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Short Note

The Berlage wavelet

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INTRODUCTION

Symmetric wavelets are commonly used in seismic modeling studies. However, accurate simulation of many physical wave propagation phenomena requires a causal waveform possessing a certain degree of differentiability. The purpose of this note is to quantitatively describe the characteristics of a relatively unfamiliar wavelet, called the Berlage wavelet, that is appropriate for such studies.

Recent investigations have emphasized the advantages (Hubral and Tygel, 1989) as well as some of the drawbacks (Hosken, 1988) of using a simple, easily computable waveform for synthetic seismogram modeling. Like the Rayleigh and Ricker pulses discussed by these authors, the Berlage wavelet is unambiguously specified by a straightforward mathematical formula. Together with its Hilbert transform and phase-shifted versions, it offers sufficient flexibility for modeling a wide range of seismic waveforms observed in both field and processed data. It is particularly valuable in theoretical or computational studies where causality of the incident seismic signal is judged to be an important condition.

DEFINITION AND TIME-DOMAIN PROPERTIES

The Berlage wavelet (Berlage, 1932, p. 369) is defined by the following formula:

$$w(t) = AH(t)t^n e^{-\alpha t} \cos (2\pi f_0 t + \phi_0), \qquad (1)$$

where $H(t)$ is the Heaviside unit step function [$H(t) = 0$ for $t \leq 0$ and $H(t) = 1$ for $t > 0$]. The waveform shape is controlled by four adjustable parameters. The exponential decay factor α and the time exponent n are considered to be nonnegative real constants. In general, n is not restricted to be an integer, although several derivations in the subsequent analysis are simplified by adopting this assumption.

The inclusion of an initial phase angle ϕ_0 in the defining equation (1) represents a generalization of the Berlage pulse used by Kulhánek and Klíma (1970) and Farnbach (1975). The present wavelet reduces to their form in the special case $\phi_0 = -\pi/2$. This generalization is significant since the value of the initial phase angle exerts a strong influence on waveform shape. Figure 1 depicts a suite of Berlage pulses where the initial phase angle ranges from $-\pi/2$ to $+\pi/2$ radians.

Several time-domain attributes of the Berlage wavelet make it extremely useful for seismological modeling purposes. It is causal as well as continuously differentiable to at least order $[n - 1]$. If $\phi_0 = \pm\pi/2$, then the first $[n]$ derivatives are continuous, where $[n]$ is the smallest integer greater than or equal to n. More importantly, a suitable selection of the parameter values in equation (1) yields a waveform shape very similar to that of recorded seismic wavelets. In particular, the common "cycle and a half and not much more" character is easily replicated.

ENERGY, MOMENTS, AND LENGTH

The energy of a time-domain function $w(t)$ is defined as

$$E = \int_{-\infty}^{+\infty} w(t)w(t)^* \, dt = \int_{-\infty}^{+\infty} |w(t)|^2 \, dt. \qquad (2)$$

The nonnegative function $|w(t)|^2$ is referred to as the energy density waveform. Substituting equation (1) for the Berlage wavelet into equation (2) and performing the integration yield

$$E = \frac{A^2\Gamma(2n+1)}{2^{2n+2}} \left\{ \frac{1}{\alpha^{2n+1}} + \frac{\cos [(2n+1)\psi + 2\phi_0]}{\beta^{2n+1}} \right\}, \qquad (3)$$

where

$$\psi = \tan^{-1} \frac{2\pi f_0}{\alpha},$$

$$\beta = \sqrt{\alpha^2 + (2\pi f_0)^2},$$

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and $\Gamma(n)$ is the gamma function. Obviously, energy depends on all five wavelet parameters. However, it is useful to explicitly denote its dependence upon the time exponent n only, viz., E_n. Note that the Berlage wavelet has finite energy except in the limiting case $\alpha \to 0$.

The mth order moment, about the time t_0, of the normalized energy density waveform $|w(t)|^2/E$, is defined as

$$M_m(t_0) = \frac{1}{E}\int_{-\infty}^{+\infty} (t - t_0)^m |w(t)|^2\, dt, \qquad m = 0, 1, 2, 3, \ldots \tag{4}$$

The second moment of the normalized energy density waveform is often used to quantify wavelet length in the time domain. This length measure is defined as $L(t_0) = \sqrt{M_2(t_0)}$. An advantage of this particular length measure is that it incorporates the whole of the waveform, rather than just the primary lobe. Note that $L(t_0)$ has dimension of time and is independent of the physical dimension and scaling of the wavelet $w(t)$.

The moments of the Berlage energy density waveform are easy to calculate. Substituting equation (1) into equation (4) and taking $t_0 = 0$ immediately yield $M_m(0) = E_{(n+m)/2}/E_n$. Thus the mth moment about the origin can be conveniently evaluated from the known expression for the wavelet energy. Using this result, the length of the Berlage wavelet, relative to the specified time t_0, is

$$L(t_0) = \left\{\frac{t_0^2 E_n - 2t_0 E_{(n+1)/2} + E_{n+1}}{E_n}\right\}^{1/2}. \tag{5}$$

Different choices for the reference time t_0 yield different values for the wavelet length. Using an elementary theorem regarding moments (from either mechanics or probability), the *minimum* second-order moment is obtained by choosing $t_0 = M_1(0)$. The time $\bar{t} \equiv M_1(0)$ may be referred to as the mean arrival time of the wavelet. Hence, for the Berlage pulse, the smallest time-domain length is given by

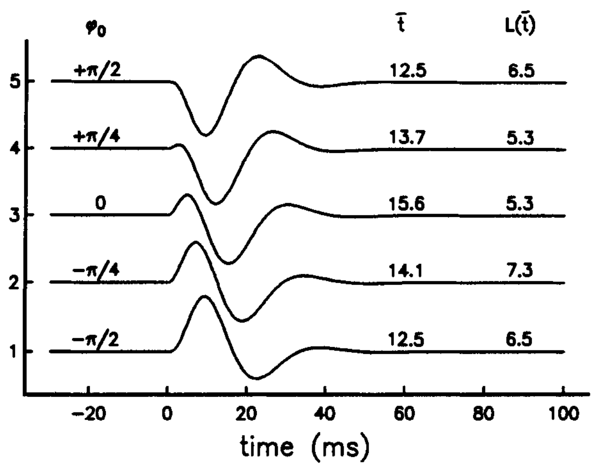

FIG. 1. A set of Berlage wavelets with constants $n = 2$, $\alpha = 180$ rad/s, and $f_0 = 30$ Hz. Values of the initial phase angle ϕ_0 (radians), mean arrival time $\bar{t}$ (ms), and length $L(\bar{t})$ (ms) are labeled. All of these wavelets have continuous first derivatives; traces 1 and 5 have continuous second derivatives.

$$L(\bar{t}) = \left\{\frac{E_n E_{n+1} - E_{(n+1)/2}^2}{E_n^2}\right\}^{1/2}. \tag{6}$$

Since $L(\bar{t})$ is analogous to a standard deviation, most of the wavelet energy is concentrated within two or three such "lengths" about the mean time $\bar{t}$.

FREQUENCY-DOMAIN PROPERTIES

The Fourier transform of the Berlage wavelet can be calculated by standard integration techniques:

$$W(f) = \frac{A\Gamma(n+1)}{2(i2\pi)^{n+1}}\left[\frac{e^{+i\phi_0}}{(f-F_1)^{n+1}} + \frac{e^{-i\phi_0}}{(f-F_2)^{n+1}}\right], \tag{7}$$

where

$$F_1 = +f_0 + i\frac{\alpha}{2\pi},$$

and

$$F_2 = -f_0 + i\frac{\alpha}{2\pi}.$$

The Fourier amplitude spectrum has a single peak in the vicinity of f_0 and decays asymptotically to zero at high frequencies with order $\sim f^{-(n+1)}$.

The spectrum $W(f)$ is analytically continued into the complex plane by defining a complex-valued frequency $F = f + ig(f, g$ real). If integer n is used, then equation (7) indicates that $W(F)$ possesses poles of order $n + 1$ in the upper half-plane at $F = F_1$ and $F = F_2$. This pole distribution is characteristic of the casual nature of the time-domain pulse $w(t)$. The location of the zeros of $W(F)$ is determined by rewriting the partial fraction expansion in equation (7) as a rational function. The numerator of this expression is a polynomial of degree $n + 1$ in the complex frequency F (an exception occurs if $\phi_0 = \pm\pi/2$; the degree of the numerator then reduces to n). Extraction of the roots of this polynomial yields the zeros of $W(F)$. Designating these zeros as G_k, we obtain

$$G_k = i\left[\frac{\alpha}{2\pi} - f_0 \cot \delta_k\right].$$

If $\phi_0 \neq \pm\pi/2$, then the angles δ_k are given by

$$\delta_k = \frac{k\pi - (\phi_0 + \pi/2)}{n+1}, \qquad k = 1, 2, \ldots, n+1.$$

If $\phi_0 = \pm\pi/2$, then δ_k simplifies to $k\pi/(n + 1)$ for $k = 1, 2, \ldots, n$.

The zeros of the Berlage spectrum are pure imaginary. If all zeros of $W(F)$ reside in the upper half-plane, then the Berlage wavelet is a minimum-phase function of continuous time t. Hence, the minimum-phase condition is guaranteed if $\alpha > 2\pi f_0 \cot \delta_k$ for all k. Selection of a particular phase property for the wavelet (minimum, mixed, or maximum) is easily achieved by adjusting the four defining parameters. Berlage pulses created in this manner are useful test wavelets for deconvolution processing schemes (Boland, 1989, p. 60–67) or forward modeling programs (Aldridge, 1989).

HILBERT TRANSFORM

Hilbert transformation of a time-domain wavelet corresponds to multiplying its Fourier spectrum by $+i$ sgn (f), where the signum function is given by sgn $(f) = f/|f|$ for $f \neq 0$ and sgn $(0) = 0$. Hence, an expression for the Hilbert-transformed Berlage wavelet $\hat{w}(t)$ can be derived by employing the Fourier inversion integral. The integral is evaluated by performing contour integration on the complex frequency plane; the residue theorem yields the contribution from the two poles in the upper half-plane. The result of a lengthy derivation is

$$\hat{w}(t) = \hat{w}_1(t) + \hat{w}_2(t), \qquad (8)$$

where the two component functions $\hat{w}_1(t)$ and $\hat{w}_2(t)$ are given by

$$\hat{w}_1(t) = -AH(t)t^n e^{-\alpha t} \sin(2\pi f_0 t + \phi_0), \qquad (9)$$

and

$$\hat{w}_2(t) = \frac{A}{\pi}\tau^n \sum_{k=0}^{n-1} (n-1-k)! \cos[(n-k)\psi + \phi_0]\left(\frac{t}{\tau}\right)^k - \frac{A}{\pi} t^n e^{-\alpha t} \mathcal{R}\left\{e^{+i(2\pi f_0 t + \phi_0)} Ei\left[\left(\frac{t}{\tau}\right)e^{-i\psi}\right]\right\}. \qquad (10)$$

In equation (10), $\mathcal{R}\{\ \}$ indicates the real part of $\{\ \}$, τ is a characteristic time given by $\tau = 1/\beta$, and $Ei(z)$ is the exponential integral with complex argument z (Lebedev, 1965, p. 30–42). If the time exponent n equals zero, then the summation term is omitted from the definition of $\hat{w}_2(t)$.

Farnbach (1975) approximated the Hilbert-transformed Berlage wavelet with only the causal function $\hat{w}_1(t)$. However, the acausal term $\hat{w}_2(t)$ can contribute a significant amount to the wavelet amplitude, both before and after time zero. This acausal feature is shared by the transforms of other one-sided waveforms.

Evaluating $\hat{w}_2(t)$ can present some problems. For times $|t| \gg \tau$, the asymptotic series representation of the exponential integral can be used to convert equation (10) to

$$\hat{w}_2(t) \simeq \frac{-A\tau^n}{\pi} \sum_{k=1}^{K} (n+k-1)! \cos[(n+k)\psi + \phi_0]\left(\frac{\tau}{t}\right)^k, \qquad (11)$$

where the infinite sum is truncated at the Kth term. Hence, the Hilbert-transformed pulse decays as $\sim|t|^{-1}$ at large positive or negative times. For smaller times $|t| \sim \tau$, a truncated convergent series for $Ei(z)$ is substituted into equation (10):

$$Ei(z) = \gamma + \log(-z) + \sum_{k=1}^{K} \frac{z^k}{k!k'} \qquad (12)$$

valid everywhere on the complex plane cut along the positive real axis (γ is Euler's constant). Finally, analysis indicates that $\hat{w}_2(t)$ is finite at $t = 0$ except when $n = 0$ and $\phi_0 \neq \pm\pi/2$. Thus, when the Berlage pulse possesses a step discontinuity at the origin, its Hilbert transform has a singularity there.

PHASE-SHIFTED WAVELETS

Wavelets possessing an arbitrary constant phase shift θ can be generated from a linear combination of the original waveform and its Hilbert transform:

$$v(t) = \cos\theta w(t) + \sin\theta \hat{w}(t). \qquad (13)$$

Substituting the relevant expressions for the Berlage wavelet and reducing yield

$$v(t) = AH(t)t^n e^{-\alpha t} \cos(2\pi f_0 t + \phi_0 + \theta) + \sin\theta \hat{w}_2(t). \qquad (14)$$

Note the similarity of the first term to the basic definition of the Berlage wavelet. Since the phase-shifted wavelets $v(t)$ are acausal (by virtue of the second term), their applicability to any real seismological situation must be carefully evaluated. Figure 2 depicts a suite of phase-shifted Berlage waveforms; the wavelet for $\theta = 0$ has a minimum-phase spectrum.

General theorems of signal analysis can be exploited to give the energy, mean arrival time, and length of the phase-shifted wavelets $v(t)$. It is well known that the energy E of a wavelet is unaltered by constant phase-shift filtering. The mean time $\bar{t}$ may change, however, if the original pulse has a nonzero dc spectral component. If $\bar{t}_w$ and $\bar{t}_v$ refer to the mean arrival times before and after phase shifting, respectively, then

$$\bar{t}_v = \bar{t}_w - \frac{\sin 2\theta |W(0)|^2}{2\pi E}. \qquad (15)$$

Berkhout's (1984, p. 19) conclusion regarding the invariance of signal length under Hilbert transformation is readily generalized to an arbitrary phase-shift angle θ: $L_v(t_0) = L_w(t_0)$. However, proof of this requires that the original wavelet $w(t)$ have a zero valued dc frequency component; otherwise, the length of the phase-shifted pulse $v(t)$ becomes infinite (e.g., Hubral and Tygel, 1989). Since it is permissible for certain geophysical signals (like the particle displacement

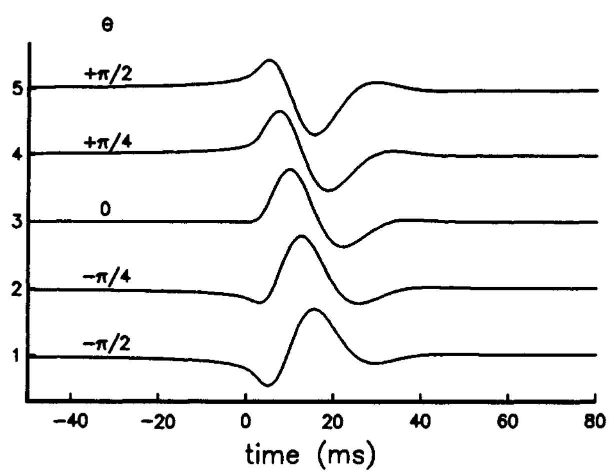

FIG. 2. Phase-shifted Berlage wavelets defined by the constants $n = 3$, $\alpha = 250$ rad/s, $f_0 = 30$ Hz, and $\phi_0 = -\pi/2$. Values of the applied phase-shift angle θ (radians) are annotated. The wavelet on trace 3 is a minimum-phase pulse; all other wavelets have an infinite length measure $L(\bar{t})$.

waveform) to have a nonzero dc frequency component, this proviso is an important qualification.

CONCLUSION

The comprehensive analysis of the Berlage wavelet presented here should be of interest to those conducting modeling experiments with this useful waveform. Wavelets with specific properties (causality, differentiability, minimum phase, constant phase shifts, etc.) are easy to create. The quantitative description of various attributes such as energy, mean arrival time, and length also facilitates comparison of the Berlage wavelet with other candidate waveforms. Finally, a general theorem regarding signal length has been clarified so that a rapid estimation of the lengths of phase-shifted pulses (Berlage or otherwise) is possible.

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