Some Effects of Topography on the Tidal Flow in a River Estuary

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(Received 1968 January 8)

Summary

A mathematical model is used to investigate the effect of topographical changes on the structure of the tidal flow and residual current system in a river estuary. Calculations are performed for an estuary bounded at its landward end by a tidal barrier. It is suggested that the resulting general conclusions may be of value in an appraisal of the consequences of proposed estuarial engineering schemes.

1. Introduction

The overall structure of the tidal flow and residual current system (mass transport velocity) in the Humber estuary has been investigated by Johns (1966, 1967). In both of these studies, the effect of vertical turbulence was parametrically represented by a coefficient of eddy viscosity increasing with height above the river bed. In the second investigation, it was found that variations in the gross topography had a critical effect upon the structure of the tidal dynamics and residual current system. In order to obtain realistic results, it was therefore necessary to regard the river as consisting of a sequence of sub-divisions in each of which the theory was separately applicable. Although of considerable success in explaining some of the observed dynamical features, the predicted flow is necessarily discontinuous at the juncture of two regions—in particular with regard to the tidal mass flow of water. It is the purpose of the present paper to partially remove this deficiency and to construct a mathematical model to take account of variations in the gross topography. Broadly speaking, the methods of solution involved are the same as in the previous study. However, it is ensured that both the tidal elevation and the tidal mass flow are continuous throughout the estuary. It is therefore unnecessary to prescribe the amplitude and phase of the tide at a sequence of locations in the river (cf. Johns (1967)).

The technique is introduced and applied to an estuary in which there is a tidal barrier at the landward extremity. Seaward of this position, limited variations are permitted in the topography and the value of the eddy viscosity. Calculations are then undertaken to assess the effect of changes in depth and width on the tidal dynamics and residual current system. The numerical process has been programmed for automatic computation and may be expected to give a guide in an appraisal of the consequences of the introduction of engineering structures into a tidal estuary.

2. Formulation

Referring spatial conditions to rectangular axes Ox, Oz, in which Ox is located within the undisturbed free surface and is directed landward along the estuary, the
linearized equations of motion governing the averaged tidal flow (cf. Johns (1967)) are

\[
\frac{\partial u}{\partial t} = -g \frac{\partial \zeta}{\partial x} + \frac{\partial}{\partial z} \left( N_{xx} \frac{\partial u}{\partial z} \right),
\]

(2.1)

\[
\frac{\partial}{\partial x} \left\{ b(x) \int_{-h(x)}^{0} u \, dz \right\} + b(x) \frac{\partial \zeta}{\partial t} = 0.
\]

(2.2)

The quantities \( N_{xx}, b(x) \) and \( h(x) \) denote respectively the eddy viscosity, the width of the river and the mean water depth at the position \( x \). These equations are to be solved subject to the usual conditions

\[
\begin{align*}
    u &= 0 \text{ at } z = -h(x) \\
    \frac{\partial u}{\partial z} &= 0 \text{ at } z = 0.
\end{align*}
\]

(2.3)

In order to allow for limited topographical changes in the landward direction, the estuary is divided into three reaches defined by

\[ x_{n-1} < x < x_n; \quad (n = 1, 2, 3; \quad x_0 = 0). \]

The same methods, however, are applicable to a greater number of sub-divisions. The seaward extremity is located at \( x = 0 \) whilst we prescribe a tidal barrier at \( x = x_3 \). Within each reach, the depth is constant, the width is given by an assumed exponential variation whilst the eddy viscosity is horizontally uniform. Each of the parameters involved, however, may differ in different reaches. Accordingly, we represent the solution for \( x_{n-1} < x < x_n \) by the real parts of

\[
\zeta_n = A_n(x) e^{-i\sigma t},
\]

\[
u_n = -\frac{ig}{\sigma} A_n'(x) U_n(x) e^{-i\sigma t}.
\]

(2.4)

(2.5)

3. Solution of equations

A variable \( \xi_n \) is defined by

\[
\xi_n = 1 + \alpha_n \left( 1 + \frac{z}{h_n} \right),
\]

(3.1)

and the eddy viscosity is prescribed in the form

\[
N_{xx} = v_n \xi_n^2.
\]

(3.2)

The quantities \( \alpha_n, h_n \) and \( v_n \) are therefore the analogues of \( \alpha, h \) and \( v \) in Johns (1967). The width of the estuary is given by

\[
b = b_0 e^{-2z_n x}, \quad (n = 1, 2, 3).
\]

(3.3)

Consequently, the form of the solutions for \( n = 1, 2, 3 \) is identical to that given in the earlier paper. In the solution for each reach, there will be two constants of integration. The values of the resulting six unknown quantities are determined by invoking the following requirements:

1. Continuity of the tidal elevation and mass flow at points of juncture.
2. The vanishing of the tidal current at the tidal barrier.
3. Specification of the tidal amplitude and phase at the estuary mouth.
The rather artificial specification of a discontinuous depth and rate of estuarial convergence will of necessity lead to structural changes in the vertical current profile in different reaches. The present model, although being compatible with a continuity of tidal elevation and total mass flow, will accordingly yield discontinuities in the other flow variables. Although these might possibly be minimized by an appropriate choice of values for \( v \) (the author is indebted to a referee for this suggestion), it is felt that the root cause of the anomaly is not being attacked. It is suggested that a refinement would be to consider a greater number of sub-divisions, thus removing the need for such abrupt changes in topographical conditions. Alternatively, a mathematical model incorporating continuous changes might be developed. The author is now in the early stages of such an investigation.

The continuity of tidal elevation leads to the set of relations

\[
A_n(x_n) = A_{n+1}(x_n), \quad (n = 1, 2),
\]

(3.4)

which are equivalent to two linear equations involving the before-mentioned constants of integration. The continuity of mass flow at \( x = x_n \) yields

\[
\frac{h_n}{\alpha_n} A'_n(x_n) \int_1^{1 + \sigma_n} U_n d\xi = \frac{h_{n+1}}{\alpha_{n+1}} A'_{n+1}(x_n) \int_1^{1 + \sigma_{n+1}} U_{n+1} d\xi, \quad (n = 1, 2). \tag{3.5}
\]

(Bartholomeuze (1958) has discussed the suitability of a similar condition in connection with long wave propagation over a step). By virtue of equation (3.5), we therefore obtain an additional pair of relations between the unknown constants.

Finally, the vanishing of the tidal current at \( x = x_3 \) is assured if

\[
A'_3(x_3) = 0, \tag{3.6}
\]

thus making a total of five relations between six complex unknowns. These equations may then be solved in terms of the prescribed amplitude and phase at \( x=0 \).

Upon taking the real part of equation (2.4), we may write the surface elevation \( \zeta \) in the form

\[
\zeta = |A(x)| \cos(\sigma t + \phi), \tag{3.7}
\]

where

\[
A(x) = A_n(x) \quad \text{for} \quad x_{n-1} < x < x_n, \tag{3.8}
\]

and

\[
e^{-i\phi} = A(x)/|A(x)|. \tag{3.9}
\]

Likewise, from equation (2.5), the averaged tidal current \( u \) is given by

\[
u = C(x, z) \cos(\sigma t + \psi), \tag{3.10}
\]

where

\[
C(x, z) = \frac{g}{\sigma} |A'(x)| U(z), \tag{3.11}
\]

and

\[
U(z) = U_n(z) \quad \text{for} \quad x_{n-1} < x < x_n. \tag{3.12}
\]

The stress \( \tau_b \) at the estuary bed is given by

\[
\tau_b = S(x) \cos(\sigma t + \beta), \tag{3.13}
\]

where

\[
S(x) = \frac{\rho v(x)}{\sigma} g |A'(x)| U'(-h), \tag{3.14}
\]
and

\[\begin{align*}
n(x) &= n_x \\
h &= h_n
\end{align*}\]

for \(x_{n-1} < x < x_n\).  

Finally, on using equation (4.10) of Johns (1967), the Lagrangian residual current, or mass transport current, just beyond the shear layer is given by

\[U = -\frac{g^2}{4\sigma} \left|U(-h+\delta)\right|^2 \text{Re}\{3-5i\} A'(x) A''(x)^*\]. \hspace{1cm} (3.16)

The quantity \(\delta\) is representative of the thickness of the shear layer (and is presumably a function of the total depth which the present analysis does not yield). The asterisk denotes the complex conjugate.

4. Numerical evaluation

The numerical scheme for the evaluation of the formulae of Section 3 has been programmed for automatic computation. At the estuary mouth the tidal amplitude has a specified value \(A\) whilst the phase angle \(\phi\) is zero. Computations were then undertaken to investigate the influence of several topographies on the overall structure of the tidal dynamics and Lagrangian residual current. In the present paper, calculations were performed for a semi-diurnal tidal flow in an estuary, of total length 75 000 ft (22 860 m) bounded at its landward extremity by a tidal barrier. The three reaches are each of length 25 000 ft (7620 m) and, throughout, the eddy viscosity is given a value suggested by earlier investigation. The relevant quantities are:

\[v = 2 \times 10^{-2} \text{ ft}^2 \text{ s}^{-1} \quad (\approx 18.6 \text{ cm}^2 \text{ s}^{-1})\]

\[(N_x)^{\text{surface}} = 5.0 \text{ ft}^2 \text{ s}^{-1}. \quad (\approx 4650 \text{ cm}^2 \text{ s}^{-1}).\]

The parameters describing the topography in the calculations are given in Table 1.

<table>
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<th>2</th>
<th>3</th>
<th>4</th>
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<td>40</td>
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<td>40</td>
<td>35</td>
<td>40</td>
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<td>(a_3) (ft(^{-1}))</td>
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<td>(5 \times 10^{-6})</td>
<td>(5 \times 10^{-6})</td>
<td>(1 \times 10^{-6})</td>
</tr>
</tbody>
</table>

The four cases are representative of the following topographies:
1. A parallel-sided estuary of constant depth 40 ft \((\approx 12.19 \text{ m})\).
2. A uniformly-converging estuary of constant depth 40 ft.
3. A uniformly converging estuary with a decreasing depth in the landward direction—the average depth being 40 ft.
4. An estuary of constant depth 40 ft, the mid-reach of which is more rapidly convergent than the landward and seaward reaches.

The results of the computations of the amplitude \(C\) of the surface tidal current are given in Fig. 1. It is apparent that a decrease in the rapidity of convergence,
Fig. 1. Representation of the amplitude $C$ of the surface tidal current as a function of position along the estuary.

Fig. 2. Representation of the Lagrangian residual current $U$ at 5–8 ft (1.52–2.44 m) above the river bed as a function of position along the estuary.
or depth, results in an increase in the tidal current. Although the tidal mass flow is continuous throughout the estuary, the tidal current may be discontinuous at $x = 25,000$ ft and $x = 50,000$ ft ($15,240$ m). The position of a discontinuity is represented by a broken line.

The direction of the Lagrangian residual current above the bed was found to be landward in each case—suggesting an overall landward movement of sediment along the estuary. The value $U$ of the residual current, evaluated at 5-8 ft ($1.52-2.44$ m) above the bed, is given in Fig. 2. An important aspect of the calculation is that the overall magnitude of the residual current is reduced by an increase in the rapidity of estuarial convergence. This applies also to a local increase since, in case 4, the residual current within the mid-reach rapidly adjusts itself to a value only slightly greater than that predicted in case 2. In reality, of course, there would not be a discontinuity at $x = 25,000$ ft, but the general tendencies are expected to be similar.

The value of the amplitude of the bottom stress $S$ is given in Fig. 3. From this, it is readily seen that the bottom stress is decreased by an increase in the depth or rapidity of convergence. Moreover, case 2 gives a good approximation to the dynamical conditions in cases 3 and 4 within the mid-reach where the topographies are again identical.

Since the suspension of loose bed material is likely to result from a high bed shear, and since this circumstance is accompanied by a high residual current, it is

* The validity of this conclusion, and those that follow, requires that the tidal amplitude $A$ at the estuary mouth be unaltered for different estuarial topographies. The computed tidal elevations within the estuary are found to be so weakly dependent upon the topography that an almost uniform value of $A$ seems plausible.
suggested that overall movements of bed material will be minimized by an increase in the rapidity of estuarial convergence.

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1968 February.

References

