Does quantum mechanics need imaginary numbers? FREE

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Johanna L. Miller

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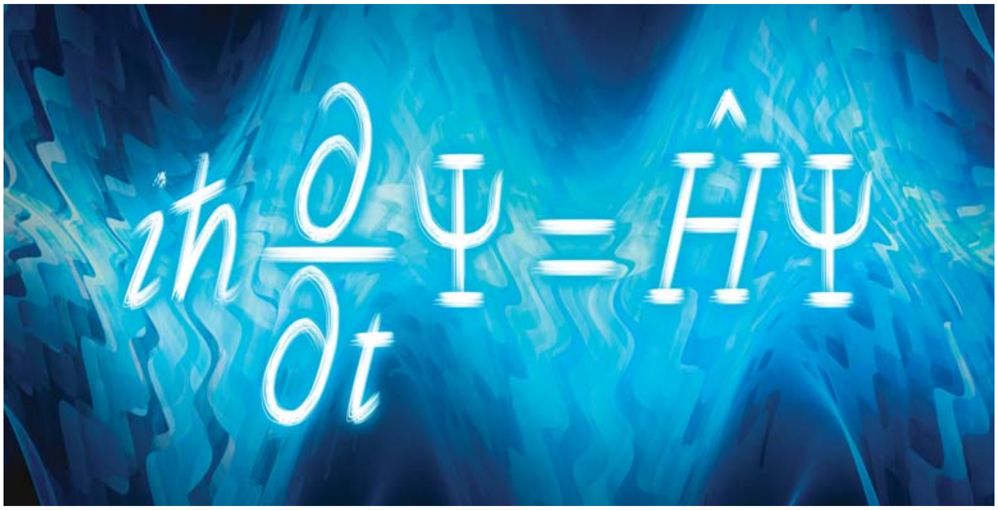
The square root of negative one doesn't correspond to any physical quantity, but that doesn't mean it has no place in the physical sciences. For example, putting an imaginary number in an exponent changes the behavior of the exponential from rapid growth or decay to a steady sinusoidal oscillation. The result is a useful description of the physics of waves. (See, for example, the Quick Study by Iñigo Liberal and Nader Engheta on page 62 of this issue.)

In electromagnetism and most other fields of physics, imaginary numbers are merely a mathematical convenience. All the relevant phenomena can still be described using nothing but real numbers. Quantum mechanics is an exception: The observable quantities and probabilities are by necessity all real, but the underlying quantum states and governing equations involve imaginary numbers, and there's no simple way to remove them. But are they just an artifact of the way the theory was written down, or do they really need to be there?

In their new theoretical work, Miguel Navascués of the Institute for Quantum Optics and Quantum Information in Vienna and colleagues shed some light on that question.[1] They find that, subject to some postulates about how a quantum theory must be mathematically structured, no real-valued version of quantum theory can duplicate all the predictions of the familiar complex-valued formulation. Moreover, they designed an experimentally feasible test capable of ruling out real-valued quantum theories. In the time since their proposal was made public in January 2021, two groups carried out the experiment—and both found results in favor of standard complex-valued quantum theory.[2]



UNLIKE MOST physics equations, the time-dependent Schrödinger equation features the imaginary unit *i*. Purging the imaginary numbers from quantum mechanics would require major changes to the theory's mathematical structure. (Image by iStock.com/sakkmesterke.)

By any other name

Ever since its advent a century ago, the quantum world has challenged classical intuitions in many ways, with even prominent physicists bristling against quantum weirdness. A quantum state, for example, doesn't contain enough information to prescribe the outcome of every possible measurement on the state; rather, for most measurements, it offers only a probability distribution among the possible outcomes.

Could it be that the theory's pioneers unluckily happened on an incomplete description of the quantum world, just waiting to be supplemented by a system of local hidden variables that do, in fact, preordain every measurement outcome?

Thanks to the work of John Bell and others, that idea has been laid to rest. An experiment can be designed in which quantum mechanics and any possible theory of local hidden variables predict different results. The description of the experiment alone is enough to establish that no complete set of local hidden variables could possibly be lurking beneath the veneer of quantum theory. But when the experiment is actually performed, quantum mechanics emerges triumphant every time.

The question of the necessity of complex numbers has a lot in common with the question of quantum theory's inherent uncertainty, but it's much more subtle. One can always devise new mathematical constructs that behave in all the same ways as complex numbers even though they're called something else. As early as 1960, Ernst Stueckelberg did essentially just that with his real-valued formulation of quantum mechanics.[3] For the question to make sense, it's therefore necessary to establish some ground rules that exclude real-valued quantum theories that restate the standard complex-valued theory by other names.

Too many dimensions

A complex number, $a + bi$, can be described by an ordered pair (a, b) of real numbers—that is, a vector in the two-dimensional space of real numbers. But quantum states themselves are multidimensional vectors of complex numbers, and the compounding of dimensions on dimensions gets complicated. A spin-½ qubit is represented by a vector in two complex dimensions. It could also be written as a vector in four real dimensions, but those dimensions aren't naturally all equivalent.

The cracks start to show when one considers how to construct multiparticle states. One of Navascués and colleagues' ground rules, which they say they consider to be a fundamental mathematical property of a quantum theory, is that the combination of two quantum systems is represented by their tensor product. (Stueckelberg's formulation violates that rule.) For example, standard quantum theory says that the combination of two qubits, each with two complex dimensions, has 2 × 2 = 4 complex dimensions, equivalent to 8 real dimensions. But in a real-valued formulation, the same two qubits each have 4 real dimensions, and their tensor product has 4 × 4 = 16 real dimensions—twice as many as necessary to describe the system.

Having too many dimensions doesn't seem as though it would be a fatal problem for a theory—and it isn't. Previous work has shown that for all manner of Bell-like experiments, in which two or more entangled particles emerge from a central source and are measured by spatially separated observers, real-valued theories can be formulated to mimic all the predictions of standard quantum mechanics, even with the constraint of the tensor-product rule.

But what about when the number of dimensions is made to decrease, not increase? That can happen in an entanglement-swapping experiment, as sketched in the figure below. Rather than originating from a single source, two sets of entangled qubits are created by separate sources. One observer, Bob, receives one qubit from each pair (B_1 and B_2), and the other two, A and C, go to Alice and Charlie, respectively.

Bob then makes a joint measurement on his two qubits, with four possible outcomes. In complex-valued quantum mechanics, that measurement halves the number of dimensions of the system and cuts the number of entangled pairs from two to one. That is, it transfers the entanglement to qubits A and C. But in a real-valued formulation, Bob's four-outcome measurement doesn't cut the dimensionality by enough to fully swap the entanglement—he'd need an eight-outcome measurement to do that—so qubits A and C don't end up fully entangled.

The dimension mismatch still doesn't mean that the real-valued theory can't describe the system, especially if the extra dimensions didn't need to be there in the first place. And Navascués and colleagues spent a lot of time trying to make the real-valued description work before they turned to trying to prove that it couldn't.

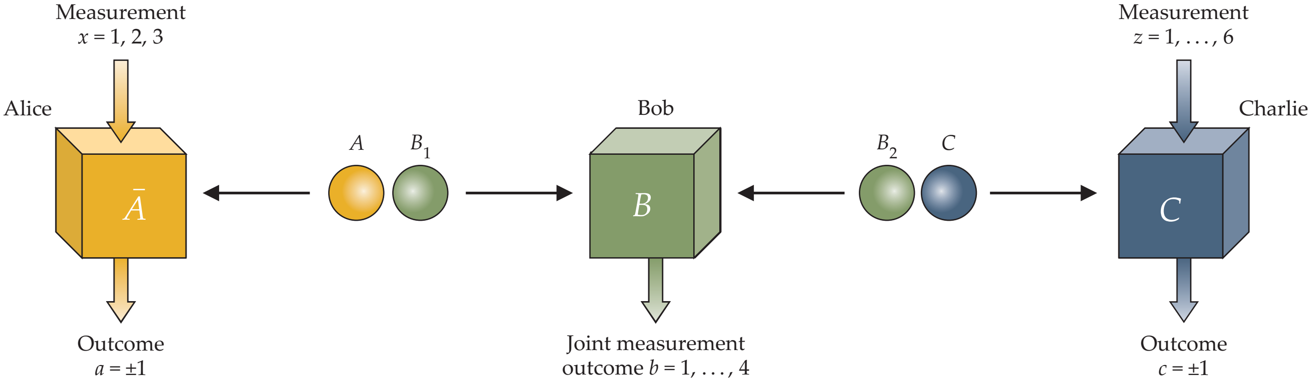
Real complex

Mathematical proofs of impossibility can be much more difficult than constructions of what's possible. To show that quantum mechanics (subject to the tensor-product rule) needs complex numbers, Navascués and colleagues had to prove not just that the most obvious real-valued formulation doesn't work, but that none of them do. Perhaps it's most natural to represent each complex dimension by two real dimensions, but real-valued theories need not be so limited. There could be three real dimensions per complex dimension—or four, or even infinitely many.

Accounting for all the possibilities was daunting. Help came in the form of a recent paper by Antonio Acín (also a coauthor on the new paper) and colleagues on certifying entanglement for quantum information networks.[4] By piggybacking on that work, Navascués and colleagues found a function of measurement correlations for the entanglement-swapping experiment that could reach $6\sqrt{2} = 8.49$ in standard quantum theory but that could never exceed 7.66 in a real-valued formulation.

That's not a lot of wiggle room, and Navascués suspects that the real-valued bound could be significantly improved. When the researchers first tried to calculate it numerically, their computer ran out of memory. In the end, they had to make some approximations that gave them a significantly looser bound than they'd hoped for.

Still, when Jian-Wei Pan and colleagues at the University of Science and Technology of China in Hefei carried out the experiment using superconducting qubits, they observed a value of 8.09, comfortably in the realm of complex quantum theory. And when Jingyun Fan (of Southern University of Science and Technology in Shenzhen, China) and colleagues



AN EXPERIMENT on entanglement swapping can distinguish real-valued from complex-valued quantum theories. Two pairs of entangled qubits (A and B_1; B_2 and C) are produced at separate sources. When an observer, Bob, makes a joint measurement on B_1 and B_2, he transfers the entanglement to a pair of qubits, A and C, that never interacted. Two other observers, Alice and Charlie, measure those qubits: Alice chooses from among three measurements to make on her qubit, and Charlie chooses from among six. The correlations between their measurements predicted by standard quantum mechanics—and observed when the experiment is performed—are inconsistent with real-valued theories. (Adapted from ref. 1.)

used photons to measure a related quantity, they too found a vindication for complex-valued quantum mechanics.[2]

Like Bell tests, the experiments are subject to some fine print. The measurements should be close enough to simultaneous to ensure that no classical information can pass between the observers that could influence their outcomes. And few enough of the measurement trials should go undetected to ensure that the correlation threshold is met not just by the detected trials, but by all of them. If either of those loopholes is not closed, it's possible for quantum-like correlations to be mimicked not just by a real-valued theory but by a classical one. (See PHYSICS TODAY, December 2011, page 20.) Closing the loopholes in Bell tests themselves was a decades-long effort that came to fruition only in 2015. (See PHYSICS TODAY, January 2016, page 14.)

Neither Pan's nor Fan's group has yet closed the loopholes in their experiments. Technically, therefore, the jury is still out on whether real or complex numbers are the better descriptors of the quantum world. Still, it seems likely that future students of quantum mechanics will have no choice but to continue to grapple with the mathematics of imaginary numbers.

Johanna Miller

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Krypton isotopes tell the early story of Earth's life-giving elements

Since its infancy, our planet has accumulated volatiles from more than one source.

The Galápagos Islands' extraordinary biodiversity famously helped inspire Charles Darwin to formulate his theory of the evolution of life on Earth. But the volcanic islands also offer a window into our planet's even deeper past. The volcanoes, including the one in figure 1, sit atop a mantle plume that channels deep-mantle material to Earth's surface. And the portion of the mantle tapped by the plume has been unusually stagnant over the planet's 4.5-billion-year history.

Over geologic time, Earth's crust and much of its mantle are in constant, albeit slow, motion, as tectonic plates are recycled from the crust to the mantle and back again. Like the churning of butter, the churning of the planet's thickest layer serves not to homogenize its components but to separate them based on their density, volatility, and chemical properties. As a result, almost nothing we encounter on Earth's surface bears any relation to the planet's average composition.

But some pockets of the mantle seem to have been immune to that mixing and have instead remained undisturbed by geological processes since at least the first 100 million years of the planet's history. (For more on the analysis that makes that conclusion possible, see PHYSICS TODAY, October 2010, page 16.) When bits of those primitive materials make their way to the surface—as they do in the Galápagos, Iceland, and a few other volcanic regions—they provide scientists with a valuable look back in time to reveal what the infant planet was originally made of.

FIGURE 1. FERNANDINA VOLCANO in the Galápagos Islands is one of several sites around the world where geoscientists can discern Earth's original composition. (Photo by tomowen/Shutterstock.com.)

Now Sandrine Péron (a postdoc at the University of California, Davis, at the time she did the work, now at ETH Zürich) and colleagues have used a newly developed technique to analyze some primordial mantle samples for their krypton, an element present only at the parts-per-trillion level.[1] The findings paint a picture not only of krypton itself but of carbon, hydrogen, nitrogen, and oxygen—all the building blocks of life.

Mantle fingerprints

Early Earth was a hot place, as the young planet was frequently enduring energetic collisions with the planetesimals that it