New Formulations of Dual-Doppler Wind Analysis

ALAN SHAPIRO AND JOHN J. MEWES
School of Meteorology, University of Oklahoma, Norman, Oklahoma

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ABSTRACT

New formulations of three-dimensional dual-Doppler wind analysis are presented. The new formulations are conceptually simple, preserve the radial nature of the wind observations, involve only one analysis step (i.e., all constraints are imposed in one functional), and are in a form in which the well-posed condition can most readily be checked. These techniques minimize functionals that incorporate the anelastic mass conservation equation and the radial wind observations as strong or weak constraints. The minimizations are accomplished by appealing directly to the Euler–Lagrange equations and proceed most naturally in the “coplane” cylindrical polar coordinate system. In one method, the anelastic mass conservation equation is applied as a weak constraint, while the radial wind observations are imposed as strong constraints. This results in an algorithm similar to the Armijo wind analysis but with provision for vertical velocity data specification on both upper and lower boundaries (as in the O’Brien adjustment). In another method, the anelastic mass conservation equation is imposed as a strong constraint, while the radial wind observations are used as weak constraints. In a third method, both mass conservation and the radial wind observations are used as weak constraints. In each of the latter two formulations, the analysis reduces to solving a second-order linear partial differential equation, the solution of which is unique. As in Armijo’s research, the shape of the analysis domains must be suitably restricted if the problems are to be well posed.

1. Introduction

Improved dual-Doppler wind analyses (and associated improvements in thermodynamic fields derived from wind estimates) have a potentially wide-ranging impact on a variety of meteorological research and on operational meteorology. Improved wind and thermodynamic field estimates can lead to improved understanding of short timescale mixing processes and complex structures in the atmospheric boundary layer (ABL) (Doviak and Jobson 1979; Reinking et al. 1981; Rabin et al. 1982; Gal-Chen and Kropfli 1984; Eymard and Weill 1988) and can potentially lead to improved ABL parameterizations in mesoscale, regional, and climate models. Dual-Doppler analyses can also aid in the identification and characterization of clear-air ABL structures associated with the onset of severe weather and lead to improved conceptual models of convective phenomena, such as squall lines, bow echoes, mesoscale cellular convection, supercell and multicell thunderstorms, microbursts, and tornadoes (Ray et al. 1975; Ray et al. 1981; Kropfli and Miller 1976; Brandes 1977; Brandes et al. 1988; Carbone 1982, 1983; Knupp and Cotton 1982; Ziegler et al. 1983; Roux et al. 1984; Kessinger et al. 1987; Parsons and Kropfli 1990; Lemaître and Scialom 1992; Dowell and Bluestein 1997). Dual-Doppler wind and thermodynamic analyses can potentially be used as high-resolution data sources for convective scale and mesoscale numerical weather prediction models (Lin et al. 1993; Sun and Crook 1996a; Weygandt et al. 1998). Finally, we note that an emerging new technology, the single-Doppler velocity retrieval (in which winds are estimated from single-Doppler radar data and dynamical constraints), relies on dual-Doppler wind analyses as ground truth for validation (Tuttle and Foote 1990; Sun and Crook 1994; Xu et al. 1995; Laroch & Zawadzki 1995; Shapiro et al. 1995; Zhang and Gal-Chen 1996).

The roots of most modern day dual-Doppler analysis techniques can be traced to Armijo’s (1969) pioneering investigation. Armijo derived the (unique) solution for a velocity field that satisfies the anelastic mass conservation equation and has radial components (wind projections in the directions of the two Doppler radars) that equal the observed Doppler radial wind components. Simple formulas were obtained for the horizontal wind components, while the vertical velocity component was found to satisfy a linear inhomogeneous partial differential equation. Armijo introduced a special cylindrical polar coordinate system (later termed the “coplane” coordinate system) in which the vertical velocity was
obtained by integrating a quasi-horizontal mass flux divergence along coplane azimuthal coordinate lines, the characteristics of the partial differential equation.

In practice, several factors can lead to large errors in the exact Armijo solution. These factors include the nonsimultaneous nature of the data acquisition, spatial interpolation errors, discretization errors, contamination of radial wind data (due to sidelobes and ground clutter), and errors in terminal velocity estimates. In particular, biased errors in the divergence field can accumulate in the integration process and lead to severe errors in the vertical velocity field. Thus, despite its theoretical appeal, the Armijo vertical velocity solution is usually not applied in its original form. This runaway error problem is a potential drawback of any analysis technique—not just dual-Doppler analysis—in which the horizontal mass flux divergence is integrated upward vertically or along coplane azimuthal coordinate lines.

To ameliorate the integrated divergence error problem, O’Brien (1970) introduced a method in which provisional estimates of the horizontal divergence are variationally adjusted so as to satisfy the anelastic mass conservation equation as a strong constraint (i.e., exactly) with vertical velocity data specified on upper and lower boundaries (e.g., zero vertical velocity). In a related technique the vertical velocity field is obtained as the solution of a second-order mass conservation equation, that is, the mass conservation equation differentiated with respect to height (pressure) (Lateef 1967; O’Brien 1970). This second-order equation admits more solutions than the original first-order equation and can be solved with vertical velocity data imposed on both upper and lower boundaries. It is straightforward to show that this latter technique is equivalent to imposing the mass conservation equation as a weak constraint (i.e., in a least squares error sense): the Euler–Lagrange equation stemming from this weak constraint is just the second-order mass conservation equation. The vertical velocity problem and other practical dual-Doppler analysis issues (filtering, interpolation, advection corrections for nonsimultaneous data, etc.) are discussed in Miller and Strauch (1974), Ray et al. (1975), Ray et al. (1980), Gal-Chen (1982), Matejka and Srivastava (1982), Testud and Chong (1983), Chong et al. (1983), Chong and Testud (1983), Ziegler et al. (1983), Scialom and Lemaitre (1990), and in numerous other studies. The extension of the Armijo methodology to airborne dual-beam Doppler radar data (including correction for storm advection) is discussed in Chong and Testud (1996).

In the present investigation three new dual-Doppler wind analysis techniques are formulated using concepts from the calculus of variations (Sasaki 1970; Courant and Hilbert 1953). In each method we define a functional whose value we seek to minimize. The wind analysis consists of finding the velocity field associated with the functional minimum. Because of the radial (spherical) nature of the observations, the formulations proceed most naturally in the coplane cylindrical polar coordinate system (section 2). The first method (section 3) imposes mass conservation as a weak constraint and the radial wind observations as strong constraints. This method is similar to the Armijo procedure (1969), but the weak mass conservation constraint allows vertical velocity data to be specified on both upper and lower boundaries. In section 4 we impose mass conservation as a strong constraint and the radial wind observations as weak constraints. In the third formulation (section 5) we impose both mass conservation and the radial wind observations as weak constraints. In each of the latter two formulations, the analysis reduces to solving a second-order linear partial differential equation, the solution of which is unique. All three analyses are well posed when, for each point in the analysis domain, radial velocity data from both radars are available along the coplane azimuthal coordinate line running through that point and extending to lower and upper boundaries on which vertical velocity (or coplane azimuthal velocity) data are specified. Compared with traditional iterative methods (e.g., Ray et al. 1980; Ziegler et al. 1983; Kessinger et al. 1987), our new formulations preserve the radial nature of the observations, involve only one analysis step (all constraints are imposed in one functional), and are in a form in which the “well-posed condition” can most readily be checked. A summary and discussion of these methods are presented in section 6.

2. Governing equations and geometrical considerations

Consider a right-handed rectangular Cartesian x, y, z coordinate system in which the z-axis points in the direction opposite the gravity vector. The position vector of an analysis point with respect to the origin (0, 0, 0) of this coordinate system is given by \( \mathbf{r} = (x, y, z) \). The position vectors of radars 1 and 2 with respect to the origin are denoted by \( \mathbf{r}_1 = (x_1, y_1, z_1) \) and \( \mathbf{r}_2 = (x_2, y_2, z_2) \). The position vectors of an analysis point with respect to radars 1 and 2 are therefore \( \mathbf{R}_1 = \mathbf{r} - \mathbf{r}_1 \) and \( \mathbf{R}_2 = \mathbf{r} - \mathbf{r}_2 \). These position vectors can be written as \( \mathbf{R}_1 = R_1 \hat{\mathbf{R}}_1 \) and \( \mathbf{R}_2 = R_2 \hat{\mathbf{R}}_2 \), where \( \hat{\mathbf{R}}_1 \) and \( \hat{\mathbf{R}}_2 \) are unit vectors, and \( R_1 \) and \( R_2 \) are the magnitudes of \( \mathbf{R}_1 \) and \( \mathbf{R}_2 \). We also define \( \mathbf{R}_1 \times \mathbf{R}_2 \). Since \( \hat{\mathbf{R}}_1 \) and \( \hat{\mathbf{R}}_2 \) are not necessarily orthogonal, \( \mathbf{R}_1 \) is not necessarily a unit vector, although we can write it as \( \mathbf{R}_3 = R_1 \hat{\mathbf{R}}_1 \), where \( \hat{\mathbf{R}}_1 \) is a unit vector and \( R_3 \) is the magnitude of \( \mathbf{R}_1 \).

The velocity \( \mathbf{v}_s \) of the scattering particles in a volume sampled by the radars is related to the air velocity \( \mathbf{v} \) and the terminal velocity of the scatterers \( \mathbf{W}_s \) by \( \mathbf{v}_s = \mathbf{v} - \mathbf{W}_s \mathbf{k} \). Here \( \mathbf{k} \) is the unit vector directed opposite the (local) gravity vector and \( \mathbf{W}_s \) is considered positive. Assuming the radar beams are straight, the radial velocities observed by the first and second radars are \( v_{s1} = \hat{\mathbf{R}}_1 \cdot \mathbf{v}_s = \hat{\mathbf{R}}_1 \cdot (\mathbf{v} - \mathbf{W}_s \mathbf{k}) \) and \( v_{s2} = \hat{\mathbf{R}}_2 \cdot \mathbf{v}_s = \hat{\mathbf{R}}_2 \cdot (\mathbf{v} - \mathbf{W}_s \mathbf{k}) \). Assuming geopotential surfaces are flat (so \( \mathbf{k} \)
is everywhere parallel to the z-axis), the radial velocities can be written as

\[ R_1 u_x = (x - x_1) u + (y - y_1) v + (z - z_1)(w - W), \]

\[ R_2 u_y = (x - x_2) u + (y - y_2) v + (z - z_2)(w - W). \]

A number of semiempirical power laws are available to parameterize the terminal velocity of hydrometeor scatterers in terms of the reflectivity field (Atlas et al. 1973; Doviak and Zrnić 1993). Such parameterizations, however, are rather crude and should be used with caution, particularly at mid- and upper levels of the atmosphere. With \( W \) regarded as known, (1) and (2) comprise two equations in three unknowns: \( u, v, \) and \( w \). To close the system, we introduce the anelastic mass conservation equation,

\[ \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0, \tag{3} \]

where \( \rho = \rho(z) \) is a height-dependent background density field.

In the following sections it will be convenient to use a special cylindrical polar coordinate system (coplane system) in which the line connecting radars 1 and 2 (the “baseline”) comprises the central axis of the cylinder (see Fig. 1). The axial (or baseline) coordinate \( s \) measures the projection of the position vector of an analysis point (drawn from radar 1) on the baseline. The radial coordinate \( r \) measures the distance along the perpendicular from the baseline to an analysis point. The azimuthal angle \( \alpha \) is the angle between the plane containing the two position vectors \( \hat{R}_1 \) and \( \hat{R}_2 \) and a reference plane containing the baseline (for a flat earth, the reference plane can conveniently be taken as ground level). This azimuthal angle is not to be confused with the azimuthal angle of a spherical polar coordinate system (radar coordinates) associated with the individual radars. The unit basis vectors associated with the \( s, r, \) and \( \alpha \) coordinate lines in the coplane cylindrical polar coordinate system are denoted by \( \hat{s}, \hat{r}, \) and \( \hat{\alpha} \), respectively. If the cylindrical polar coordinate system is right-handed (so that \( \hat{\alpha} = \hat{s} \times \hat{r} \)), then \( \hat{\alpha} \) must equal \( \hat{R}_3 \).

Although we will sometimes refer to \( \alpha \) as a “quasivertical” coordinate, a curve of \( \alpha \) increasing traces a circle in which \( z \) increases on one side of the baseline and decreases on the other side of the baseline. In cases where radar data are present on both sides of the baseline (i.e., two dual-Doppler lobes are present), it may be convenient to perform a separate analysis for each lobe, with the number designation of radars 1 and 2 reversed for each lobe. In other words, we may use the criterion that increasing values of \( \alpha \) are to be associated with increasing values of \( z \) to establish, for a given lobe, the number designations for the two radars.

We will later have occasion to use the vector identities (Hay 1953),

\[ \hat{R}_3 \times \hat{r} = -\hat{s}, \quad \hat{R}_3 \times \hat{s} = \hat{r}, \quad \hat{s} \times \hat{r} = \hat{R}_3. \tag{4} \]

\[ \hat{R}_1 \cdot (\hat{R}_3 \times \hat{R}_2) = \hat{R}_1, \quad (\hat{R}_1 \times \hat{R}_3) = \hat{R}_1 \cdot \hat{R}_3 = R_1, \tag{5} \]

\[ (\hat{R}_1 \cdot \hat{s})(\hat{R}_2 \cdot \hat{r}) - (\hat{R}_1 \cdot \hat{r})(\hat{R}_2 \cdot \hat{s}) = (\hat{s} \times \hat{r}) \cdot (\hat{R}_1 \times \hat{R}_2) = \hat{R}_3 \cdot \hat{R}_1 R_3 = R_3. \tag{6} \]

Also, for future reference, we note that the velocity vector can be decomposed into its coplane components as, \( \mathbf{v} = (\mathbf{v} \cdot \hat{s})\hat{s} + (\mathbf{v} \cdot \hat{r})\hat{r} + (\mathbf{v} \cdot \hat{\alpha})\hat{\alpha} \). Or, letting \( U = (\mathbf{v} \cdot \hat{s}), \quad V = (\mathbf{v} \cdot \hat{r}), \) and \( W = (\mathbf{v} \cdot \hat{\alpha}) \) denote the baseline, radial, and azimuthal components of the wind field, respectively, we get \( \mathbf{v} = U\hat{s} + V\hat{r} + W\hat{\alpha} \).

In the coplane system, the mass conservation equation (3) becomes

\[ \frac{\partial \rho U}{\partial s} + \frac{\partial \rho V}{\partial r} + \frac{1}{r} \frac{\partial \rho W}{\partial \alpha} = 0. \tag{7} \]

The wind components in the directions of the two radars, \( \mathbf{v} \cdot \hat{R}_1 \) and \( \mathbf{v} \cdot \hat{R}_2 \), can be expressed in terms of the baseline and radial wind components as

\[ \mathbf{v} \cdot \hat{R}_1 = U(\hat{s} \cdot \hat{R}_1) + V(\hat{r} \cdot \hat{R}_1), \]

\[ \mathbf{v} \cdot \hat{R}_2 = U(\hat{s} \cdot \hat{R}_2) + V(\hat{r} \cdot \hat{R}_2). \tag{8} \]

3. Dual-Doppler analysis based on mass conservation as a weak constraint and radial wind observations as strong constraints

Here we seek a velocity field such that (1) and (2) are satisfied as strong constraints (i.e., as exact relations), while (3) is satisfied as a weak constraint (i.e., approximately, in a least squares error sense). Thus, we seek to minimize the functional
\[ J = \int \left[ \left( \frac{\partial \mu}{\partial x} + \frac{\partial \nu}{\partial y} + \frac{\partial \omega}{\partial z} \right)^2 \right. \]
\[ \left. + \lambda_1 \bar{R}_1 \bar{v}_n - (x - x_1)u - (y - y_1)v \right] - (z - z_1)(w - W) \]
\[ + \lambda_2 \bar{R}_2 \bar{v}_n - (x - x_2)u - (y - y_2)v \]
\[ - (z - z_2)(w - W) \right] dV, \quad (9) \]

where \( \lambda_1 \) and \( \lambda_2 \) are Lagrange undetermined multipliers and \( dV = dx \, dy \, dz \) is an element of the analysis volume. Here we have implicitly assumed that the weighting factor for the mass conservation constraint is constant, and without further loss of generality have chosen this constant to be unity (since there is only one weak constraint, division by the constant yields an equivalent functional with a weak constraint weight of unity).

To minimize \( J \), take the first variation of \( J \) with respect to \( u \), \( v \), and \( w \), integrate by parts, and set the integrands of the volume and surface (boundary) integrals separately equal to zero. In this manner we obtain the Euler–Lagrange equations as

\[ \frac{\partial}{\partial x} \left( \frac{\partial \mu}{\partial x} + \frac{\partial \nu}{\partial y} + \frac{\partial \omega}{\partial z} \right) + \lambda_1 (x - x_1) = 0, \quad (10) \]
\[ \frac{\partial}{\partial y} \left( \frac{\partial \mu}{\partial x} + \frac{\partial \nu}{\partial y} + \frac{\partial \omega}{\partial z} \right) + \lambda_1 (y - y_1) = 0, \quad (11) \]

subject to the boundary conditions that either (i) \( \bar{\partial} u/\bar{\partial} x + \bar{\partial} v/\bar{\partial} y + \bar{\partial} w/\bar{\partial} z = 0 \) (the so-called natural condition) or (ii) the normal velocity component is specified.

Equations (10)–(12) can be written concisely in vector form as

\[ 2\nabla D + \lambda_1 \bar{R}_1 \bar{\hat{R}}_1 + \lambda_2 \bar{R}_2 \bar{\hat{R}}_2 = 0, \quad (13) \]

where \( D = \bar{\partial} u/\bar{\partial} x + \bar{\partial} v/\bar{\partial} y + \bar{\partial} w/\bar{\partial} z \). Taking the dot product of (13) with \( \bar{\alpha} \), we obtain \( \partial D/\partial \alpha = 0 \), which integrates to

\[ D = C(s, r), \quad (14) \]

where \( C \) is a function of integration. That is, \( D \) is constant along coplane azimuthal coordinate lines.

The development of our least squares analysis technique (for which \( \partial D/\partial \alpha = 0 \) now parallels that of the exact formulation (for which \( D = 0 \)) as described by Armijo (1969). We obtain a single equation for one of the velocity components (say, \( w \)) by making use of (1), (2), and (14). Toward that end, solve (1) and (2) for \( u \) and \( v \) in terms of \( w \):

\[ u = \frac{[(y - y_1)\bar{R}_2 \bar{v}_n - (y - y_2)\bar{R}_1 \bar{v}_n] + (w - W)[(y - y_2)(z - z_1) - (y - y_1)(z - z_2)]}{(x - x_2)(y - y_1) - (x - x_1)(y - y_2)}, \]
\[ v = \frac{[(x - x_2)\bar{R}_1 \bar{v}_n - (x - x_1)\bar{R}_2 \bar{v}_n] + (w - W)[(x - x_1)(z - z_2) - (x - x_2)(z - z_1)]}{(x - x_2)(y - y_1) - (x - x_1)(y - y_2)}. \quad (15) \]

For the particular case where the origin of our coordinate system coincides with radar 1 (so \( x_1 = y_1 = z_1 = 0 \)), the \( x \)-axis is oriented such that it pierces radar 2 (\( y_2 = 0 \)) and the elevation of radar 1 is the same as that of radar 2 (\( z_2 = 0 \)) then (15) reduces to Armijo’s (3.1).

In general, we can rewrite (15) in terms of the dot products of \( \bar{\alpha} \) with the Cartesian basis vectors. We consider \( \bar{R}_1 \bar{\hat{R}}_2 = \bar{R}_1 \times \bar{R}_2 \) with each of the Cartesian basis vectors is then found to be proportional to a numerator or denominator in (15), and we obtain

\[ u = \Gamma + w \frac{\hat{j} \cdot \bar{\alpha}}{\hat{k} \cdot \bar{\alpha}}, \quad v = \psi + w \frac{\hat{j} \cdot \bar{\alpha}}{\hat{k} \cdot \bar{\alpha}}, \quad (16) \]

where \( \Gamma \) and \( \psi \) are the known functions,

\[ \Gamma = \frac{[(y - y_1)\bar{R}_2 \bar{v}_n - (y - y_2)\bar{R}_1 \bar{v}_n] - W[(y - y_2)(z - z_1) - (y - y_1)(z - z_2)]}{(x - x_2)(y - y_1) - (x - x_1)(y - y_2)}, \]
\[ \psi = \frac{[(x - x_2)\bar{R}_1 \bar{v}_n - (x - x_1)\bar{R}_2 \bar{v}_n] - W[(x - x_1)(z - z_2) - (x - x_2)(z - z_1)]}{(x - x_2)(y - y_1) - (x - x_1)(y - y_2)}. \quad (17) \]
Applying (16) in the definition of $D$, we obtain

$$D = \frac{\partial}{\partial x} \left( \frac{p_w}{k \cdot \mathbf{\hat{a}}} \right) + \frac{\partial}{\partial y} \left( \frac{p_w}{k \cdot \mathbf{\hat{a}}} \right) + \frac{\partial p_w}{\partial z} + \frac{\partial p \Gamma}{\partial x}$$

$$+ \frac{\partial p \psi}{\partial y}.$$  \hspace{1cm} (18)

Equation (18) can be written compactly in terms of a “pseudo-vertical velocity” $\Omega$ as

$$D = \nabla \cdot (\hat{\mathbf{a}} \Omega) - \beta,$$

$$\Omega = \frac{p_w}{k \cdot \hat{\mathbf{a}}},$$

$$\beta = -\left( \frac{\partial p \Gamma}{\partial x} + \frac{\partial p \psi}{\partial y} \right).$$  \hspace{1cm} (19)

Since $\nabla \cdot \hat{\mathbf{a}} = 0$, and $\mathbf{\hat{a}} \cdot \nabla = r^{-1}\partial / \partial \alpha$, (19) becomes

$$D = \frac{1}{r} \frac{\partial \Omega}{\partial \alpha} - \beta.$$  \hspace{1cm} (20)

Applying (20) in (14) yields

$$\frac{\partial \Omega}{\partial \alpha} = r\beta(s, r, \alpha) + rC(s, r).$$  \hspace{1cm} (21)

Since $\Omega$ is differentiated only with respect to $\alpha$, the partial differential Eq. (21) can be treated as an ordinary differential equation with independent variable $\alpha$ and other position dependencies treated as parameters. The solution of (21) is

$$\Omega = \int_{\alpha_0}^{\alpha} (C + \beta) r \, d\alpha' + E,$$  \hspace{1cm} (22)

where $E = E(s, r)$ is another function of integration and $\alpha_0$ is the lower limit of integration (the lowest value of $\alpha$ for which there are data). The two functions of integration can be determined by applying boundary conditions on $w$ at the two ends of an arc of constant radius (i.e., the upper and lower azimuthal angles $\alpha$). In other words, two conditions on $w$ are required to evaluate (22).

It can be noted, however, that if we only wanted to specify $w$ on one of these $\alpha$ boundaries, then we would have to impose the natural boundary condition $D = 0$ along the other of these boundaries and (14) would yield $D = 0$ throughout the domain. In this case the problem becomes one of purely strong constraint analysis, with exact solution described by Armijo (1969).

4. Dual-Doppler analysis based on mass conservation as a strong constraint and radial wind observations as weak constraints

Here we seek a velocity field such that (1) and (2) (divided by their respective radii) are satisfied as weak constraints, while (3) is satisfied exactly. Division of (1) and (2) by the radii prevents data at large radii from dominating the solution. Thus, we seek to minimize the functional

$$J = \int_v \left[ \lambda \left( \frac{\partial p \mathbf{\mu}}{\partial x} + \frac{\partial p \mathbf{\nu}}{\partial y} + \frac{\partial p \mathbf{\psi}}{\partial z} \right) ight.$$  

$$+ \frac{\mu_1 p^2}{R_1^2}[R_1 v_1 - (x - x_1)u - (y - y_1)v$$  

$$- (z - z_1)(w - W_1)^2]$$  

$$+ \frac{\mu_2 p^2}{R_2^2}[R_2 v_2 - (x - x_2)u - (y - y_2)v$$  

$$- (z - z_2)(w - W_2)^2] \right] \, dV,$$  \hspace{1cm} (23)

where $\lambda$ is a Lagrange undetermined multiplier, $\mu_1$ and $\mu_2$ are weights (Gauss Precision Moduli), and other symbols are as defined previously. Again, taking the first variation of $J$ with respect to $u$, $v$, and $w$, integrating by parts, and setting the integrands of the volume and surface integrals to zero, we obtain the Euler–Lagrange equations,

$$\frac{\partial \lambda}{\partial x} + 2\mu_1 p \frac{(x - x_1)}{R_1^2}$$  

$$\times [R_1 v_1 - (x - x_1)u - (y - y_1)v - (z - z_1)(w - W_1)]$$

$$+ 2\mu_2 p \frac{(x - x_2)}{R_2^2} [R_2 v_2 - (x - x_2)u - (y - y_2)v$$

$$- (z - z_2)(w - W_2)] = 0,$$  \hspace{1cm} (24)

$$\frac{\partial \lambda}{\partial y} + 2\mu_1 p \frac{(y - y_1)}{R_1^2}$$  

$$\times [R_1 v_1 - (x - x_1)u - (y - y_1)v - (z - z_1)(w - W_1)]$$

$$+ 2\mu_2 p \frac{(y - y_2)}{R_2^2} [R_2 v_2 - (x - x_2)u - (y - y_2)v$$

$$- (z - z_2)(w - W_2)] = 0,$$  \hspace{1cm} (25)

$$\frac{\partial \lambda}{\partial z} + 2\mu_1 p \frac{(z - z_1)}{R_1^2}$$  

$$\times [R_1 v_1 - (x - x_1)u - (y - y_1)v - (z - z_1)(w - W_1)]$$

$$+ 2\mu_2 p \frac{(z - z_2)}{R_2^2} [R_2 v_2 - (x - x_2)u - (y - y_2)v$$

$$- (z - z_2)(w - W_2)] = 0,$$  \hspace{1cm} (26)

subject to the boundary conditions that either (i) $\lambda = 0$ or (ii) the normal velocity component is specified. Equations (24)–(26) can be written in vector form as

$$\n \nabla \lambda + 2\mu_1 p \hat{\mathbf{R}}_1 [v_1 - \hat{\mathbf{R}}_1 \cdot (v - W_1 \hat{\mathbf{k}})]$$

$$+ 2\mu_2 p \hat{\mathbf{R}}_2 [v_2 - \hat{\mathbf{R}}_2 \cdot (v - W_2 \hat{\mathbf{k}})] = 0.$$  \hspace{1cm} (27)

For later use, we note that the dot product of (27) with $\hat{\mathbf{a}}$ yields
Applying (8) in (27), we get
\[ \nabla \lambda + 2g - 2\mu_1 \overline{\nabla} \lambda_1[(\hat{\mathbf{R}}_1 \cdot \hat{\mathbf{R}}_1) V + (\hat{\mathbf{R}}_1 \cdot \hat{\mathbf{R}}_1) U] \\
- 2\mu_2 \overline{\nabla} \lambda_2[(\hat{\mathbf{R}}_2 \cdot \hat{\mathbf{R}}_2) U + (\hat{\mathbf{R}}_2 \cdot \hat{\mathbf{R}}_2) V] = 0, \] (29)
where \( g \) is a vector containing the observational data,
\[ g = \overline{\nabla} \lambda_1(v_{11} + W_1 \hat{\mathbf{R}}_1 \cdot \hat{\mathbf{R}}_1) \]
\[ + \overline{\nabla} \lambda_2(v_{22} + W_2 \hat{\mathbf{R}}_2 \cdot \hat{\mathbf{R}}_2). \] (30)
The radial and baseline components of (29) can be written as
\[ \frac{\partial \lambda}{\partial r} + 2g \cdot \hat{r} + 2\overline{\nabla} \lambda_1 [(ac + bd)U - (c^2 + d^2)V] = 0, \]
\[ \frac{\partial \lambda}{\partial s} + 2g \cdot \hat{s} + 2\overline{\nabla} \lambda_2 [-(a^2 + b^2)U + (ac + bd)V] = 0, \] (31)
where
\[ a = -\frac{\hat{\mathbf{R}}_1 \cdot \hat{\mathbf{R}}_1}{R_1 \sqrt{\mu_1}}, \quad b = \frac{\hat{\mathbf{R}}_1 \cdot \hat{\mathbf{R}}_1}{R_1 \sqrt{\mu_2}}, \]
\[ c = \frac{\hat{\mathbf{R}}_2 \cdot \hat{\mathbf{R}}_1}{R_2 \sqrt{\mu_1}}, \quad d = -\frac{\hat{\mathbf{R}}_2 \cdot \hat{\mathbf{R}}_2}{R_2 \sqrt{\mu_2}}. \] (32)
Solving (31) for \( u \) and \( v \), we get
\[ \overline{W} = \mu \frac{\partial \lambda}{\partial r} + \xi \frac{\partial \lambda}{\partial s} + \mu(\hat{\mathbf{r}} \cdot \mathbf{g}) + \xi(\hat{\mathbf{s}} \cdot \mathbf{g}), \]
\[ \overline{V} = \mu \frac{\partial \lambda}{\partial s} + \nu \frac{\partial \lambda}{\partial r} + \mu(\hat{\mathbf{s}} \cdot \mathbf{g}) + \nu(\hat{\mathbf{r}} \cdot \mathbf{g}), \] (33)
where
\[ \nu = a^2 + b^2, \quad \mu = ac + bd, \]
\[ \xi = c^2 + d^2, \] (34)
and we have used \((bc - ad)^{-1} = R_1 \sqrt{\mu_1 \mu_2}\), which can be proved with the aid of (6).

Applying (33) in the mass conservation Eq. (7) results in
\[ -2 \frac{\partial \overline{W}}{\partial r} = \nu \frac{\partial^2 \lambda}{\partial r^2} + \xi \frac{\partial^2 \lambda}{\partial s^2} + 2\mu \frac{\partial^2 \lambda}{\partial s \partial r} \]
\[ + \frac{\partial \lambda}{\partial r} \left( \frac{\partial \nu}{\partial s} + \frac{\partial \nu}{\partial r} \right) + \frac{\partial \lambda}{\partial s} \left( \frac{\partial \xi}{\partial s} + \frac{\partial \mu}{\partial r} \right) \]
\[ + \frac{2}{2} \frac{\partial}{\partial s} [\mu(\hat{\mathbf{r}} \cdot \mathbf{g}) + \xi(\hat{\mathbf{s}} \cdot \mathbf{g})] \]
\[ + \frac{2}{2} \frac{\partial}{\partial r} [\mu(\hat{\mathbf{s}} \cdot \mathbf{g}) + \nu(\hat{\mathbf{r}} \cdot \mathbf{g})]. \] (35)
Integrating (35) with respect to \( \alpha \) upward from the lower data surface, \( \alpha_0 = \alpha_0(s, r) \), and noting from (28) that \( \lambda \) is independent of \( \alpha \), we get
\[ 2\overline{P}W = 2\overline{P}W_0 - A_a \frac{\partial^2 \lambda}{\partial \alpha^2} - B_a \frac{\partial^2 \lambda}{\partial \alpha \partial \sigma} - C_a \frac{\partial^2 \lambda}{\partial \sigma^2} \]
\[ - D_a \frac{\partial \lambda}{\partial r} - E_a \frac{\partial \lambda}{\partial s} - F_a, \] (36)
where
\[ A_a = \int_a^{a_0} \nu r \, d\alpha', \quad B_a = 2 \int_a^{a_0} \mu r \, d\alpha', \]
\[ C_a = \int_a^{a_0} \xi r \, d\alpha', \quad D_a = \int_a^{a_0} \left( \frac{\partial \lambda}{\partial s} + \frac{\partial \nu}{\partial r} \right) r \, d\alpha', \]
\[ E_a = \int_a^{a_0} \left( \frac{\partial \xi}{\partial s} + \frac{\partial \mu}{\partial r} \right) r \, d\alpha', \]
\[ F_a = 2 \int_a^{a_0} \left[ \frac{\partial}{\partial s} [\mu(\hat{\mathbf{r}} \cdot \mathbf{g}) + \xi(\hat{\mathbf{s}} \cdot \mathbf{g})] \right. \]
\[ + \left. \frac{\partial}{\partial r} [\mu(\hat{\mathbf{s}} \cdot \mathbf{g}) + \nu(\hat{\mathbf{r}} \cdot \mathbf{g})] \right] r \, d\alpha'. \] (37)
Here \( \overline{P}W_0 \) is \( \overline{P}W \) evaluated at \( \alpha = \alpha_0(s, r) \). If we evaluate (36) at a top data surface \( \alpha = \alpha_t(s, r) \) on which \( \overline{P}W \) is assumed known, \( \overline{P}W = \overline{P}W_1 \), we obtain a second-order partial differential equation for \( \lambda \):
\[ A_a \frac{\partial^2 \lambda}{\partial \alpha^2} + B_a \frac{\partial^2 \lambda}{\partial \alpha \partial \sigma} + C_a \frac{\partial^2 \lambda}{\partial \sigma^2} + D_a \frac{\partial \lambda}{\partial r} + E_a \frac{\partial \lambda}{\partial s} + F_a = 0. \] (38)
Here the coefficients \( A_a, F_a \) are just the functions \( A_a, F_a \) of (37) evaluated at \( \alpha = \alpha_0(s, r) \).

If boundary data for \( w \) (instead of \( W \)) are known, \( w = w_0(s, r) \) on the lower boundary and \( \overline{P}W = \overline{P}W_0 \) on the upper boundary, (38) can be suitably modified. By considering \( w = \hat{\mathbf{k}} \cdot \nabla \), with \( \mathbf{v} = U\hat{\mathbf{s}} + V\hat{\mathbf{r}} + \hat{\mathbf{k}} \), and \( U \) and \( V \) obtained from (33), we find that
\[ \overline{P}W = \frac{\overline{P}w}{\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}} - \eta - \theta - 2\eta \hat{\mathbf{r}} \cdot \mathbf{g} - 2\theta \hat{\mathbf{s}} \cdot \mathbf{g}. \] (39)
where \( \eta = \frac{\mu(\hat{\mathbf{k}} \cdot \hat{\mathbf{s}}) + \nu(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}})}{2\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}} \), \( \theta = \frac{\xi(\hat{\mathbf{k}} \cdot \hat{\mathbf{s}}) + \mu(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}})}{2\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}} \).

Evaluating (39) at the upper and lower boundaries, and substituting in (38), we obtain an equation of the same form as (38) but with \( \overline{P}W_0 = \overline{P}w_0 \) in place of \( \overline{P}W_1 = \overline{P}w_0 \), and with \( D_a, E_a, F_a \) redefined. The coefficients of the second derivative terms are not affected by this procedure.

To show that (38) is elliptic we must show that the
discriminant $\Delta = B^2_{\alpha_1}/A_{\alpha_1} - A_{\alpha_1}^{\alpha_1}$ is negative. Toward that end, write

$$\Delta = \left( \int_{a_1}^{b_1} (ac + bd)r \, d\alpha \right)^2 - \int_{a_1}^{b_1} (a^2 + b^2)r \, d\alpha' \int_{a_1}^{b_1} (c^2 + d^2)r \, d\alpha',$$

and apply Schwarz's inequality to the expanded form of the first term on the right-hand side:

$$\left( \int_{a_1}^{b_1} (ac + bd)r \, d\alpha \right)^2 = \left( \int_{a_1}^{b_1} acr \, d\alpha' \right)^2 + \left( \int_{a_1}^{b_1} bdr \, d\alpha' \right)^2 + 2 \int_{a_1}^{b_1} acr \, d\alpha' \int_{a_1}^{b_1} bdr \, d\alpha'
\leq \int_{a_1}^{b_1} a^2r \, d\alpha' \int_{a_1}^{b_1} c^2r \, d\alpha' + \int_{a_1}^{b_1} b^2r \, d\alpha' \int_{a_1}^{b_1} d^2r \, d\alpha'
+ 2 \left( \int_{a_1}^{b_1} a^2r \, d\alpha' \int_{a_1}^{b_1} c^2r \, d\alpha' \int_{a_1}^{b_1} d^2r \, d\alpha' \right)^{1/2}.
\tag{41}$$

Then, applying (41) in (40), we find that

$$\Delta \leq 2 \left( \int_{a_1}^{b_1} a^2r \, d\alpha' \int_{a_1}^{b_1} c^2r \, d\alpha' \int_{a_1}^{b_1} b^2r \, d\alpha' \int_{a_1}^{b_1} d^2r \, d\alpha' \right)^{1/2}
- \int_{a_1}^{b_1} a^2r \, d\alpha' \int_{a_1}^{b_1} d^2r \, d\alpha' - \int_{a_1}^{b_1} b^2r \, d\alpha' \int_{a_1}^{b_1} c^2r \, d\alpha'
= - \left( \int_{a_1}^{b_1} a^2r \, d\alpha' \int_{a_1}^{b_1} d^2r \, d\alpha' \right)^{1/2}
- \left( \int_{a_1}^{b_1} b^2r \, d\alpha' \int_{a_1}^{b_1} c^2r \, d\alpha' \right)^{1/2}
\leq 0.
\tag{42}$$

Thus (38) is elliptic except for the special points for which the equalities in (42) hold, for example, along the baseline ($r = 0$). In our application of the Schwarz inequality, we have assumed that our weights $\mu_1$ and $\mu_2$ are positive (the weights are proportional to the reciprocal error variances, which are positive) so that $a^2$, $b^2$, $c^2$, and $d^2$ are real and positive. Solution uniqueness follows from a maximum principle for linear elliptic equations (Smirnov 1964, p. 694).

The foregoing results can be used as the basis for a dual-Doppler wind analysis algorithm. First solve (38) for $\lambda$ on $r, s$ planes with lateral boundary conditions that arose from the minimization procedure (in practice we would apply the natural boundary condition $\lambda = 0$). The $U$ and $V$ (baseline and radial) components of the wind field follow from (33) and $W$ is obtained from (36). As in the previous formulation, we may specify boundary conditions on $W$ (or $W$) at the upper and lower $\alpha$ boundaries. However, if we only specified $w$ on one of these boundaries, then we would impose the natural boundary condition $\lambda = 0$ along the other of these boundaries and (28) would yield $\lambda = 0$ throughout the domain. In this case (27) shows that the radial wind constraints would be satisfied exactly and our problem would again become that of strong constraint analysis.

5. Dual-Doppler analysis with mass conservation and radial wind observations imposed as weak constraints

Now consider a dual-Doppler wind analysis in which mass conservation and the radial wind observations are imposed as weak constraints. Here we seek to minimize the functional

$$J = \int \mu \left( \frac{\partial \tilde{\mathbf{u}}}{\partial x} + \frac{\partial \tilde{\mathbf{v}}}{\partial y} + \frac{\partial \tilde{\mathbf{w}}}{\partial z} \right)^2
+ \frac{\mu_1}{R_1^2} [R_1 v_1 - (x - x_1)u - (y - y_1)v - (z - z_1)(w - W_1)]^2
+ \frac{\mu_2}{R_2^2} [R_2 v_2 - (x - x_2)u - (y - y_2)v - (z - z_2)(w - W_2)]^2 \, dV,
\tag{43}$$

where $\mu_1$ is the weight for the mass conservation constraint (for convenience we suppose it is constant) and other symbols are as defined previously. Weak constraints for mass conservation, radial wind observations, and the impermeability condition on $w$ at ground level were adopted by Scialom and Lemaître (1990) within the context of their Multiple Analytical Doppler analysis. In that study, each of the wind components was expanded in a series of orthonormal functions in each of the three Cartesian coordinates. The resulting Euler-Lagrange equations led to a matrix equation for the amplitudes of the orthonormal functions. In our present formulation we seek to minimize the functional without imposing a spatial structure on the solution. Accordingly, we minimize the functional in gridpoint space rather than in spectral space. The development of our minimization procedure parallels that of the previous section and we again find that use of the coplane system reduces the problem to two-dimensional complexity.
The Euler–Lagrange equations associated with (43) can be cast in vector form as
\[ \mu_3 \nabla D + \mu_1 \tilde{\nabla} \left[ (a c + b d) U - (c^2 + d^2) V \right] \]
where \( D = \partial \tilde{\psi} / \partial x + \partial \tilde{\psi} / \partial y + \partial \tilde{\psi} / \partial z \). The boundary conditions are that either (i) \( D = 0 \) or (ii) the normal velocity component is specified.

It is straightforward to show that \( \partial D / \partial \alpha = 0 \), while the radial and baseline components of (44) satisfy
\[ \frac{\partial D}{\partial r} + g \cdot \hat{r} + \nabla \cdot \mu_1 \mu_2 \bar{R}^2 [(a c + b d)U - (c^2 + d^2)V] \]
\[ = 0, \]
\[ \frac{\partial D}{\partial s} + g \cdot \hat{s} + \nabla \cdot \mu_1 \mu_2 \bar{R}^2 [-(a c + b d)U + (a^2 + b^2)V] \]
\[ = 0, \]
where \( g \) is defined in (30) and \( a, b, c, d \) are defined in (32). These equations are analogous to (31) [note the factor of 2 difference and the presence of \( \mu_3 \) in (45)].

We obtain \( \bar{p} U \) and \( \bar{p} V \) from (45) as
\[ \bar{p} U = \mu_3 \left( \frac{\partial D}{\partial r} + \xi \frac{\partial D}{\partial s} + \mu (\hat{r} \cdot g) + \xi (\hat{s} \cdot g) \right), \]
\[ \bar{p} V = \mu_3 \left( \frac{\partial D}{\partial s} + \nu \frac{\partial D}{\partial r} + \mu (\hat{s} \cdot g) + \nu (\hat{r} \cdot g) \right), \]
where \( \mu, \nu, \) and \( \xi \) are defined in (34). Applying these expressions for \( \bar{p} U \) and \( \bar{p} V \) in the definition of \( D \) in coplane coordinates \( D = \partial \tilde{\psi} / \partial s + \partial \tilde{\psi} / \partial r + r^{-1} \partial \tilde{\psi} / \partial \alpha \) and integrating with respect to \( \alpha \) upward from the lower data surface, \( \alpha_0 = \alpha_0(s, r) \), (noting that \( D \) is independent of \( \alpha \)), we get
\[ \bar{p} W = \bar{p} W_0 - A_1 \frac{\partial D}{\partial r} - B_1 \frac{\partial D}{\partial s} - C_1 \frac{\partial D}{\partial r} - D_1 \frac{\partial D}{\partial r} \]
\[ - E_1 \frac{\partial D}{\partial s} + r(\alpha - \alpha_0)D - F_1, \]
where
\[ A_1 = \int_{\alpha_0}^{\alpha} v r d \alpha', \quad B_1 = 2 \mu_3 \int_{\alpha_0}^{\alpha} \mu r d \alpha', \]
\[ C_1 = \int_{\alpha_0}^{\alpha} \xi r d \alpha', \quad D_1 = \mu_3 \int_{\alpha_0}^{\alpha} \left( \frac{\partial \mu}{\partial s} + \frac{\partial \nu}{\partial r} \right) r d \alpha', \]
\[ E_1 = \int_{\alpha_0}^{\alpha} \left( \frac{\partial \xi}{\partial s} + \frac{\partial \mu}{\partial r} \right) r d \alpha', \]
\[ F_1 = \int_{\alpha_0}^{\alpha} \frac{\partial}{\partial s} \left[ \mu (\hat{r} \cdot g) + \xi (\hat{s} \cdot g) \right] r d \alpha' + \frac{\partial}{\partial r} \left[ \mu (\hat{s} \cdot g) + \nu (\hat{r} \cdot g) \right] r d \alpha'. \]
Evaluating (47) on a top data surface \( \alpha = \alpha(s, r) \) on which \( \bar{p} W \) is assumed known, \( \bar{p} W = \bar{p}_0 W_0 \), we obtain a second-order linear partial differential equation for \( D \):
\[ A_1 \frac{\partial^2 D}{\partial r^2} + B_1 \frac{\partial^2 D}{\partial s^2} + C_1 \frac{\partial^2 D}{\partial r^2} + D_1 \frac{\partial D}{\partial r} + E_1 \frac{\partial D}{\partial s} + F_1 = 0. \]
Here \( A_1, B_1, C_1, D_1, E_1, F_1 \) are just the functions \( A_1 = F_1 \) of (48) evaluated at \( \alpha = \alpha(s, r) \).

Equations (47)–(49) are analogous to (36)–(38). If boundary data for \( w \) (but not \( W \) ) are known, (49) can be modified as in the previous section. The ellipticity of (49) (and solution uniqueness) can be established through a slight modification of the proof in the previous section. It is interesting to note that if the coefficient of the undifferentiated \( D \) term in (49) had been positive instead of negative, solution uniqueness would not be guaranteed (Smirnov 1964).

To obtain the velocity field, first solve (49) for \( D \) on \( r, s \) planes with lateral boundary conditions that arose from the minimization procedure (in practice we would impose the natural condition \( D = 0 \) ). Then \( U \) and \( V \) follow from (46) and \( W \) is obtained from (47). Again, conditions on \( w \) (or \( W \) ) are required at the upper and lower \( \alpha \) boundaries to evaluate the solution. If we only specify \( w \) on one of these boundaries, then we would impose the natural condition \( D = 0 \) along the other of these boundaries. In this case \( D = 0 \) throughout the domain, the radial wind constraints would also be satisfied exactly [from (44)], and our problem would again become that of strong constraint analysis.

6. Concluding remarks

Three new formulations of three-dimensional dual-Doppler radar wind analysis are proposed. In each method we define a functional incorporating radial wind observations and the anelastic mass conservation equation as either strong (exact) or weak (least squares error) constraints. We obtain the velocity field associated with the functional minimum by solving the corresponding Euler–Lagrange equations. In our first method, mass conservation is applied as a weak constraint, while the radial wind observations are imposed as a strong constraint (i.e., the projected wind vectors are forced to agree with the observed radial winds). In our second technique, mass conservation is imposed as a strong constraint, while the radial wind observations are used as weak constraints. In our third technique, both mass conservation and the radial wind observations are used as weak constraints. As in the purely strong constraint analysis of Armijo (1969), transformation to the coplane \( (r, s, \alpha) \) cylindrical polar coordinate system greatly reduces the complexity of our analyses. The first method yields the vertical velocity field in the form of an integral along coplane azimuthal \( (\alpha) \) coordinate lines and provides simple formulas for the horizontal wind compo-
ments. The second method yields formulas for the coplane velocity components in terms of derivatives of a Lagrange multiplier \( \lambda \), while the third method yields formulas for the coplane velocity components in terms of derivatives of the three-dimensional mass divergence \( D \). The \( \lambda \) and \( D \) are independent of the coplane azimuthal coordinate, and each satisfies a two-dimensional planar \((r, s)\) second-order linear elliptic partial differential equation, the solution of which is unique.

These new formulations are of practical value in that they can be used for fairly simple dual-Doppler analysis algorithms: evaluating simple formulas and, in the second and third methods, solving a two-dimensional second-order linear partial differential equation. For each method, the well-posed condition is obtained by examining the data dependencies in these formulas and in the coefficients of the second-order partial differential equations. The new formulations are well posed when, for each point in the analysis domain, radial velocity data from both radars are available along the coplane azimuthal coordinate line running through that point and extending to lower and upper boundaries on which vertical velocity \( w \) (or coplane \( W \)) data are specified. This imposes a restriction on the shape of the domains in which the problems are well posed. If vertical velocity data are specified on only one of these boundaries, then the problem becomes one of purely strong constraint analysis, with exact solution described by Armijo (1969). Regions of the domain that fail to satisfy the well-posed condition are associated with functions of integration or coefficients that cannot be evaluated. Such points are “orphaned” by these procedures.

Our new formulations can be compared with traditional dual-Doppler techniques that also combine radial wind observations with the mass conservation equation (e.g., Ray et al. 1980; Ziegler et al. 1983; Kessinger et al. 1987). The traditional methods obtain provisional estimates of \( u, v, \) and \( w \) by iterating between formulas for \( u \) and \( v \) [e.g., our (15) or similar expressions obtained from a least squares approach] and the formula for \( w \) obtained by integrating the anelastic mass conservation equation upward or downward. These estimates are then variationally adjusted using the vertically integrated mass conservation equation (in anelastic or incompressible form) as a strong constraint with vertical velocity data specified on upper and lower boundaries. An obvious advantage of these traditional methods is that they proceed on Cartesian grids and are easy to implement. On the other hand, since the three Cartesian velocity components are adjusted in the second step without considering the radial nature of the observations, the radial wind constraints imposed in the first step will no longer be satisfied in the forms originally imposed. In contrast, the analyzed wind fields in our one-step (one functional) formulations satisfy the radial wind constraints (strong or weak) as specified. Perhaps more importantly, solution uniqueness cannot be guaranteed for the iterative methods since they typically proceed without considering the well-posed condition. Furthermore, we note that the iterative methods become unstable when the ratio of the vertical grid spacing \( \Delta z \) to the smallest of the horizontal grid spacings (\( \Delta x \) or \( \Delta y \)) exceeds the cotangent of the elevation angle (Ray et al. 1985). Geometrically, this instability condition is associated with grid boxes that are so tall that no coplane azimuthal coordinate lines piercing the top of the box can exit through the bottom of the box (i.e., the lines leave the grid box through its sides).

As an alternative to the minimization procedures described herein involving transformations to the coplane system, one could seek to minimize the same functionals with a conjugate gradient algorithm or other iterative descent algorithm (Navon and Legler 1987). In general, these descent algorithms can only be used to find local (in state space) functional minima. However, if well-posed considerations are used to reject orphaned analysis points, a local minimum should be the global minimum. A drawback of these descent algorithms is that they are fully three-dimensional and can be computationally expensive. Furthermore, care must be taken when specifying a convergence threshold as large solution errors can result if the iterations stop prematurely on a relatively flat functional topography. However, an attractive feature of the descent algorithms is that they can be readily generalized to account for data from a third (or more) radars or additional constraints (e.g., smoothness constraints). In contrast, our proposed formulations are not readily generalizable. Any modification to our original functionals would change the Euler–Lagrange equations and necessitate a new derivation. In the case of three (or more) radars, the simplifications afforded to our formulations by the coplane system would disappear.

In both the traditional and newly proposed dual-Doppler analysis techniques, one must contend with the problem of boundary conditions on the vertical velocity field (or coplane \( W \) field) on the lower and/or upper boundaries. The natural boundaries for the problem are the irregular lower/upper boundaries of the data region itself. If the region of dual-Doppler data coverage extended all the way to ground level, then the impermeability condition could safely be applied. Unfortunately, the lower data boundary often lies hundreds to thousands of meters above ground level, where it is often inappropriate to apply the impermeability condition. A “weak echo region” or “bounded weak echo region” in severe storm updrafts (Marwitz et al. 1972) provides an extreme example of this difficulty. Furthermore, since horizontal velocity divergence can be strong (often the strongest) near the ground, it is often dangerous to extrapolate dual-Doppler analyzed velocity divergence down to ground level. This is especially true for downbursts, microbursts, cold outflows, and the shallow low-level inflows into severe and tornadic thunderstorms. In cases where high-resolution wind data are available at ground level, for example, from a mesonetwork, it
would presumably be safe to interpolate the divergence across the data void, but in general, these data will not be available.

Special mention should be made of Sun and Crook’s (1996b) adjoint approach to the vertical velocity and thermodynamic retrieval problem. Their method imposes dynamical constraints (the dry, Boussinesq equations of motion and thermodynamic energy) and methods of optimal control (the “adjoint” technique) to obtain the vertical velocity field as well as thermodynamic variables from the horizontal wind field. Although the Sun and Crook method was applied to simulated data that extended down to the lower physical boundary, the use of the dynamical equations to aid in the recovery of the vertical velocity field is an exciting new development.

Planned work with the new dual-Doppler formulations includes testing the algorithms with real and simulated radar data. We will also investigate the use of dynamical constraints to provide additional information about the vertical velocity field, for example, by “inverting” the Boussinesq form of the vertical vorticity equation to derive the boundary condition for $w$ on the lower boundary of the data coverage.

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