Estimating Spatial Velocity Statistics with Coherent Doppler Lidar

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ABSTRACT

The spatial statistics of a simulated turbulent velocity field are estimated using radial velocity estimates from simulated coherent Doppler lidar data. The structure functions from the radial velocity estimates are processed to estimate the energy dissipation rate $\varepsilon$ and the integral length scale $L_i$, assuming a theoretical model for isotropic wind fields. The performance of the estimates are described by their bias, standard deviation, and percentiles. The estimates of $\varepsilon^{2/3}$ are generally unbiased and robust. The distribution of the estimates of $L_i$ are highly skewed; however, the median of the distribution is generally unbiased. The effects of the spatial averaging by the atmospheric movement transverse to the lidar beam during the dwell time of each radial velocity estimate are determined, as well as the error scaling as a function of the dimensions of the total measurement region. Accurate estimates of $L_i$ require very large measurement domains in order to observe a large number of independent samples of the spatial scales that define $L_i$.

1. Introduction

Coherent Doppler lidars are attractive instruments for high-resolution measurements of wind fields (Frehlich et al. 1994, 1997, 1998; Henderson et al. 1991, 1993; Huffaker and Hardesty 1996; Kavaya et al. 1989; Menzies and Hardesty 1989; Mayor et al. 1997). In the weak signal regime, performance is improved by accumulating (averaging) the basic estimation statistics (spectra or covariance estimates) over multiple pulses (Frehlich et al. 1994, 1998; Mayor et al. 1997; Rye and Hardesty 1993a,b). Liders with high pulse repetition frequencies (PRFs) (100 Hz–10 KHz) and a Gaussian pulse temporal profile produce high-resolution measurements of the spatial and temporal variations of the radial velocity. The trade-off between pulse energy and PRF has been investigated with simulations (Frehlich and Yadlowsky 1994; Frehlich 1996; Rye and Hardesty 1993a,b) and verified with actual lidar data (Frehlich et al. 1998).

The simplest measurement geometry is a fixed lidar beam with a slowly advecting wind field. Robust algorithms (Frehlich 1997) can determine the statistical performance of velocity estimates generated by computer simulations for both ideal conditions with no wind turbulence and typical cases of wind turbulence. Algorithms have also been developed for extracting these parameters from actual lidar data (Frehlich et al. 1994, 1997, 1998). The effects of variations of backscatter over the range gate have been shown to be small (Rye 1990). The statistical description of the wind field has been determined using the spatial statistics of the Doppler lidar velocity estimates (Banakh and Smalikho 1997; Frehlich 1997; Frehlich et al. 1998). This requires a model for the spatial structure function of the radial velocity. If the atmosphere is homogeneous and stationary, many statistically independent realizations of the wind field can be sampled to accurately determine the correct model.

The interpretation of data is more difficult for a scanning lidar and/or moving platforms (aircraft and satellite) because of increased spatial averaging of the wind field. Various scan patterns have been used to investigate atmospheric conditions. The velocity–azimuth display (VAD), defined as a variable azimuth angle for fixed zenith angle, has been used to produce profiles of the mean wind vector (Eberhard et al. 1989). The VAD scan with a zenith angle near 90$^\circ$ approximates a two-dimensional plane that is attractive for measurements of the spatial statistics of the wind field.

The spatial statistics of the velocity field are described by the structure function. In many cases, such as a typical boundary layer, the structure function has a simple description in terms of two parameters: the energy dissipation rate $\varepsilon$ and an integral length scale $L_i$. The structure function of Doppler lidar radial velocity measurements can be expressed in terms of the model parameters $\varepsilon$ and $L_i$, and the lidar pulse parameters, which describe the spatial averaging by the lidar measurement (Frehlich 1997; Frehlich et al. 1998). In addition, the effects of the random error of the lidar velocity measurements must be removed. The performance of various algo-
algorithms to estimate the parameters of the spatial statistics of the velocity field will be determined using computer simulations of coherent Doppler lidar data and the turbulent velocity field. The bias, probability density functions, and accuracy of the estimates of $\epsilon$ and $L$, will be determined for various lidar and atmospheric parameters and various estimation algorithms.

2. Theory

a. Coherent Doppler lidar signal

The coherent Doppler lidar signal $z(\tau)$, where $\tau$ denotes the time referenced to the transmitted pulse, can be a real signal or a complex signal (complex data is produced by an analog complex receiver (Doviak and Zrnic 1993; Van Trees 1992) or by digital processing of the real signal (Gold et al. 1970; Cizek 1970)). Complex data is more convenient for advanced velocity estimation algorithms like the maximum likelihood estimator and is also more numerically efficient for correcting the frequency reference for zero velocity resulting from the frequency difference $\Delta f$ between the transmit laser and reference laser. The coherent Doppler lidar signal consists of an aerosol target component $s(\tau)$ and a detector noise component $N(\tau)$. Doppler lidar data is denoted by $z(\tau_{ref} + kT)_{s}$, where $\tau_{ref}$ is the reference time for the range gate, $k$ is the data index, and $T_{s}$ is the sampling interval of the data that defines the maximum velocity search space $V_{search} = \lambda/(4T_{s})$ for real data and $V_{search} = \lambda/(2T_{s})$ for complex data, where $\lambda$ is the laser wavelength. Coherent Doppler lidar data is well approximated as a zero-mean Gaussian random process because the total received laser field is the superposition of many backscattered fields with random phase from the aerosol targets in the sensing volume. For many coherent Doppler lidars, the sensing volume for a single pulse is a narrow pencil-shaped volume approximately 10–50 cm wide and 20–50 m long. For a given observation time $T_{obs} = MT_{s}$ per velocity estimate (Frehlich and Yadlowsky 1994; Frehlich 1997), the size of the range gate $\Delta p$ is defined by the distance that the laser-illuminated aerosol region (the pencil-shaped volume) travels during $T_{obs}$; that is,

$$\Delta p = MT_{s} c/2 = T_{obs} c/2,$$

where $M$ is the number of complex data for a velocity estimate and $c$ is the speed of light.

The lidar signal is normalized such that the average noise power $\langle |N(\tau)|^{2} \rangle = 1$ where $\langle \rangle$ denotes ensemble average and $|z|$ denotes the complex modulus of $z$. The signal-to-noise ratio (SNR) is a key performance parameter defined by (Frehlich et al. 1994, 1997)

$$\text{SNR}(\tau) = \frac{\langle |s(\tau)|^{2} \rangle}{\langle |N(\tau)|^{2} \rangle} = \langle |s(\tau)|^{2} \rangle.$$

For a narrow lidar beam (Frehlich 1997),

$$\text{SNR}(\tau) = \frac{\eta_{d}U_{s}}{hvB} \int_{-\infty}^{\infty} H(r) \left| A_{l}(\tau - 2r/c) \right|^{2} \, dr,$$

where $\eta_{d}$ (electrons/photon) is the detector quantum efficiency, $U_{s}$ is the laser pulse energy, $h$ is Planck’s constant, $v$ is the laser frequency, $B$ is the detector bandwidth, $A_{l}(\tau)$ is the normalized complex amplitude of the laser pulse, $r$ denotes the distance along the lidar beam axis, and the system gain $H(r)$ is

$$H(r) = K^{2}(r)B(r)C(r),$$

where $K(r)$ is the one-way irradiance extinction, $B(r)$ is the aerosol backscatter coefficient, and $C(r)$ is the coherent responsivity (Frehlich and Kavaya 1991) of the coherent Doppler lidar. The complex laser pulse amplitude is normalized such that $\int_{-\infty}^{\infty} |A_{l}(\tau)|^{2} \, dt = 1$. Recent coherent Doppler lidars using a solid-state laser have a Gaussian pulse with 1/e intensity radius $\sigma_{r}$ and linear frequency chirp $\phi$ (Frehlich et al. 1994, 1997)

$$A_{l}(\tau) = \frac{1}{\pi^{1/2} \sigma_{r}^{1/2}} \exp \left( -\frac{r^{2}}{2\sigma_{r}^{2}} + \pi i \phi r^{2} \right).$$

The transmitted pulse defines the origin of the time axis, that is, $\tau = 0$ corresponds to the maximum of the lidar pulse at the telescope aperture located at range coordinate $r = 0$.

The function

$$W(r) = |A_{l}(\tau - 2r/c)|^{2}$$

is the lidar pulse range-weighting function that indicates the mapping from time $\tau$ to aerosol target range $r$, that is, $\tau = 2r/c$. For the Gaussian lidar pulse and fixed time $\tau$, the range weighting (Frehlich and Yadlowsky 1994) is a Gaussian function with a 1/e radius

$$r_{p} = c\sigma_{r}e/2$$

and a full width at half maximum (FWHM) of

$$\Delta r = \sqrt{\ln2}c\sigma_{r}.$$

The effective range resolution $\Delta R$ of the measurement is given by $\Delta R = \Delta r + \Delta p$ (Frehlich and Yadlowsky 1994).

b. Single-pulse Doppler velocity estimates

For single-pulse lidar velocity estimates, only the radial velocity $v(r, t)$ of the aerosol particles along the lidar beam axis $r$ are sampled for a given measurement time $t$. The statistical performance of the velocity estimates depends on the lidar parameters and the statistics of $v(r, t)$. The accuracy of a velocity estimate $\hat{R}(R, t)$ requires a definition of the desired measurement, which is typically chosen as the conditional ensemble average $m_{\text{ave}}(R, t)$ of the velocity estimate for a given realization of the atmosphere $v(r, t)$ (Frehlich 1997). The statistical performance of the estimates is defined as an ensemble
average of the conditional statistics over many realizations of the wind field. For example, the bias of the velocity estimate is defined as

$$\text{bias}(R, t) = \langle [\hat{v}(R, t) - m_{ave}(R, t)] \rangle,$$  

(9)

and the variance of the random error of the velocity estimate is defined as

$$\sigma^2 = \langle [\hat{v}(R, t) - m_{ave}(R, t)]^2 \rangle,$$  

(10)

where $\langle \rangle$ denotes an ensemble average over many realizations of the random wind field. It is common practice to assume that the statistics are stationary in time $t$ and the ensemble average is produced by a time average. The effect of wind turbulence on the performance of single-pulse Doppler lidar velocity estimators has been determined with computer simulations (Banakh and Smalikho 1997; Frehlich 1997), and the conditional average velocity $m_{ave}(R, t)$ is well approximated by the pulse-weighted velocity over the range gate (Frehlich 1997), that is,

$$m_{wgt}(R, t) = \frac{1}{\Delta p} \int_{R-\Delta p/2}^{R+\Delta p/2} v_{\text{pulse}}(r, t) \, dr,$$  

(11)

where

$$v_{\text{pulse}}(r, t) = \int_{-\infty}^{\infty} v(s, t) I_s(r - s) \, ds,$$  

(12)

$$I_s(r) = W(r) \int_{-\infty}^{\infty} W(s) \, ds.$$  

(13)

The pulse-weighted velocity $m_{wgt}(R, t)$ and the conditional average velocity $m_{ave}(R, t)$ is a spatial convolution of the instantaneous radial velocity $v(r, t)$ with an effective spatial filter given in terms of the lidar pulse weighting function $I_s(r)$ and the range-gate length $\Delta p$. Therefore, spatial statistics of the wind field can be recovered by careful deconvolution methods and signal processing algorithms (Banakh and Smalikho 1997; Frehlich 1997; Frehlich et al. 1998; Mayor et al. 1997).

c. Multiple-pulse Doppler velocity estimates

The accuracy of Doppler lidar measurements is improved by fixing the laser beam and accumulating the signal statistics from $N_p$ lidar pulses for each range gate (Frehlich and Yadowsky 1994; Frehlich 1996; Frehlich et al. 1998; Rye and Hardesty 1993a,b). For multiple-pulse lidar measurements, an important parameter is the total measurement time or dwell time $T = N_p/\text{PRF}$. If the random radial velocity $v(r, t)$ does not vary much over the dwell time $T$, then the single-pulse analysis of the previous section is valid for multiple-pulse data.

For general operation, a lidar samples a parallelogram of the atmosphere in a plane defined by the fixed laser beam and fixed mean velocity vector. To simplify the analysis, we assume that the random atmosphere is a frozen homogeneous velocity field and the laser beam axis is transverse to the mean horizontal velocity. For a vertically pointed lidar beam, the radial velocity $v(r, t)$ as a function of range $r$ is related to the vertical velocity field $u(r, h)$ by

$$v(r, t) = u(r, V_H t),$$  

(14)

where $V_H$ is the mean horizontal velocity and $h$ denotes the horizontal distance of the measurement plane defined by the lidar shots. A horizontally pointed lidar beam transverse to the mean horizontal vector also has the same geometry and samples a rectangular plane. Spatial averaging of the wind field may result from the transverse motion of the atmosphere or, equivalently, the transverse motion of the lidar platform, which is the geometry for space-based or aircraft lidar measurements (Frehlich 2000, 2001a,b). The horizontal distance sampled by the lidar for each velocity estimate is

$$\Delta h = V_H T$$  

(15)

when the lidar beam axis is transverse to the horizontal velocity vector, for example, for a vertically pointed beam. The general case is a small extension of these results.

For a fixed lidar beam and multiple-pulse data products, the ensemble average velocity $m_{ave}(R, t)$ for a given radial velocity field is well approximated by the spatial average of the radial velocity $u(r, h)$ over the two-dimensional observation plane, that is,

$$m_{wgt}(R, H) = \int_{-\infty}^{\infty} \int_{-\Delta h/2}^{H+\Delta h/2} u(r, h) I_s(R - r) \, dh \, dr,$$  

(16)

where $H$ denotes the center of the measurement region in the horizontal direction. This is also a good approximation for space-based measurements and typical velocity estimators (Frehlich 2000).

For two in situ point measurements separated vertically by $\Delta s$, the important parameter to describe the horizontal spatial averaging is $\Delta h/\Delta s$. For Doppler lidar, the spatial average along the lidar beam is described by the pulse length $\Delta r$, the range-gate length $\Delta p$, and the separation of the velocity measurements $\Delta s$. For typical operation $\Delta r \approx \Delta p$ and $\Delta s \approx \Delta p$. The velocity estimates should behave like the single-pulse estimates when the horizontal averaging distance $\Delta h$ is much less than the range-gate length $\Delta p$. This assumption will be evaluated by computer simulations. We will adopt the parameter $\Delta h/\Delta p$ as a measure of the horizontal averaging by the transverse motion of the atmosphere or, equivalently, the transverse motion of a moving lidar platform such as an aircraft or scanning lidar.

d. Description of turbulent wind fields

The statistical description of Doppler lidar velocity estimates requires a statistical description of the radial velocity $v(r, t)$ of the vector velocity field (also called
the longitudinal velocity). For stationary conditions, the structure function of \( u(r, t) \) is independent of time \( t \) and defined as

\[
D_u(s) = \langle [u(r_0) - u(r_0 + s)]^2 \rangle.
\]

A simple model for the structure function is

\[
D_u(s) = 2\alpha_u^2 \Lambda(s/L_0),
\]

where \( \alpha_u^2 \) is the variance of the radial velocity, and \( L_0 \) is the outer scale of turbulence, which is proportional to other length scales such as the integral length scale \( L_* \), which is defined as the integral of the normalized correlation of the radial velocity.

For the von Kármán model (Hinze 1959; Lenschow and Kristensen 1988; Frehlich 2000; Frehlich et al. 2001),

\[
\Lambda(x) = 1 - \frac{2^{2/3}}{\Gamma(1/3)} x^{1/3} K_{1/3}(x) = 1.0 - 0.592 \times 5.548 x^{1/3} K_{1/3}(x),
\]

where \( K_{1/3}(x) \) is the modified Bessel function of order 1/3, and

\[
L_* = \sqrt{\frac{\pi}{\Gamma(5/6)}} L_0 = 0.746 \times 834 \times 3 L_0.
\]

If the outer scale is very large (\( L_0 \gg s \)) then the Kolmogorov model is valid; that is,

\[
D_u(s) = C_s \epsilon^{2/3} x^{2/3},
\]

where \( \epsilon \) is the energy dissipation rate, the Kolmogorov constant \( C_s = 2 \) (Monin and Yaglom 1975, p. 485), and

\[
\epsilon = \frac{2^{4/3} \pi}{\sqrt{3} \Gamma(1/3) \Gamma(4/3) C_s} \frac{\sigma_u^3}{\sigma_x^3} \frac{L_0}{L_*} = 0.933 \times 668 \times \frac{\sigma_u^3}{\sigma_x^3} \frac{L_0}{L_*} = 0.697 \times 295 \times \frac{\sigma_u^3}{\sigma_x^3} \frac{L_0}{L_*}.
\]

The von Kármán model also describes the homogeneous isotropic turbulence for the two-dimensional velocity field \( u(r, h) \) (Frehlich 2000; Frehlich et al. 2001).

e. Estimating parameters of the turbulent wind field

When the Doppler lidar signal power is high, the velocity estimates at range gate \( R \) and measurement time \( t \) can be represented as

\[
\hat{u}(R, t) = m_{vel}(R, t) + e(R, t),
\]

where \( e(R, t) \) is a zero-mean random estimation error due to the random fluctuations of the lidar signal (Frehlich and Yadołwsky 1994; Frehlich 1997, 2001b). The estimation error is typically uncorrelated with the atmospheric velocity component \( m_{vel}(R, t) \) (Frehlich 2001b). The structure function of the mean Doppler lidar velocity estimates is given by

\[
D_{vel}(R_1, R_2) = \langle [m_{vel}(R_1, t) - m_{vel}(R_2, t)]^2 \rangle.
\]

For a stationary and homogeneous wind field \( D_{vel}(R_1, R_2) = m_{vel}(R_1, t) - m_{vel}(R_2, t) \), and for a Gaussian lidar pulse with a von Kármán model for the turbulent wind field

\[
D_{vel}(s, \sigma_v, L_0) = 2\alpha_v^2 G(s/\Delta p, \Delta p/L_0, \sqrt{2 \ln \Delta p/\Delta r}),
\]

where \( G(m, \mu, \chi) \) is given by Eq. (39) of Frehlich et al. (1998).

When the estimation error \( e(R, t) \) is uncorrelated with the pulse-weighted velocity \( m_{vel}(R, t) \), an unbiased estimate for the velocity structure function of the mean Doppler lidar velocity estimates \( m_{vel}(R, t) \), which is approximately equal to the conditional mean \( m_{ave}(R, t) \) is given by (Frehlich et al. 1998)

\[
\hat{D}_{vel}(R_1 - R_2) = \hat{D}_{ave}(R_1 - R_2) - \delta_x^2(R_1 - R_2),
\]

where \( \hat{D}_{ave}(k\Delta s) \)

\[
= \frac{1}{N_s(N_s - k)} \sum_{j=1}^{N_s} \sum_{j=1}^{N_s-k} [\hat{u}(j\Delta s, IT) - \hat{u}((j + k)\Delta s, IT)]^2
\]

is the raw estimate of the velocity structure function, \( \Delta s \) is the spacing between adjacent Doppler lidar velocity estimates, \( N_s \) is the number of range gates for a single line of sight (LOS), \( T \) is the dwell time for each LOS measurement, \( N_r \) is the number of LOS velocity measurements, and \( \delta_x^2(R_1 - R_2) \) is an unbiased correction for the estimation error \( e(R, t) \) of the velocity estimates.

The estimation errors \( e(R_1, t) \) and \( e(R_2, t) \) are correlated for nearby range gates \( R_1 \) and \( R_2 \) because the velocity estimates are produced with sliding range gates with \( \Delta s < \Delta p \), or the spatial extent of the lidar pulse extends into the adjacent range gate. For range gates with larger separation \( \Delta p \) the estimation error \( e(R_1, t) \) is uncorrelated with \( e(R_2, t) \), and the estimation function correction for the estimation error can be based on estimates of the variance of \( e(R_1, t) \). However, the small separations of the structure function sample the inertial region \( s \ll L \), which is critical for accurate estimates of \( e \). Therefore, various algorithms have been proposed for \( \delta_x^2 \) based on the statistics of

\[
\Delta e(R_1, R_2, t) = e(R_1, t) - e(R_2, t),
\]

which does not require uncorrelated errors for \( e(R, t) \) as a function of range \( R \) but assumes that \( \Delta e(R_1, R_2, t) \) is uncorrelated in time \( t \) (this is a good assumption for coherent Doppler lidar because the speckle fluctuations and the additive detector noise are uncorrelated in time). The estimation algorithms for \( \delta_x^2 \) include the temporal spectrum of \( \Delta e(R_1, R_2, t) \) (Frehlich 1996; Frehlich et al. 1994, 1997, 1998; Lenschow et al. 2000) and the temporal autocovariance of \( \Delta e(R_1, R_2, t) \) (Mayor et al. 1997; Lenschow et al. 2000). Another algorithm based on the difference of velocity estimates using data from
even and odd numbered lidar shots is attractive for a larger range of conditions (Frehlich 2001a,b). These algorithms were compared for estimation of the variance of \( e(R, t) \) (Frehlich 2001b), and similar performance is produced for estimating the variance of \( \Delta e(R_1, R_2, t) \).

The spectral (spec) method uses the constant level of the temporal spectrum of \( \Delta e(R_1, R_2, t) \) at high frequency (Frehlich 2001b; Frehlich et al. 1998) to estimate

\[
\sigma_{\Delta e}^2(R_1, R_2) = \langle \Delta e(R_1, R_2, t) \rangle^2.
\]

An unbiased estimate for \( \sigma_{\Delta e}^2(R_1 - R_2) \) in Eq. (26) is given by

\[
\hat{\sigma}_{\Delta e}^2(k\Delta s) = \frac{1}{N_k - k} \sum_{j=1}^{N_k - k} \hat{\sigma}_{\Delta e}^2[j\Delta s, (j + k)\Delta s],
\]

where \( \hat{\sigma}_{\Delta e}^2(R_1, R_2) \) is the average of the high-frequency region of the temporal spectrum of \( \Delta e(R_1, R_2, t) \) that is within 5% of constant value (see Eq. (36) of Frehlich et al. 1998). This method is successful when there is a well-defined constant level at high frequencies (Frehlich 2001b).

The covariance (cov) method uses the discontinuity of the temporal covariance function of \( \Delta e(R_1, R_2, t) \) at zero lag as an estimate for \( \sigma_{\Delta e}^2(R_1, R_2) \). The most robust estimator is the difference between the covariance estimate at zero lag and the linear interpolation of the covariance estimates of the first two lags to zero lag [see Eq. (27) of Frehlich 2001b]. This method is successful when there is a well-defined discontinuity at zero lag.

In some cases the spectral and covariance methods produce biased estimates of the structure functions. The velocity difference (vel diff) method produces unbiased structure function estimates for most operating conditions by producing two Doppler lidar velocity estimates, \( \hat{v}_e(R, t) \) and \( \hat{v}_o(R, t) \), generated from the even and odd numbered lidar pulses, respectively. These two estimates have the same conditional mean value \( m_{ave}(R, t) \) (which is approximately equal to \( m_{ave}(R, t) \)) and statistically independent random error \( e(R, t) \), even for adjacent range gates. The unbiased structure function estimate (see Eq. 26) is produced with

\[
\hat{D}_{ave}(k\Delta s) = \frac{1}{2N_f(N_k - k)} \sum_{j=1}^{N_f} \sum_{i=1}^{N_k - k} \left( \hat{v}_e(j\Delta s, iT) - \hat{v}_o((j + k)\Delta s, iT) \right)^2
\]

\[+ \left( \hat{v}_o(j\Delta s, iT) - \hat{v}_e((j + k)\Delta s, iT) \right)^2
\]

(31)

and

\[
\hat{\sigma}_{\Delta e}^2(k\Delta s) = \frac{1}{N_k - k} \sum_{j=1}^{N_k - k} \hat{\sigma}_{\Delta e}^2(j\Delta s)
\]

\[+ \hat{\sigma}_{\Delta e}^2((j + k)\Delta s),
\]

where

\[
\hat{\sigma}_{\Delta e}^2(R) = \frac{1}{2N_f} \sum_{j=1}^{N_f} \left( \hat{v}_e(R, l\Delta t) - \hat{v}_o(R, l\Delta t) \right)^2
\]

(33)

is an unbiased estimate for the variance of \( e(R, t) \) for the velocity estimates \( \hat{v}_e(R, t) \) and \( \hat{v}_o(R, t) \) (Frehlich 2001b). The estimates for \( \sigma_{\Delta e}^2(k\Delta s) \), Eqs. (30) and (32), are also unbiased when the variance of \( e(R, t) \) depends on the range \( R \), which is the typical operating condition.

The parameters of the wind field, \( \epsilon \) and \( L_\epsilon \), are determined by minimizing the weighted error \( \chi^2 \) between the structure function estimates \( D_{ave}(s) \) and the model predictions \( D_{ave}(s, \sigma_s, L_\epsilon) \); that is,

\[
\chi^2 = \frac{1}{N_s} \sum_{k=1}^{N_s} \left( \frac{\hat{D}_{ave}(k\Delta s) - D_{ave}(k\Delta s, \sigma_s, L_\epsilon)}{\left[kG_{ave}(k\Delta s, \sigma_s, L_\epsilon)\right]^2} \right)^2,
\]

where \( \Delta s \) is the range separation between adjacent velocity estimates and \( N_s \) is the number of lags used for the fit. The variance of the estimates \( \hat{D}_{ave}(k\Delta s) \) is inversely proportional to the number of independent samples \( N_i \) of the velocity difference at separation \( k\Delta s \). Because \( N_i \) is proportional to \( L_\epsilon/N_s \), the denominator of Eq. (34) is approximately proportional to the variance of the estimates \( \hat{D}_{ave}(k\Delta s) \). The estimates \( \hat{D}_{ave}(k\Delta s) \) have a Gaussian distribution under the central limit theorem of a large number of samples \( N_{total} = N_sN_i \). Then \( \chi^2 \) is approximately the chi-squared statistic, which is the optimal choice for the best-fit parameters \( L_\epsilon \) and \( \sigma_s \) using Eq. (22). The estimates of \( L_\epsilon = 0.7468343L_0 \) [see Eq. (20)].

The statistical description of the Doppler lidar velocity estimates \( v(R, t) \) is given in terms of the velocity field \( v(r, t) \), which is related to a frozen velocity field \( u(r, h) \) by Eq. (14). The important lidar and atmospheric parameters are the sampling interval \( T_s \) for the raw lidar data, the number of accumulated lidar pulses per velocity estimate, the range-gate length \( \Delta p = MT_s/2 \) for each velocity estimate, the spacing \( \Delta s \) between velocity estimates in range, the horizontal translation \( \Delta h \) of the atmosphere during a velocity estimate, the number of velocity estimates in range and time \( (N_sN_r) \), the corresponding sampling length in range and horizontal distance \( L_r = \Delta p + (N_r - 1)\Delta s \) and \( L_\epsilon = \Delta hN_r \), respectively, the integral length scale \( L_\epsilon \), and the energy dissipation rate \( \epsilon \) for the von Kármán turbulence spectrum.

3. Results

The velocity field \( u(r, h) \) is simulated as an isotropic velocity field with correct spatial statistics that satisfy the von Kármán model (Frehlich 2000; Frehlich et al. 2001). The algorithm is valid for any other model for homogeneous velocity fields. The radial velocity \( v(r, t) \) for the simulation of the Doppler lidar signals is generated using Eq. (14), that is, \( v(r, t) = u(r, V_s) \), assuming constant SNR over the simulation domain (Frehlich 1997, his appendix C; see also Salamitou et al. 1995). Velocity estimates \( \hat{v}(\Delta s, l\Delta t) \) are produced with the maximum likelihood estimator, which assumes a constant velocity over the range gate and assumes that the SNR is known (the SNR can be accurately deter-
mined in high SNR conditions and a large number $N_P$ of lidar pulses per estimate (Frehlich and Yadlowsky 1994). The lidar parameters are chosen to approximate a 2-µm lidar (Henderson et al. 1991, 1993; Frehlich et al. 1994, 1997, 1998), that is, $\lambda = 2.0$ µm, $T_0 = 0.02$ µs, $\Delta r = 24.0$ m, and $v_{\text{search}} = 25.0$ m s$^{-1}$. The other lidar and atmospheric parameters are listed in Table 1. The turbulence parameters for the benchmark case are $\epsilon^{2/3} = 0.01$ m$^{4/3}$ s$^{-2}$, $\epsilon_\text{m} = 0.001$ m$^2$ s$^{-3}$, $L_i = 100$ m, and $\sigma_\text{e} = 0.5234$ m s$^{-1}$. This is a low-turbulence case.

A high spatial resolution case was considered first with $M = 8$ complex data samples per velocity estimate, a pulse length of $\Delta r = 24.0$ m, a range-gate length $\Delta p = 24.0$ m, a range-gate separation $\Delta s = 24.0$ m (no overlapping data), $N_p = 20$ lidar pulses per velocity estimate, SNR = 2.5, $N_R = 16$ range gates, and, for the structure functions estimates, $N_s = 15$ lags and $N_T = 512$ LOS velocity estimates. The dimensions of the measurement plane are $L_H = 384$ m along the lidar beam and $L_R = 1538$ m transverse to the lidar beam. The horizontal averaging distance per LOS measurement is $\Delta h = 3.0$ m, which is equivalent to a vertically pointed lidar with a PRF $= 20$ Hz, a dwell time of $T = 1.0$ s for each LOS measurement, and a mean horizontal velocity $v_H = 3.0$ m s$^{-1}$.

An example of a structure function estimate and the best-fit model parameters is shown in Fig. 1. The correction for the estimation error $[\sigma^2_{\text{est}}(R_2, R_1)]$ in Eq. (26) is important for producing a good fit in this parameter regime, especially for the small lags, which have a large contribution from the estimation error and are critical for adequate sampling of the inertial region $s \ll L_i$. The good agreement of the structure function estimate at the first lag is verification of the accuracy of the noise correction as well as verification of the uncorrelated nature of the random error $e(R, t)$ for the velocity estimates of the even and odd numbered lidar pulses. The estimate for the integral length scale $L_i$ is poor because there are few independent samples of the velocity fluctuations on scales comparable to $L_i$. For example, at a separation of $s = 300$ m there are approximately $L_H L_R / s^2 \approx 6$ independent samples of the wind field. This is reflected in the large spatially correlated error in the structure function estimates at large lag. For a lag of $s = 24$ m there are approximately 1000 independent samples of the wind field, which results in a more accurate estimate of $\epsilon^{2/3}$ if the large contribution from the estimation error can be accurately determined.

The probability density function (PDF) or histogram of 500 estimates of $e^{2/3}$ and $L_i$ are shown in Figs. 2 and 3, respectively, for the same conditions as Fig. 1. The estimates for $e^{2/3}$ are well behaved and symmetrically distributed around the true value. The cov method for noise correction has a small negative bias because of the error in estimating $\sigma^2_{\text{est}}(R_2, R_1)$ for $\Delta h = 3.0$ m. This has also been observed for the measurement of the random velocity error of the radial velocity estimates (Frehlich 2001b); however, the error approaches zero as $\Delta h$ approaches zero. The spec and vel diff correction algorithms are unbiased for this case.

Because of the higher correlated error in the structure function estimates at large separation, the distribution of the estimates for $L_i$ are skewed to large values, and the mean and standard deviation do not adequately describe the PDF. Therefore, we will use the median $L_i^\text{m}$ and error bars based on the $1\sigma$ percentiles of a Gaussian distribution around its median to describe the statistical performance of the estimates for $L_i$. The $+1\sigma$ value is the difference between the value of $L_i$ at the 84.1345 percentile (the percentile of a Gaussian distribution at $1\sigma$ above the median) minus the median. Similarly, the

### Table 1. Lidar and atmospheric parameters.

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<th>Fig.</th>
<th>$\Delta p$ (m)</th>
<th>$\Delta s$ (m)</th>
<th>$L_H$ (m)</th>
<th>$L_R$ (m)</th>
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<th>$L_i$ (m)</th>
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Fig. 2. PDFs of estimates of $e^{2/3}$ with the same parameters as Fig. 1 and the three methods of noise correction: covariance (cov), spectral (spec), and velocity difference (vel diff).

Fig. 3. PDFs of estimates of $L_i$ with the same parameters as Fig. 1 and the three methods of noise correction (cov, spec, vel diff). The median ($\overline{1} s$), $1\sigma$, and $-1\sigma$ values are listed.

Fig. 4. Average $\langle e^{2/3} \rangle$ and standard deviation SD($e^{2/3}$) of the estimates of $e^{2/3}$ from the best-fit to the structure function measurements vs the true value $e^{2/3}$ for the three methods of noise correction (cov, spec, vel diff) with $\Delta \rho = 24$ m, $\Delta s = 24$ m, and $\Delta h = 3$ m. The solid line in the upper panel indicates $10\%$ error and in the lower panel indicates the input value of $e^{2/3}$.

$-1\sigma$ value is the difference between the median and the value of $L_i$ at the 15.8655 percentile. All three algorithms have similar performance.

The performance of the estimates for $e^{2/3}$ as a function of $e^{2/3}$ is shown in Fig. 4. All the estimation methods have small error ($<10\%$) when the turbulence is moderate ($e^{2/3} > 0.01$), but the cov method has a small $5\%$--$7\%$ negative bias. The spec and vel diff noise-correction algorithms are unbiased because the estimates of $\hat{\sigma}_2(R_1, R_2)$ are unbiased and therefore the corrected structure functions $\hat{D}_{e^{2/3}}(k\Delta s)$ are unbiased. The parameter $e^{2/3}$ was chosen for analysis instead of $e$ because the estimates $\hat{D}_{e^{2/3}}(s)$ are proportional to $e^{2/3}$ for $s \ll L_i$ [see Eq. (18)] and $\langle e^{2/3} \rangle \neq \langle e \rangle^{2/3}$; that is, the estimates of $e$ are biased. The spec method of noise correction has the best performance, especially in the very weak turbulence regime ($e^{2/3} < 0.002$). Here, the spectral method is the most accurate estimator of the additive noise contribution. The velocity difference algorithm has difficulty because the contribution from the estimation error is twice as large compared with the spec and cov methods, because the structure function estimate Eq. (32) is based on the velocity estimates using the even and odd numbered pulses, which have twice the error variance $\sigma^2(R, t)$ and $\sigma^2(R_1, R_2)$.

The performance of the estimates for $L_i$ as a function of $e^{2/3}$ is shown in Fig. 5. The median value $L_i^m$ is close
Fig. 5. Median $L_i^*$ (symbols), $+1\sigma$, and $-1\sigma$ percentiles (error bars) of the estimates of $L_i$ from the best-fit to the structure function measurements vs the true value $\varepsilon^{2/3}$ for the three methods of noise correction [(top) cov, (middle) spec, (bottom) vel diff] with $D_p = 24$ m, $D_s = 24$ m, and $\Delta h = 3$ m. The horizontal line indicates the true value of $L_i$.

Fig. 6. Average $\langle \varepsilon^{2/3} \rangle$ and standard deviation $SD(\varepsilon^{2/3})$ of the estimates of $\varepsilon^{2/3}$ from the best-fit to the structure function measurements vs the horizontal measurement interval $\Delta h/\Delta p$ for the three methods of noise correction (cov, spec, vel diff) with $N_T = 512$, $D_p = 24$ m, and $\Delta s = 24$ m. The horizontal line indicates the true value of $\varepsilon^{2/3}$.

Fig. 7. Median $L_i^*$ (symbols), $+1\sigma$, and $-1\sigma$ percentiles (error bars) of the estimates of $L_i$ from the best-fit to the structure function measurements vs the horizontal measurement interval $\Delta h/\Delta p$ for the three methods of noise correction [(top) cov, (middle) spec, (bottom) vel diff] with $N_T = 512$, $D_p = 24$ m, and $\Delta s = 24$ m. The horizontal line indicates the true value of $L_i$.

to the correct value for most cases. There is a small bias for the cov noise-correction method that is related to the small bias in estimating $\varepsilon^{2/3}$. The $+1\sigma$ error bars are much larger than the $-1\sigma$ error bars, reflecting the skewed PDFs for $L_i$. It is difficult to produce an accurate structure function estimate on scales comparable to $L_i = 100$ m with a sampling region $(L_R, L_H) = (384$ m, $1536$ m). A larger sampling region is required to increase the number of independent samples of the larger scales.

The estimates of the turbulence parameters ($\varepsilon^{2/3}$, $L_i$) are produced by assuming that the Doppler lidar measurements are given by the spatial average of the instantaneous radial velocity [see Eq. (11)]; that is, there is no spatial averaging in the horizontal direction ($\Delta h \ll \Delta p$). The performances of the estimates for $\varepsilon^{2/3}$ and $L_i$ as a function of $\Delta h/\Delta p$ are shown in Figs. 6 and 7, respectively, for $\varepsilon^{2/3} = 0.01$ m$^{2/3}$ s$^{-2}$ and a fixed number $N_T$ of velocity estimates. The spec method has good performance for $\Delta h < 0.2 \Delta p$; the vel diff method performs well for $\Delta h < 0.4 \Delta p$; and the cov method has a small bias for $\Delta h < 0.2 \Delta p$ then follows the performance curves of the other two methods as the effects
of the transverse spatial averaging become important. If the assumption of Taylor’s hypothesis of a frozen flow [Eq. (14)] is valid, and if the parameter \( \Delta h \) is known and constant over the measurement interval, then the structure function model can be calculated to include the spatial average of the velocity field in the transverse direction using Eq. (16).

There are many parameters that affect the estimators for \( D^{2/3} \) and \( L_{v} \). The effects of increasing the range-gate dimension \( \Delta p \) from 24.0 to 48.0 m are shown in Figs. 8–12 with \( M = 16 \) complex data samples per velocity estimate, a pulse length of \( L_{u} = 24.0 \) m, a range-gate separation \( \Delta s = 24.0 \) m (overlapping data), \( N_{p} = 20 \) lidar pulses per velocity estimate, \( \text{SNR} = 2.5 \), and 16 range gates, and, for the structure function estimates, \( N_{L} = 15 \) lags and \( N_{R} = 512 \) LOS velocity estimates. An example of the best-fit structure function is shown in Fig. 8. The magnitude of the effect of spatial averaging by the lidar pulse is larger than for the 24-m range gate (see Fig. 1), especially for the first lag of 24 m where they differ by a factor of 1.67. The PDFs of the estimates of \( D^{2/3} \) and \( L_{v} \) are similar to Figs. 2 and 3 and are not reproduced here. The standard deviations of the estimates of \( D^{2/3} \) shown in Fig. 9 are smaller than those of the 24-m range gate for smaller \( D^{2/3} \) but larger for larger \( D^{2/3} \) (see Fig. 1). The magnitude of the random error \( e(R, t) \) decreases when \( \Delta p \) increases, and the effects of velocity fluctuations over the range gate are negligible (small \( D^{2/3} \)). However, the converse is true when the velocity fluctuations over the range gate become large (Frehlich 1997).

The error in the estimates of \( L_{v} \) shown in Fig. 10 are also smaller than those of the 24-m range gate for smaller \( D^{2/3} \) (see Fig. 5). The effects of the horizontal averaging of the wind field (\( \Delta h \)) are shown in Figs. 11 and 12. The \( \Delta p = 48 \) m range gate produces good results for \( \Delta h/\Delta p < 0.1 \) or \( \Delta h < 4.8 \) m, which agrees with the performance for the \( \Delta p = 24 \) m range gate in Fig. 6, where good results are produced for \( \Delta h/\Delta p < 0.2 \) or \( \Delta h < 4.8 \) m. The effects of horizontal averaging of the wind field on the estimation of \( L_{v} \), shown in Fig. 12 for the 48-m range gate are similar to the results for the 24-m range gate shown in Fig. 7 when compared as a function of \( \Delta h \).

The error \( \text{SD}(e^{2/3}) \) in the estimates of \( D^{2/3} \) should scale as \( 1/N_{L} N_{R} \times 1/N_{L} N_{R} \) (Banakh and Smalikh 1997). This hypothesis is tested for the 24- and 48-m range gates in Figs. 13 and 14, respectively, for the most common situation of a fixed measurement range \( L_{R} \) and a variable horizontal measurement distance \( L_{H} \). The results for the 24-m range gate agree with the error scaling \( \text{SD}(e^{2/3}) \propto L_{H}^{-1/2} \); however, the results for the 48-m range gate do not agree with this error scaling for large \( L_{H} \).

4. Summary and discussion

Coherent Doppler lidar measurements of the spatial wind statistics (the parameters \( D^{2/3} \) and \( L_{v} \)) using estimates of the spatial structure function of the radial velocity are feasible for those conditions where the important scales of the velocity field are accurately resolved, that is, when \( \Delta p < L_{v} \) and \( L_{p} > L_{v} \). The parameters \( D^{2/3} \) and \( L_{v} \) are determined by the best fit between...
lidar estimates and a model for the structure functions, assuming a von Kármán spatial spectrum of the random velocity field. The techniques are applicable to any other structure function model; however, similar analyses are required to determine the performance of the estimates. Unbiased estimates of $\varepsilon^{2/3}$ require an accurate correction for the estimation error $\sigma_{\varepsilon_{\Delta r}}^2$ of the difference of the radial velocities [see Eq. (26) and Figs. 1 and 8]. The probability density functions (PDFs) of the estimates for $\varepsilon^{2/3}$ are approximately Gaussian (see Fig. 3), with little bias when there is negligible spatial averaging in the horizontal direction ($\Delta h/\Delta p \ll 1$). The PDFs of the estimates for $L_i$ are highly skewed (see Fig. 3), and the median and 1σ percentiles of the PDFs produce a better statistical description (see Figs. 5, 7, 10, 12). A large sampling area ($L_R \times L_H$) is required to reduce the skewness in the PDF($L_i$) by sampling all the scales of the process, which is required to remove the numerical instabilities of the nonlinear fitting algorithm.

When there is negligible spatial averaging in the horizontal direction, all the estimation algorithms produce unbiased estimates of $\varepsilon^{2/3}$ for a wide range of conditions (see Figs. 4, 9). The median value of the estimates for $L_i$ from the best-fit to the structure function measurements vs the horizontal measurement interval $\Delta h/\Delta p$ for the three methods of noise correction [(top) cov, (middle) spec, (bottom) vel diff] with $N_r = 512$, $\Delta p = 48$ m, and $\Delta s = 24$ m. The horizontal line indicates the true value of $L_i$.
L_i are also unbiased (see Figs. 5, 10). The accuracy of the estimates of \( \epsilon^{2/3} \) defined by SD(\( \epsilon^{2/3} \)) is proportional to \( \epsilon^{2/3} \) for large \( \epsilon^{2/3} \) where the effects of the correction for velocity estimation error are negligible (see Figs. 4 and 9). The spectral (spec) noise-correction method is the most robust method for weak turbulence \( \epsilon^{2/3} \leq 0.003 \) m \( \frac{4}{3} \) s \( \text{m}^{-2} \).

The spatial averaging of the velocity field in the horizontal direction of the lidar beam axis produces no bias when the horizontal averaging dimensions \( \Delta h \ll \Delta p \) where \( \Delta p \) is the range-gate length of each velocity estimate (see Figs. 6, 11). For the parameters of Figs. 6 and 11, this requires \( \Delta h = V_H T < 5 \) m, which implies that the horizontal velocity \( V_H < 5 \) m s\(^{-1} \) for \( T = 1 \) s dwell time for each velocity estimate. For higher velocities, the dwell time can be reduced using a high PRF lidar, or the effects of the horizontal averaging can be included in the structure function model. This requires a two-dimensional calculation of the structure function model and, for a vertically pointed lidar, a priori knowledge of the horizontal velocity \( V_H \) as a function of range. A scanning lidar does not require a priori knowledge of \( V_H \) if the velocity field in the plane of the scan is approximately frozen. Further work is required to determine the merits of these techniques.

The analysis with a \( \Delta p = 24 \) m range gate performs better than the case of a 48-m range gate for moderate turbulence conditions (\( \epsilon^{2/3} > 0.005 \) m \( \frac{4}{3} \) s \( \text{m}^{-2} \)), while the latter analysis performs better for weak turbulence conditions (\( \epsilon^{2/3} < 0.005 \) m \( \frac{4}{3} \) s \( \text{m}^{-2} \)) (see Figs. 4, 9). This is a result of the different scaling of the random velocity error for low and high turbulence (Frehlich 1997). For the 24-m range gate, the error in the estimates of \( \epsilon^{2/3} \) defined by SD(\( \epsilon^{2/3} \)) scales as \( L_H^{2/3} \) (see Fig. 13), where \( L_H \) is the total horizontal dimensions of the measurement region, and the total measurement range \( L_R \) is fixed. This scaling was also observed for estimates of \( \epsilon \) assuming infinite \( L_H \) (Banakh and Smalikho 1997). For the 48-m range gate, this scaling is only valid for smaller distances \( L_H \) (see Fig. 14). The small bias in the cov method of correcting the estimation error \( \sigma^2_{\epsilon} \) [see Eq. (26)] of the velocity differences (Frehlich 2001b) produces a small bias in the estimates of \( \epsilon^{2/3} \) in Figs. 13 and 14.

The results presented here assume a simple Cartesian lidar beam geometry and can be applied to the range height indicator (RHI) and plan-position indicator (PPI) scanning patterns for ground-based and aircraft-based platforms if the trapezoidal measurement region can be approximated as a rectangle, that is, for a large measurement range \( L_H \) and small angular extent. The effects of the RHI and PPI scan patterns can be determined for the model structure functions by using the correct statistical analysis in the appropriate angular coordinate system. Future work is required to determine the impact of these various approximations.

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