Beam Broadening Effect on Oblique MST Radar Doppler Spectrum

YEN-HSYANG CHU
Institute of Space Science, National Central University, Chung-Li, Taiwan, Republic of China

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ABSTRACT

Beam broadening effect on Doppler spectrum for an obliquely pointed meso-, strato-, troposphere (MST) radar beam is investigated theoretically in this article. The general behavior of radial velocity on the cross section of an oblique radar volume is analyzed first. It is found that the contour of radial velocity can be described perfectly by a set of concentric circles. The coordinate of the center of the circle is governed not only by the ratio of vertical w to horizontal U wind velocities, but also by the tilt angle of the radar beam. The contour of radial velocity is approximate to a straight line if the ratio of w to U is low (<0.35) and the tilt angle of the antenna beam is small (<30°). Under these conditions, the analytic expression of the corresponding beam broadening spectrum for an oblique radar beam with beamwidth of a few degrees is derived. The results show that the width of the beam broadening spectrum is not only determined by antenna beamwidth and horizontal wind velocity, but also governed by the tilt angle of antenna beam and the ratio of w to U. Quantitative calculation indicates that the spectral width at zenith direction may be greater than that at a zenith angle of 30° by about 32%, provided the ratio of vertical to horizontal wind velocities is 0.4 and the horizontal wind is parallel to the antenna beam. Moreover, it is found that horizontal wind direction plays a crucial role in determining the beam broadening width. Theoretical calculation shows that a change in horizontal wind direction from 0° to 90° with respect to direction of the oblique radar beam can lead to a 15% change in beam broadening spectral width for a 20° zenith angle of antenna beam and 0.2 of w/U. These results suggest that the effects of tilt angle of antenna beam, horizontal wind direction, and the ratio of vertical to horizontal wind velocities on Doppler spectral width of an obliquely directed MST radar beam should be taken into account in analyzing the radar returns to estimate atmospheric parameters of interest from the observed Doppler spectral width.

1. Introduction

It is recognized that beam broadening spectral width, induced by radar targets drifted with background wind across antenna beam with finite width, substantially dominates the breadth of meso-, strato-, troposphere (MST) radar Doppler spectrum for the radar returns from diffusive targets, including refractivity fluctuations (Hocking 1985; Woodman and Chu 1989) and precipitation particles (Wakasugi et al. 1986; Chu et al. 1997). Apparently, the beam broadening spectral width should be removed thoroughly from the observed Doppler spectral width before the desired atmospheric parameters are extracted from the observed one. The quantitative estimation of the beam broadening spectral width has long been a topic of interest in the atmospheric radar community. For example, Atlas (1964) suggested the expression $0.6 U \theta_{1/2}/\lambda$ to quantitatively estimate beam broadening spectral width of a vertical antenna beam caused by horizontal wind $U$ across the sampling volume, where $\lambda$ is radar wavelength and $\theta_{1/2}$ is half-power beamwidth (HPBW) of antenna beam. The expression derived by Sloss and Atlas (1968) for the estimation of beam broadening spectral width of an obliquely pointed radar beam is $U \theta_{1/2} \cos \delta/(\sqrt{2.76} \lambda)$, where $\delta$ is zenith angle of radar beam axis. Gage and Balsley (1978) proposed a different equation $2U \sin(\theta_{1/2}/\lambda)$ to calculate the vertical beam broadening spectral width. On the basis of numerical simulation method, Hocking (1985) found that the beam broadening spectral width due to horizontal wind can be estimated according to $U \theta_{1/2}/(\sqrt{2} \ln 2 \lambda \cos \delta)$. Pan and Liu (1992) established a model for oblique spaced antenna technique and proposed that the beam broadening spectral width for an off vertical antenna beam can be calculated in accordance with $U \theta_{1/2}/(\sqrt{2} \ln 2 \lambda \cos \delta)$. Recently, by considering the effects of finite beamwidth together with vertical wind shear of horizontal wind, Nastrom (1997) derived an analytical expression to quantitatively estimate their contributions to Doppler spectral width. His result shows that, by neglecting the wind shear effect, the beam broadening spectral width due to horizontal wind can be approximately estimated as $U \theta_{1/2} \cos \delta/(\sqrt{3} \lambda)$. In light of no general agreement on the quantitative estimation of beam broadening spectral width, more investigation of this issue is required to clarify the differences between theoretical results as mentioned above.
One of fundamental and crucial issues in investigating the beam broadening effect on Doppler spectrum is to ascertain the characteristics of radial velocity distributed in the radar volume. Although the behavior of radial velocity contour for a vertically pointed radar beam has been investigated (Palmer et al. 1997; Larsen and Palmer 1997), to the author’s knowledge, the characteristics of radial velocity for an oblique antenna beam are not well documented. In this article, we will show that the ratio of vertical to horizontal wind velocities, zenith angle of antenna beam, and azimuth angle of horizontal wind relative to antenna beam play crucial roles in the determination of the pattern of radial velocity for an oblique antenna beam.

In order to quantitatively estimate the contribution of beam broadening effect on oblique MST radar Doppler spectrum, the following expression is employed extensively by the radar community to model the beam broadening spectrum (Woodman and Chu 1989)

\[ S_v(\omega) = S_v e^{-\left(\frac{\omega + \mathbf{k} \cdot \mathbf{v}}{2\sigma^2(\delta)}\right)^2} \]

where \( \omega = 2\pi f \) is Doppler frequency in units of radian s\(^{-1} \), \( \mathbf{k} \) is Bragg wave vector in the direction of boresight of the radar beam and its magnitude is equal to \( 4\pi\lambda \), \( \lambda \) is radar wavelength, \( \mathbf{V} = \mathbf{u}i + \mathbf{v}j + \mathbf{w}k \) is wind vector, and \( \sigma^2(\delta) = \frac{2\pi s^2(\delta)}{\lambda} \) is the beam broadening spectral width as a function of zenith angle \( \delta \) of an obliquely steered antenna beam. In practice, (1) can be rewritten below

\[ S_v(f) = S_v \exp\left[ -\left( f + \frac{2U}{\lambda} \cos\xi \sin\delta + \frac{2w}{\lambda} \cos\delta \right)^2 \right] + 2\sigma^2(\delta), \]

where \( U = \sqrt{u^2 + v^2} \) is horizontal wind velocity, \( u \) and \( v \) are, respectively, eastward and northward components of the horizontal wind velocity, and \( \xi \) is the angle between horizontal wind vector and azimuth angle of antenna beam. Although (2) is used extensively to model beam broadening spectra for vertical and oblique antenna beams, a number of important assumptions are implicitly made in (2). First of all, the beam broadening spectral width is assumed to be insusceptible to vertical wind, that is, background vertical wind is thought to have nothing to do with the broadening of spectral width and only plays the role in the shift of mean Doppler frequency. Second, the spectral width in (2) is insensitive to the azimuth direction of horizontal wind relative to antenna beam. Moreover, although (2) is in Gaussian form, theoretical investigation shows that the shape of beam broadening spectral width may not be necessarily Gaussian, depending on the antenna beam pattern, the ratio of vertical to horizontal wind velocities, and the inhomogeneous distribution of horizontal wind in radar volume (Larsen and Palmer 1997).

An attempt is made in this article to derive analytical expression of beam broadening spectrum for an obliquely pointed radar beam. The major findings are that, aside from the horizontal wind velocity and antenna beamwidth, the oblique beam broadening spectral width is not only a function of the tilt angle of antenna beam, but also a function of azimuth angle of wind direction relative to the direction of antenna beam. Theoretical expression predicts that the beam broadening spectral width for the case of horizontal wind perpendicular to the oblique radar beam is greater than that for the case of horizontal wind parallel to the direction of the oblique radar beam. This result provides a theoretical explanation on the experimental evidences of the azimuthal anisotropy of Doppler spectral width (Nastrom and Tsuda 2001). Moreover, the ratio of vertical to horizontal wind is also a decisive factor affecting the magnitude of the spectral width. In section 2, the general behavior of radial velocity distributed in the radar volume is studied. In section 3, the analytical expression of the beam broadening spectra for vertical and oblique antenna beam is derived on the basis of the results obtained in section 2. Detailed analysis and discussion on the characteristics of the theoretical beam broadening spectrum under various conditions are made in section 4. Conclusions are drawn in section 5.

2. Characteristics of radial velocity

The configuration of radial velocity contour in resolution volume combined with antenna beam pattern are
key factors governing the behavior of beam broadening spectrum (Sloss and Atlas 1968; Larsen and Palmer 1997). Therefore, detailed examination of the characteristics of radial velocity distributed in radar volume is required before developing analytical expression of beam broadening spectrum. Figure 1 presents a schematic diagram showing the geometric relation between cross sections of an oblique antenna beam normal to the beam axis with zenith angle $\delta$ at slant range $R$ and background wind, in which $\text{AO}$ represents boresight of antenna beam; $\text{AP}$ is the vector connecting antenna A and an arbitrary point $P$ on the oblique plane; $U$ and $w$ are, respectively, horizontal and vertical wind velocities; and the line $\text{DD}'$ is the trace of horizontal wind through point $O$ on the oblique plane. Namely, $\text{DD}'$ is the intersection of the oblique plane and the vertical plane comprising horizontal wind vector through point $O$. In order to simplify mathematical manipulation, the curvature effect due to wave front is neglected. The reference axis selected for mathematical derivation, the cone angle $\phi$ of the point $P$ on the oblique plane are introduced, where $\theta$ is formed by vectors $\text{AO}$ and $\text{AP}$ and $\phi$ is measured counterclockwise from reference axis $\text{BO}$ to line $\text{PO}$. Referring to Fig. 1, the following identities can be obtained

$$\cos \gamma = \cos \delta \cos \theta + \sin \delta \sin \theta \cos \phi,$$
$$\sin \gamma = \sin \theta \sin \phi,$$
$$\cos \gamma = \sin \cos \delta - \cos \sin \theta \cos \phi.$$  \( \text{(4),(5),(6)} \)

Substituting (4), (5), and (6) into (3), we have

$$V_r = U \cos \gamma (\sin \cos \theta - \cos \sin \theta \cos \phi)$$
$$+ U \sin \gamma \sin \theta \sin \phi$$
$$+ w (\cos \cos \theta + \sin \sin \theta \cos \phi).$$  \( \text{(7)} \)

From (7), the clock angle $\phi_m$ and cone angle $\theta_m$ for the specific point $M$ at where maximum radial velocity $V_{max}$ locates can be evaluated by differentiating with respect to $\phi$ and $\theta$, respectively, and the results are

$$\phi_m = \tan^{-1} \left[ \frac{U \sin \phi}{w \sin \delta - U \cos \phi \cos \delta} \right].$$.  \( \text{(8)} \)

$$\theta_m = \tan^{-1} \left[ \frac{U \sin \phi \sin \theta_m - U \cos \phi \cos \phi_m + w \sin \phi \cos \theta_m}{w \cos \delta + U \cos \phi \sin \delta} \right].$$  \( \text{(9)} \)

Clearly, $\phi_m$ and $\theta_m$ are both the functions of $U$, $w$, $\delta$, and $\xi$. For the special case with weak $w$ and small $\delta$, $\phi_m$ and $\theta_m$ will be approximate to $-\xi$ and $\pm \pi/2$, respectively. The result of $\theta_m \approx \pm \pi/2$ implies that the contour of radial velocity can be treated as straight line owing to enormously large radius of the contour on the oblique plane. The approximation of $\phi_m \approx -\xi$ indicates that $V_{max}$ will situate on the axis of horizontal wind vector through the center of radar volume and symmetrical axis of radial velocity will coincide with the horizontal wind direction. However, it is not the case if either $\delta$ or $w$ is not small.

Figure 2 presents an example of the contour of radial velocity on the oblique plane, in which the bold arrow $\text{DD}'$ and the line $\text{BV}$ are the same as those shown in Fig. 1, and the line $\text{AA}'$ is the symmetrical axis of the radial velocities. The concentric circles marked with dashed curves are the contours of cone angle, and the solid curves attached with integers represent contours of radial velocity in unit of m s$^{-1}$. The maximum cone angle we adopt in Fig. 2 for the calculation of $V_r$ is $30^\circ$. The reason for selecting this angle is to make sure that the range of the radial velocity on oblique plane is large enough such that the general behavior of radial velocity distributed on the plane can be examined thoroughly. In fact, the maximum cone angle that we specify later in practically simulating theoretical beam broadening spectrum is less than $8^\circ$. The data employed for the computation of $V_r$, shown in Fig. 2 are $u = 10$ m s$^{-1}$, $v = 2$ m s$^{-1}$, $w = 10$ m s$^{-1}$, $\delta = 25^\circ$, and $\eta = 20^\circ$. It should be noted that the unusually large vertical wind velocity (10 m s$^{-1}$) is used to highlight the curved behavior of the radial velocity contour on the cross section of oblique radar beam. The angle $\beta$ between BV and DD’ can be computed in accordance with the following relation:
FIG. 2. An example showing contour of radial velocity (m s\(^{-1}\)) on cross section of oblique radar beam, where point M is the location of maximum radial velocity, line AA' represents the symmetrical axis of the radial velocity, and the lines BV and DD' are the same as those shown in Fig. 1. The data employed to calculate the lines and contours of radial velocity are the eastward, northward, and vertical components of wind velocity, respectively, 10, 2, and 10 m s\(^{-1}\), and the zenith and azimuth angles of antenna beam, respectively, set as 25° and 20°.

\[
\tan \beta = \tan \xi \cos \delta. \tag{10}
\]

From (7) incorporating with (10), the beam broadening Doppler spectrum for an obliquely pointed MST radar beam can be derived. Details of the derivation will be shown in the next section.

Figure 2 indicates that the contour of Doppler velocity on an oblique plane can be described by a set of concentric circles. The center of the concentric circle is located at the point M with azimuth angle \(\phi_m\) and cone angle \(\theta_m\) as indicated in (8) and (9), respectively. Figure 2 also shows that the symmetrical axis AA' of radial velocity does not coincide with the line DD'. However, we will show later that in the conditions of small \(\delta\) and low ratio of \(w\) to \(U\), AA' will be fairly approximate to DD'. In order to quantitatively estimate the angle between AA' and DD', we define an angle difference \(\Delta \epsilon\) as follows:

\[
\Delta \epsilon = |\phi_m - \beta|. \tag{11}
\]

Observably, symmetrical axis of radial velocity can be thought to coincide with the trace of horizontal wind vector on the oblique plane if \(\Delta \epsilon\) is sufficiently small. Figure 3 presents the variations of \(\Delta \epsilon\) with zenith and azimuth angles for an oblique antenna beam, in which the eastward, northward, and vertical components of wind velocity employed in the calculation are set to be 30, 20, and 2 m s\(^{-1}\), respectively. As shown in Fig. 3, \(\Delta \epsilon\) increases with the increase of \(\delta\), and its value is less than 10° if the zenith angle of antenna beam is smaller than 30°, irrespective of the azimuth angle of antenna beam. Aside from zenith and azimuth angles, (8) and (10) also demonstrate that \(\Delta \epsilon\) is also a function of the ratio of \(w\) to \(U\). Figure 4 presents the variation of \(\Delta \epsilon\) with wind direction and \(w/U\), in which 20° zenith angle
and 50° azimuth angle of antenna beam are given for the calculation. Apparently, $\Delta e$ is smaller than 0.05 if the ratio of $w$ to $U$ is less than 0.35. Therefore, the symmetrical axis of radial velocity can be thought to agree with the trace of horizontal wind vector through beam axis on the oblique plane if the ratio of $w$ to $U$ and zenith angle of antenna beam are both small.

3. Beam broadening spectrum

a. Beam broadening spectrum for vertical beam

In this section, analytical expression of beam broadening spectral width for a vertical antenna beam subject to a uniform horizontal wind is derived. It should be noted that the intensity of every spectral component of Doppler radar power spectrum is the result of integrating all radar power along the contour of corresponding Doppler frequency (or radial velocity) in the sampling volume. Because the beam broadening spectrum can be thought to be the result of radar returns from ideal targets drifting with the background wind in the scattering volume, which are equal in size and isotropic in shape and distribute uniformly without random motions, its power spectral density $S_b(f)$ can be defined as follows (Sloss and Atlas 1968):

$$S_b(f)df = \int \Pi G[\gamma(f), \Psi(f)] \, dA(\gamma, \Psi),$$  \hspace{1cm} (12)

where $G(\gamma, \Psi)$ is two-way power pattern of antenna beam; $dA$ is an infinitesimal area in the sampling volume; $f$ is Doppler frequency shift; $\gamma$ and $\Psi$ are, respectively, zenith and azimuth angles; and $\Pi$ in (12) represents reflectivity. For the present study, we assume that $\Pi$ is uniform over the sampling volume, namely, the targets are implicitly isotropic and no Fresnel-like radar returns are included in $S_b(f)df$. Figure 5 presents geometric relationship between the cross section of a vertically pointed antenna beam and horizontal wind $U$, in which $U$ is assumed in the $x$ direction and $\theta$ is the zenith angle of point $S$ in the $x$ direction. Because Fig. 5 is a special case of Fig. 1 at $\delta = 0^\circ$, $\theta$ in Fig. 5 is equivalent to cone angle defined in Fig. 1 and the angles $\beta$, $\zeta$, and $\xi$ in Fig. 1 are all equal to $0^\circ$ in Fig. 5. From Fig. 5, the area $dA$ of a infinitesimal region THH'P can be represented in terms of $h$, $\gamma$, and $\Psi$ as follows

$$dA = h^2 \tan \gamma dy d\Psi,$$  \hspace{1cm} (13)

where $h$ is the height of the cross section of vertical radar beam. In order to simplify mathematical manipulation, the rectangular coordinate is adopted in the following derivation. To proceed, the expression of two-way power pattern $G$ of a Gaussian antenna beam together with $\Pi$ is given below to facilitate the mathematical manipulation of expression (12)

$$\Pi G(x, y) = G_0 e^{-\left((x^2 + y^2)\sigma^2\right)},$$  \hspace{1cm} (14)

where $G_0 = C/h^2$, $C$ is a constant as a function of radar parameters and reflectivity, and $\sigma$ is defined as $\sigma = (h \theta_j) / (2\sqrt{2} \ln 2)$. Referring to Fig. 5, we observe that the identity $\tan \beta = \tan \gamma \cos \Psi$ exists. With the help of $\beta$, we have

$$x = h \tan \theta,$$  \hspace{1cm} (15)

$$y = h \tan \theta \tan \Psi,$$  \hspace{1cm} (16)

$$dx = h d\theta,$$  \hspace{1cm} (17)

$$dy = \frac{h \tan \theta}{\cos \Psi} d\Psi.$$  \hspace{1cm} (18)

Substituting (15)–(18) into (14) and then into (12), we have

$$S_b(f)df = \int G_0 e^{-\left((h \tan \theta)^2 + (h \tan \theta \tan \Psi)^2\right)\sigma^2} \frac{h^2 \tan \theta}{\cos \Psi^2} \, d\Psi \, d\theta.$$  \hspace{1cm} (19)

Assume that the antenna beam width is so small that the approximation $\tan \beta = \sin \beta$ is valid. Consequently, substituting (16) and (18) into (19) and performing integration with respect to $\Psi$, we have

$$S_b(f)df = G_0 h e^{-\left(h \sin \beta\right)\sigma^2} \sqrt{\pi} \sigma d\beta.$$  \hspace{1cm} (20)

To proceed, we see from Fig. 5 that the radial velocity at point $S$ can be expressed as

$$V_r = -U \sin \beta,$$  \hspace{1cm} (21)

where radial velocity $V_r$ is positive (negative) if its direction is away (toward) radar. Again, because the antenna beam is narrow, the approximation $\sin \beta = \beta$ is valid. As a result, the following relation is obtained accordingly:

$$\beta \approx \frac{\lambda}{2U} f.$$  \hspace{1cm} (22)

Substituting (22) into (20) and rearranging the resulting
expression, we obtain the power density of beam broadening spectrum

\[ S_b(f) = \frac{C\sqrt{\pi}\sigma_b}{2U} e^{-f^2/(2U\sigma_b^2)}, \quad (23) \]

where \( \sigma_b \) is defined as the ratio of \( \sigma/h \). From (23), the beam broadening spectral width \( \sigma_b \) can be readily obtained with

\[ \sigma_b = \frac{U\sqrt{2}\sigma_b}{\lambda}. \quad (24) \]

Because (24) is obtained by defining the spectral width as the second moment of the expression (23), different definitions of the spectral width will result in different expressions of the beam broadening spectral width.

b. Beam broadening spectrum for oblique beam

In this section, an analytical expression of beam broadening spectrum for an obliquely pointed antenna beam subject to horizontal in combination with vertical winds is derived to investigate the effects of the ratio \( w/U \), vertical wind velocity, horizontal wind direction, and tilt angle of antenna beam on the beam broadening spectral width. Moreover, the characteristics of the oblique beam broadening spectrum are also analyzed to validate the applicability of conventional beam broadening spectrum as shown in (2). In order to simplify the mathematical manipulation, the following assumptions are made: (a) Gaussian antenna beam pattern, (b) low ratio of \( w/U \) (<0.35), (c) small zenith angle of antenna beam (<30°), (d) narrow antenna beam width (HPBW < 10°), and (e) uniform wind velocity in radar volume. As discussed in section 2, the contour of radial velocity on the cross section of oblique radar beam can be reasonably treated as a straight line if the assumptions (b), (c), and (d) are valid. Moreover, assumptions (b) and (c) further imply that the trace of horizontal wind through beam axis on oblique plane can be thought to coincide with the symmetrical axis of the radial velocity and be perpendicular to the radial velocity contour. Under these circumstances, the characteristics of radial velocity contour for an oblique beam will be very similar to that for a vertical beam. Therefore, we can take advantage of the theoretical results for vertical beam developed in the previous section to evaluate the beam broadening spectrum for an oblique beam. The mathematical relation between Doppler frequency of the beam broadening spectrum, background wind field, and oblique antenna beam can be established from Fig. 6, in which line DD’ the same as the one shown in Fig. 1 is the intersection of the oblique plane and the vertical plane comprising horizontal vector through point O. If we designate point O in Fig. 6 as the origin point and the line DD’ as the x axis, the cone angle \( \theta \) of point S in Fig. 5 is analogous to cone angle \( \Omega \) of point S in Fig. 6. As a result, in analog to (20) for vertical beam, the beam broadening spectrum for oblique beam can be expressed as

\[ S_b(f) df = G_0 \text{Re} \left( \frac{R^2}{f \sin \Omega^2} \sqrt{\pi} \rho \right) d\Omega, \quad (25) \]

where \( R \) is the range of oblique plane, \( G_0 \) now is defined as \( C/R^2 \), and \( \Omega \) is the angle between \( AO \) and \( AS \). Apparently, \( \Omega \) will be very small if a narrow antenna beam with a few degrees width is adopted in the calculation. From (25), \( S_b(f) \) can be obtained once \( d\Omega/df \) is evaluated. Before deriving the mathematical relation between \( \Omega \) and \( f \), a number of auxiliary expressions should be established. Referring to Fig. 6, radial velocity \( V_r \) at point S on the oblique plane can be formulated as

\[ V_r = U \cos(\rho + \xi) \sin\Sigma + w \cos\Sigma, \quad (26) \]

where \( \Sigma \) is the zenith angle of point S, \( \rho \) is the angle between vectors \( AO' \) and \( AS' \), \( \xi \) is the angle between horizontal vector and \( AO' \), and \( AO' \) and \( AS' \) are, respectively, the projections of vectors \( AO \) and \( AS \) on the ground. Following the same procedures of deriving the relations between \( \xi, \gamma, \delta, \theta, \) and \( \phi \) as shown in (4)–(6), we have

\[ \sin\Sigma = \frac{\sin\delta \cos\Omega - \cos\delta \sin\Omega \cos\beta}{\cos\rho}, \quad (27) \]

\[ \cos\Sigma = \cos\Omega \cos\delta + \sin\Omega \sin\delta \cos\beta, \quad (28) \]

where \( \beta \) is the angle between lines BV and DD’. As a matter of fact, (26) will reduce to (7) if expressions (27) and (28) are substituted into (26) and let \( \beta = -\phi \). Substituting (27) and (28) into (26) and rearranging the expression, we have
\[ \Gamma_1 \Omega^2 + \Gamma_2 \Omega - \Gamma_3 = 0, \]  
\[ \Gamma_1 = \frac{U \cos(\rho + \xi) \sin \delta}{\cos \rho} + \frac{w}{\lambda} \cos \delta, \]  
\[ \Gamma_2 = \frac{2U \cos(\rho + \xi) \sin \delta \cos \beta}{\cos \rho} - \frac{2w}{\lambda} \sin \delta \cos \beta, \]  
\[ \Gamma_3 = \frac{2U \cos(\rho + \xi) \sin \delta}{\cos \rho} + \frac{2w}{\lambda} \cos \delta + f, \]

where

\[ S_B(f) = \frac{\sqrt{\pi} C \sigma_b}{\sqrt{\Gamma_1^2 + 4\Gamma_2(2\Gamma_1 + f)}} e^{-\left(\Gamma_1^2 + 4\Gamma_2(2\Gamma_1 + f)^{-1}\right)} \sigma_b^2, \]

where \( C \) is a constant related to radar parameters and reflectivity, and \( \sigma_b = (\theta_{12})/(2\sqrt{2 \ln 2}) \).

Equation (33) shows that the oblique beam broadening spectrum is the function of numerous factors, including three-dimensional wind velocity, zenith and azimuth angles of antenna beam, wind direction, and antenna beamwidth. Detailed analysis on the characteristics of the mean Doppler velocity and spectral width will be given later.

Because of its complex form, the complete investigations of the general behavior of \( S_B(f) \) shown in (33) can be done only in terms of numerical computation. Before doing that, we note that parameter \( \rho \) in (33) is a function of dependent variable \( \Sigma \) in accordance with (27), producing an insufficient condition for the computation. In order to resolve this problem, another expression relating \( \rho \) with independent variables \( \beta, \Omega, \) and \( \delta \) is required. From Fig. 6, the following relation can be obtained:

\[ \sin \beta \sin \Omega = \sin \Sigma \sin \rho. \]

Substituting (34) into (27) and rearranging the resulting expression, we have

\[ \rho = \tan^{-1} \left[ \frac{\sin \beta \sin \Omega}{\sin \delta \cos \Omega - \cos \delta \cos \beta \sin \Omega} \right]. \]

Incorporating with (35), the numerical analysis of the characteristics of the oblique beam broadening spectrum \( S_B(f) \) is possible by using the expressions (33) in combination with (30), (31), and (32).

Figure 7 displays an example of oblique beam broadening spectrum (solid curve) compared to conventional Gaussian beam broadening spectrum (dashed curve), where the former is calculated from (33) and the latter is obtained from (2) with \( \sigma(\delta) = \sigma_b \cos \delta \). The parameters employed for the calculation are that \( \lambda = 6 \text{ m}, \theta_{12} = 5^\circ, \delta = 20^\circ, U = 40 \text{ m s}^{-1}, w = 5 \text{ m s}^{-1}, \) and \( \xi = 0^\circ \).

As indicated in Fig. 7, the shape of exact beam broadening spectrum calculated from (33) is very close to Gaussian form and the mean Doppler frequency (defined as the first moment) of the former is almost identical to that of the latter. However, the width (defined as the second moment) of the exact beam broadening spectrum is slightly different from that of conventional one. Detailed investigations on their differences in mean Doppler velocities and spectral widths are given below.

Figure 8 depicts the azimuth angle variations of mean Doppler velocities of exact beam broadening spectrum at selected zenith angles \( 5^\circ, 10^\circ, 15^\circ, \) and \( 20^\circ, \) in which mean Doppler frequencies (marked with solid curves) of conventional beam broadening spectrum defined in (2) are also presented for comparison. The azimuth angle in Fig. 8 is defined as the horizontal wind direction relative to the azimuth angle of antenna beam and positive counterclockwise. As one would expect, the maximum mean Doppler velocity occurs when the horizontal wind is in the direction of antenna beam, while the minimum appears as the horizontal wind is perpendicular to antenna beam. Figure 8 presents that the mean Doppler velocity of the exact beam broadening spectrum is almost identical to that of the conventional one, with a discrepancy of about 0.28% in the two. This result validates the applicability of wind velocity estimation made with conventional Doppler spectrum. However, it is not the case for the spectral width as shown below.
Figure 8. Azimuth angle variations of mean Doppler velocity estimated from conventional expression (2) (marked with solid curves) and exact equation (33) (marked with different symbols) at selected zenith angles 5°, 10°, 15°, and 20°, where radar wavelength is 6 m, half-power beamwidth of antenna beam is 5°, and eastward, northward, and vertical wind velocities are 40, 10, and 5 m s⁻¹, respectively.

Figure 9. Azimuth angle variations of beam broadening spectral width (multiplied by √2) for different zenith angles δ of antenna beam, in which the bold straight line is the result of δ = 0° and identical to that calculated from (3). Note that the azimuth angle is the horizontal wind direction relative to beam direction. The data employed for computation are eastward, northward, and vertical wind speeds, respectively, 40, 10, and 5 m s⁻¹; a half-power beamwidth of antenna beam of 5°, and a radar wavelength of 6 m (corresponding to 50 MHz).

Figure 9 presents the azimuth angle variations of the width of the exact beam broadening spectrum at selected zenith angles, where the data employed for the calculation are the same as those employed in Fig. 8 and the spectral widths have been multiplied by a factor of √2. The horizontal straight line in Fig. 9 represents the spectral width at zenith direction. As shown, the spectral width varies not only with zenith angle, but also with azimuth angle. It indicates that the spectral width at 90° azimuth angle (corresponding to horizontal wind perpendicular to the radar beam) is always greater than that at 0° azimuth angle (corresponding to horizontal wind parallel to the radar beam). Calculation reveals that for 10° zenith angle and 3.6° half-power beamwidth the difference in the squares of theoretical beam broadening spectral widths between 90° and 0° azimuth angles is about 0.0269 m² s⁻², where 40 m s⁻¹ horizontal wind velocity and 2 m s⁻¹ vertical wind speed are employed for the calculation. Although this value is smaller than experimental data (about 0.05 m² s⁻²) obtained by Nastrom and Tsuda (2001) by a factor of 1.7, the theoretical result derived in this article can reasonably explain the experimental results. Figure 9 also shows that when the zenith angle is exceedingly small (<5°), the variation of spectral width with azimuth angle follows a quasi-sinusoidal function, maximum at around 180° and minimum at 0°. However, as the zenith angle is greater than 10°, the pattern of azimuth angle variation of the spectral width deviates considerably from the sinusoidal form and, instead, two salient peaks occur at specific azimuth angles. The distance between the peaks increases with the increase of zenith angle. In addition, Fig. 9 also shows that at azimuth angles of around 90° and 270°, the spectral widths do not change with the zenith angle and are equal to σ₀.

Figures 10a and 10b present the zenith angle variations of spectral width for different ratios of w to U, where the azimuth angles for Figs. 10a and 10b are, respectively, 0° and 90°. As indicated, the spectral width decreases monotonically with zenith angle, irrespective of azimuth angle. Additionally, the spectral width at given zenith and azimuth angles decreases with the increase of the ratio of w to U. Moreover, Figs. 10a and 10b show that the decreasing rate of spectral width with zenith angle for the case of j = 0° is much greater than that for j = 90°. Calculation indicates that for the case of w/U = 0.4 the difference in the spectral widths between δ = 0° and δ = 30° is about 32% for j = 0°, while less than 1% for j = 90°. An analytical expression of beam broadening spectral width as a function of δ, j, β, and w/U will be derived in the next section to illustrate the features shown in Figs. 9, 10a, and 10b.

4. Analysis and discussion
To begin with, assume that Γ,Ω ≤ Γ,Ω. Normally, this assumption is valid for the conditions that the an-
Figure 10. Zenith angle variations of beam broadening spectral width calculated from expression (37) multiplied by a factor of $\sqrt{2}$ for different ratios of vertical to horizontal wind velocities. The data employed for the calculations are $U = 40 \text{ m s}^{-1}$, $w = 5 \text{ m s}^{-1}$, and $\theta_{\alpha\beta} = 5^\circ$. (a) For the case of $\xi = 0^\circ$, (b) for the case of $\xi = 90^\circ$.

- **Fig. 10.** Zenith angle variations of beam broadening spectral width calculated from expression (37) multiplied by a factor of $\sqrt{2}$ for different ratios of vertical to horizontal wind velocities. The data employed for the calculations are $U = 40 \text{ m s}^{-1}$, $w = 5 \text{ m s}^{-1}$, and $\theta_{\alpha\beta} = 5^\circ$. (a) For the case of $\xi = 0^\circ$, (b) for the case of $\xi = 90^\circ$.

- **Fig. 10.** Zenith angle variations of beam broadening spectral width calculated from expression (37) multiplied by a factor of $\sqrt{2}$ for different ratios of vertical to horizontal wind velocities. The data employed for the calculations are $U = 40 \text{ m s}^{-1}$, $w = 5 \text{ m s}^{-1}$, and $\theta_{\alpha\beta} = 5^\circ$. (a) For the case of $\xi = 0^\circ$, (b) for the case of $\xi = 90^\circ$.

- **Fig. 10.** Zenith angle variations of beam broadening spectral width calculated from expression (37) multiplied by a factor of $\sqrt{2}$ for different ratios of vertical to horizontal wind velocities. The data employed for the calculations are $U = 40 \text{ m s}^{-1}$, $w = 5 \text{ m s}^{-1}$, and $\theta_{\alpha\beta} = 5^\circ$. (a) For the case of $\xi = 0^\circ$, (b) for the case of $\xi = 90^\circ$.
Fig. 11. Comparison of azimuth angle variation of beam broadening spectral widths estimated from exact expression (33) (dashed curve) and parabolic model (40) (solid curve), in which the spectral widths have been multiplied by $\sqrt{2}$. The data employed for the calculation are the same as Fig. 9 for the case of $\delta = 10^\circ$.

Comparison of the spectral widths obtained from complete solution (33) and from approximate model (40) for the case of $\delta = 10^\circ$. Evidently, the model result coincides accurately with the exact data, validating the applicability of (40). Employing (40) and incorporating (10), the azimuth angle $\xi_m$ where the maximum value of beam broadening spectral width locates can be evaluated as follows:

$$\xi_m = \cos^{-1} \left[ \frac{w/\delta}{U} \left( \frac{1}{2} - \frac{1}{\delta} \right) \right]. \quad (42)$$

Obviously, $\xi_m$ is dependent on $w/U$ and $\delta$ in accordance with $\cos^{-1}$ function. Figure 12 presents variation of $\xi_m$ with zenith angle, where exact data (open circles), scaled from Fig. 9 and model results (solid curves) calculated from (42) are compared, showing excellent agreement between them.

Figures 10a and 10b have demonstrated that the spectral width varies monotonically with zenith angle, in which the ratio of $w$ to $U$ governs the rolloff rate of the spectral width. In order to derive a more concise expression to highlight key parameters governing the spectral width, we assume that the direction of horizontal wind is parallel to that of antenna beam, that is, $\xi = 0^\circ$. In this special case, the value of $\beta$ will be equal to zero in accordance with (10). Substituting $\xi = 0^\circ$ and $\beta = 0^\circ$ into (31) and (32), and then resubstituting the resulting expressions into (38), $S_{bm}(f)$ becomes

$$S_{bm}(f) = \frac{\sqrt{\pi} \sigma_b}{2U} \cos \delta - \frac{2w}{\lambda} \sin \delta \times \exp \left[ - \left( \frac{f + 2U}{\lambda} \sin \delta + \frac{2w}{\lambda} \cos \delta \right)^2 \left( \frac{2U}{\lambda} \cos \delta - \frac{2w}{\lambda} \sin \delta \right)^2 \sigma_b^2 \right]. \quad (43)$$

Obviously, the mean Doppler frequency $f_m$ and spectral width $\sigma_{ab}$ of $S_{bm}(f)$ for this condition will be

$$f_m = -\frac{2}{\lambda} (U \sin \delta + w \cos \delta), \quad (44)$$

$$\sigma_{ab} = \sigma_b \left( \cos \delta - \frac{w}{U} \sin \delta \right). \quad (45)$$

In fact, (44) and (45) can also be obtained by substituting $\beta = 0^\circ$ and $\xi = 0^\circ$ into (39) and (40), respectively. We further assume that $\delta$ is so small that the approximations of $\cos \delta \sim 1 - \delta^2/2$ and $\sin \delta \sim \delta$ are valid. Consequently, expression (45) becomes

$$\sigma_{ab} \approx \sigma_b \left( 1 - \frac{\delta^2}{2} - \frac{w}{U} \delta \right). \quad (46)$$

Expression (46) states that $\sigma_{ab}$ is a function of $\delta$ in a parabolic form, in which parameter $w/U$ in (46) determines the width of the parabolic function. Figure 13
shows a comparison of zenith angle variations of spectral width, where exact data (marked with open circles) are calculated from (33) and parabolic curve (marked with solid line) is the realization of (46). The parameters $\theta_{1/2} = 5^\circ$, $U = 40 \text{ m s}^{-1}$, and $w/U = 0.125$ are employed for the calculation. As depicted, the parabolic model is in excellent agreement with the exact data within the zenith angle range of $0^\circ$–$15^\circ$. The discrepancy between the model and exact data increases gradually as the zenith angle is greater than $15^\circ$. Equation (46) also indicates that $\sigma_{sw}$ will be identical to $\sigma_s$ as $\delta = 0^\circ$, consistent with the results from (41). As the results of (45) or (46), it concludes that the effect of zenith angle of antenna beam on Doppler spectral width is significant and should be considered when the spectral widths at different zenith angle are analyzed.

As shown in Fig. 9, the spectral widths around azimuth angles of $90^\circ$ and $270^\circ$ (corresponding to the case that horizontal wind is normal to oblique antenna beam direction) are independent of zenith angle. It should be noted that the condition of $\xi = \pm \pi/2$ cannot be substituted directly into (33) to obtain corresponding $S_{Bc}(f)$. This is because the corresponding $\beta$ value will be equal to $\pi/2$ in accordance with the expression (10) and, as a result, the value of $\Gamma_c$ as shown in (31) will be equal to zero. In order to derive the corresponding beam broadening spectrum for this special case, the source equations (26)–(28) should be modified appropriately. Substituting $\xi = \pi/2$ and $\beta = \pi/2$ into (27) and (28), we have

$$\sin \Delta = \frac{\sin \delta \cos \Omega}{\cos \rho}, \quad (47)$$
$$\cos \Delta = \cos \Omega \cos \delta. \quad (48)$$

Employing (47) and (48) and incorporating (34), we have

$$\tan \Omega = \tan \rho \sin \delta. \quad (49)$$

Substituting (47) and (48) into (26) and rearranging the expression in terms of (49), we have

$$\frac{w \cos \delta}{2} \frac{\Omega^2}{\Omega^2} + \left( U \cos \delta + \frac{f \lambda}{2} \right) = 0. \quad (50)$$

From (50), the solution to $\Omega$ can be obtained accordingly and its differential with $f$ can subsequently be derived. In conditions of low ratio of $w/U$ and small zenith angle of antenna beam, it is reasonable to assume that $2w \cos \delta(w \cos \delta + f \lambda/2) \ll U^2$. Consequently, the resultant beam broadening spectrum reduces to

$$S_{Bc}(f) = \frac{\sqrt{\pi} C \sigma_{\lambda}}{2U} \exp \left\{ \frac{(f + 2w \cos \delta \lambda)^2}{(2U \sigma_{\lambda} \lambda)^2} \right\}. \quad (51)$$

Apparently, the mean Doppler frequency shift of $S_{Bc}(f)$ will be $-2w \cos \delta \lambda$ and independent of horizontal wind velocity $U$. This result explains the feature that the mean Doppler velocities at azimuth angles of $90^\circ$ and $270^\circ$ are equal to $w \cos \delta$ shown in Fig. 8. Aside from mean Doppler frequency, (51) also shows that the corresponding spectral width of $S_{Bc}(f)$ is equal to $\sigma_{s}$, exactly identical to that of the conventional beam broadening spectrum for a vertically pointed antenna beam. Comparing (51) to (43) shows that the spectral width of the former is larger than that of the latter, provided $\delta$ is not equal to zero. This result explains the feature that the spectral widths at $\xi = 90^\circ$ are greater than those at $\xi = 0^\circ$ shown in Figs. 9, 10a, and 10b.

Recently, by considering the effects of finite beamwidth and vertical wind shear of horizontal wind, Nasstrom (1997) derived an analytical expression to quantitatively estimate their contributions to the width of Doppler radar spectrum. In the absence of vertical wind shear, his expression can be simplified as

$$\sigma_n = \frac{U \sqrt{2}}{\lambda} \left[ 1 - \frac{\sin^2 \theta_{0.5h}}{\theta_{0.5h}^2} \right] + \cos \delta \left( \frac{\sin^2 \theta_{0.5h} - \sin 2 \theta_{0.5h}}{2 \theta_{0.5h}} \right)^{1/2}, \quad (52)$$

where $\theta_{0.5h}$ is half-power half-width (equal to $0.5 \theta_{0.5}$) of antenna beam. We note that (52) is derived under the condition of horizontal wind traveling in the direction along the antenna beam. By substituting $\xi = \beta = 0^\circ$ and $w = 0$ into (40), we find that the present result is almost exactly identical to the first term of Eq. (18) of Sloss and Atlas (1968) at $\delta = 0^\circ$ if $\alpha = 2 \ln 2$, but slightly greater than (52) by about 3.8%. Except for
5. Conclusions

In this article, the characteristics of radial velocity distributed on the cross section of oblique radar volume are analyzed. We find that the contour of radial velocity can be perfectly described by a set of concentric circles. The center and the radius of the concentric circle are governed not only by horizontal wind velocity and direction, but also by the ratio of vertical to horizontal wind velocities. Numerical computation demonstrates that symmetrical axis of the radial velocity on the cross section of oblique radar beam is very close to the trace of horizontal wind on the oblique cross section, provided that the ratio of vertical to horizontal wind velocities is small, the antenna beamwidth is narrow enough, and the zenith angle of antenna beam is small. Under these conditions, the contour of radial velocity can be approximated to be straight line. With the help of this approximation, the theoretical formulation of the beam broadening spectrum for an oblique Gaussian antenna beam is derived, in which three-dimensional wind field with various direction is considered in the derivation. Analyses show that the width of the oblique beam broadening spectrum varies not only with zenith angle of antenna beam, but also with azimuth angle of horizontal wind relative to antenna beam direction. An analytical expression modeling the azimuth angle variation of the beam broadening spectral width is derived, showing that the spectral width is minimum as the horizontal wind is in the direction of antenna beam and maximum as horizontal wind is perpendicular to antenna beam direction. The difference in the magnitudes of maximum and minimum may be as large as 15%. Another important finding is that the ratio of vertical to horizontal wind velocities is a crucial factor governing the width of beam broadening spectrum. Theoretical calculation shows that the spectral width for the case of $w/U = 0.4$, which may occur in a deep convective cloud, may be smaller than that for $w/U = 0$ by 32% or more, depending on zenith angle of antenna beam and azimuth angle of horizontal wind. These results strongly suggest that a great caution should be taken in the computation of pertinent beam broadening spectral width to estimate atmospheric parameters from observed Doppler spectral width of an obliquely directed radar beam.

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