigh pulse" by P. Hubral and M. Tygel (GEOPHYSICS, 54, 654-658, May 1989)

Recently, Professors Hubral and Tygel presented an interesting analysis of the Rayleigh wavelet, emphasizing its twin attributes of simplicity and flexibility. My comments here are concerned mainly with the final section of their paper, where the length properties of the Rayleigh pulse are described. In particular, I demonstrate that the class of Rayleigh wavelets is characterized by a simple length formula, provided that a proper definition of waveform length is adopted. Although there are various useful definitions for signal length, width, breadth, or duration, the specific form that I employ here is commonly used by exploration seismologists. It has been examined in detail by Berkhout (1984, p. 16-28) and others.

Consider a wavelet $w(t)$ which may, in general, be complex valued. The energy of $w(t)$ is defined as

$$E_w = \int_{-\infty}^{+\infty} |w(t)|^2 \, dt. \tag{1}$$

The nonnegative function $|w(t)|^2$ is called the energy density waveform. The length of the wavelet, relative to a reference time t_0, is given by

$$L_w(t_0) = \left\{ \frac{1}{E_w} \int_{-\infty}^{+\infty} (t - t_0)^2 \, |w(t)|^2 \, dt \right\}^{1/2}. \tag{2}$$

Clearly, $L_w^2(t_0)$ is just the second moment, about the time t_0, of the normalized energy density waveform $|w(t)|^2/E_w$. Note that $L_w(t_0)$ has dimension of time and is independent of the physical dimension and scaling of the wavelet $w(t)$. Unfortunately, the length measure used by Hubral and Tygel does not possess these desirable properties. In the present notation, their length measure equals $E_w L_w^2(0)$.

Selection of a particular reference time t_0 is problem dependent. However, it is straightforward to demonstrate that wavelet length is minimized by choosing t_0 equal to

$$\bar{t}_w = \frac{1}{E_w} \int_{-\infty}^{+\infty} t \, |w(t)|^2 \, dt. \tag{3}$$

$\bar{t}_w$ is the first moment, about the time origin, of the normalized energy density waveform. Since $L_w(\bar{t}_w)$ is analogous to a standard deviation, most of the wavelet energy is concentrated within ± 2 or 3 such lengths about $\bar{t}_w$. For purely symmetric or antisymmetric real waveforms, the mean time $\bar{t}_w$ equals zero.

I now apply these general formulae to the Rayleigh pulse. Let $B(t)$ refer to the unnormalized analytic Rayleigh wavelet of order n:

$$B(t) = \frac{i\,(-1)^n \, n!}{\pi \, (t + i\epsilon)^{n+1}}, \tag{4}$$

where ϵ is a positive parameter with dimension of time. Substituting this expression into equation (1) and performing the integration yields

$$E_B = \frac{2\,(2n)!}{\pi \, (2\epsilon)^{2n+1}}. \qquad n = 0, 1, 2, \ldots \tag{5}$$

Using this result, a very simple expression for the length of $B(t)$ relative to $t_0 = \bar{t}_B = 0$ is obtained from equation (2):

$$L_B(0) = \frac{\epsilon}{\sqrt{2n - 1}}. \qquad n = 1, 2, 3, \ldots \tag{6}$$

If the index n equals zero, then $L_B(0) = +\infty$. Since the length measure utilized here is insensitive to multiplicative scaling of the wavelet, equation (6) also gives the lengths of the *normalized* and *generalized* analytic Rayleigh wavelets discussed by Hubral and Tygel. If ϵ is expressed in terms of the mode of the Fourier amplitude spectrum of $B(t)$, then we obtain

$$L_B(0) = \frac{n}{\sqrt{2n - 1}} \, \frac{1}{2\pi f_m}. \qquad n = 1, 2, 3, \ldots \tag{7}$$

Hence, length increases with increasing n and is inversely proportional to the characteristic frequency f_m.

The real and imaginary parts of the analytic signal $B(t)$ form a Hilbert transform pair. Let $B(t) = b(t) - ic(t)$. Then, general theorems of signal analysis can be used to determine the lengths of the two real valued wavelets $b(t)$ and $c(t)$. It is well known that energy is invariant under Hilbert transformation. Length is also invariant under Hilbert transformation, provided that the original wavelet has no dc spectral content. If the wavelet has a nonzero Fourier component at zero frequency, then the length of its Hilbert transform becomes infinite. These theorems are readily established by considering the frequency domain representations of both energy and length (Berkhout, 1984). Applying these results to the Rayleigh pulse yields

$$E_b = E_c = \frac{1}{2} E_B, \tag{8}$$

and

$$L_b(t_0) = L_c(t_0) = L_B(t_0). \qquad n = 1, 2, 3, \ldots \quad (9)$$

For $n = 0$, direct integration of the defining expression for length yields $L_b(0) = \epsilon$ and $L_c(0) = +\infty$.

Equation (9) implies that the lengths of all of the wavelets within each column of Hubral and Tygel's Figure 2 are identical. Values are easily calculated via formulas (7) and (9); there is no need to numerically evaluate any integrals. For $f_m = 10$ Hz and $n = 1, 2, 3, 4$ these lengths equal 15.9, 18.4, 21.4, and 24.1 msec, respectively, and are in accord with a quick visual examination of Fig. 2.

In conclusion, the simple characterization of Rayleigh wavelet length developed here should serve to enhance the utility of this pulse for seismogram modeling and other purposes.

David F. Aldridge
Univ. of British Columbia
129-2219 Main Mall
Vancouver, BC Canada V6T 1W5

Reference

Berkhout, A. J., 1984, Seismic resolution, a quantitative analysis of resolving power of acoustical echo techniques: Geophysical Press.

Reply by the authors to David F. Aldridge

We find the observation of David F. Aldridge concerning the length of the Rayleigh pulse valid and constructive. It should contribute to make the Rayleigh pulse for practical applications even more attractive.

Peter Hubral
Martin Tygel
Geophysical Institute
Univ. of Karlsruhe
Hertzstr. 16
7500 Karlsruhe, Germany