A Parameterization of the Depth of the Entrainment Zone

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ABSTRACT

A theory of the parameterization of the entrainment zone depth has been developed based on conservation of energy. This theory suggests that the normalized entrainment zone depth is proportional to the inverse square root of the Richardson number.

A comparison of this theory with atmospheric observations indicates excellent agreement. It does not adequately predict the laboratory data, although it improves on parcel theory, which is based on a momentum balance.

1. Introduction

At the top of the boundary layer, the interface with the stably stratified atmosphere aloft is usually convoluted. These convolutions are the result of thermals that overshoot the temperature inversion. Convolutions shape the interface and increase the area of contact and entrainment between the two adjacent layers. On a larger spatial scale, a field of thermals produces an intermediate region between the boundary layer and the atmosphere aloft in which the fractional area covered by boundary layer air increases from zero at the top of this region to one at the base of this region. This region, which has been referred to as the entrainment zone (Deardorff et al. 1980), is of prime importance in boundary layer studies. Turbulence kinetic energy is used in the entrainment zone to mix warm stable air downward into the boundary layer. The use of an accurate parameterization takes on new meaning in the case of cloud-topped boundary layers as cloud top infrared cooling is an important sink in the thermodynamic budget of a cumulus filled boundary layer and an important source of turbulence kinetic energy (Randall 1980). For accurate radiative transfer calculations a precise knowledge of cloud heights is therefore essential. Considerable effort has gone into parameterization schemes to predict the entrainment zone depth (Stull 1973; Zeman and Tennekes 1977; Mahrt 1979; Deardorff et al. 1980). In turn, an accurate parameterization might be used to infer important boundary layer parameters such as surface heat flux if accurate measurements of the entrainment zone are available. The simplest but well-known parameterization proposes a linear dependence of the normalized entrainment zone depth on the inverse convective Richardson number (Deardorff et al. 1980). However, this parameterization is not borne out by their observations, nor by those of Boers and Eloranta (1986).

Both datasets in fact suggest a much weaker dependence of the normalized entrainment zone depth on the Richardson number. In the past, either dimensional arguments and/or momentum balance has been used to describe the deceleration of a parcel as it overshoots the inversion. In this paper, we will use an energy balance. The result shows a much weaker inverse Richardson number dependence of the normalized entrainment zone depth than found previously, in agreement with the experimental findings.

2. Momentum balance

A momentum balance such as developed by Deardorff et al. (1980), and Stull (1973) equates the parcel deceleration with the negative buoyancy force as the parcel overshoots the inversion. Based on this balance it can be shown that

\[
\frac{\Delta h}{h} = \alpha \frac{T_0 (w^*)^2}{g \Delta \theta_v h} = \alpha \text{Ri}^{-1}
\]

where

- \(\Delta h\) is the overshoot distance
- \(h\) is the mean boundary layer height
- \(\alpha\) is a constant of proportionality
- \(T_0\) is mean boundary layer potential temperature
- \(g\) is gravity
- \(\Delta \theta_v\) is the virtual potential temperature jump at the inversion interface
- \(w^*\) is the convective velocity scale
- \(\text{Ri}\) is the "convective" Richardson number

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If there is no buoyancy jump but only an atmosphere with a stable potential temperature lapse rate $\gamma$, the balance reads

$$\frac{\Delta h}{h} = \alpha \frac{w^*}{\omega_B h}$$  

(2)

where $\omega_B$ is the Brunt-Väisälä frequency defined as

$$\omega_B = \left(\frac{g}{T_0} \gamma \right)^{1/2}$$  

(3)

Equations (1) to (3) assume that the vertical velocity $w$ at the base of the inversion is proportional to the convective velocity scale.

As stated in the Introduction, field experiment results have not confirmed equation (1). On close examination of the process involved, it is not difficult to see why. A momentum balance as indicated by (1) represents an unrealistic lifting of the inversion interface over a large horizontal area. Since this lifting assumes the local replacement of lower density with higher density air, it supposes a gain of potential energy integrated over height. In reality however, a local lifting of the inversion at one position needs to be compensated by a lowering of the inversion around the area of lifting by virtue of continuity. At the location where the inversion is lowered, provided no mixing takes place, high density air is replaced by low density air and the potential energy integrated over height decreases. Therefore, a local disturbance at the inversion interface creates areas of gain and loss of potential energy so that it is not clear how much potential energy is gained over the whole disturbance. In the next section, we will examine such a disturbance using an energy balance integrated over height and over horizontal distance.

3. Energy balance

This section will have three parts. For simplicity we will first consider an incompressible fluid, and derive an energy balance before and after a perturbation of the interface between the two layers. Next we repeat it for the more realistic situation of two adiabatic atmospheric layers on top of each other. We then compare the theoretical results with observations. Although entrainment could play a role in determining the depth of the entrainment zone, we will not consider it here.

a. Incompressible fluid

We consider a two-layer incompressible fluid system of depth $d + h$. The lower layer beneath level $h$ has a constant density $\rho_1$, the upper layer a density $\rho_2 = \rho_1 - \Delta \rho$. This system, at rest as in Fig. 1, represents a potential energy $P_i$ integrated over depth at an initial time $t = 0$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.pdf}
\caption{Depiction of two-layer fluid systems with a flat interface, initially. The broken line represents the fluid system when a perturbation has changed the shape of the interface.}
\end{figure}

\begin{align*}
P_i &= \int_0^{h+d} \rho g z dz = \int_0^h \rho_1 g z dz + \int_h^{h+d} (\rho_1 - \Delta \rho) g z dz \\
&= \frac{1}{2} (\rho_1 - \Delta \rho) g (h + d)^2 + \frac{1}{2} \Delta \rho g h^2 
\end{align*}

(4)

After a time $\Delta t$, the interface is convoluted and the height $h$ is a function of $x$:

\begin{equation}
h_f = h_i + \delta h(x) 
\end{equation}

(5)

At time $\Delta t$ the interface is convoluted as in Fig. 1 and

\begin{align*}
P_f &= \int_0^{h+d} \rho_1 g z dz + \int_{h+d}^{h+d} (\rho_1 - \Delta \rho) g z dz \\
&= \frac{1}{2} (\rho_1 - \Delta \rho) g (h + d)^2 + \frac{1}{2} \Delta \rho g (h + \delta h)^2 
\end{align*}

(6)

Therefore,

\begin{equation}
\Delta P = P_f - P_i = \frac{1}{2} \Delta \rho g [ (\delta h)^2 + 2h(\delta h)] 
\end{equation}

(7)

If the functional form of $\delta h(x)$ is a sine such as

\begin{equation}
\delta h(x) = \frac{1}{2} \Delta h \sin \frac{2\pi x}{\Delta x} 
\end{equation}

(8)

and we average $\delta h(x)$ over one wavelength, then

\begin{equation}
\Delta P = \Delta \rho g \frac{1}{16} \Delta h^2 
\end{equation}

(9)

Equation (9) shows that an initially flat fluid layer increases in potential energy in passing to a state with a convoluted top. If potential and kinetic energy ($K$) are conserved as:

\begin{equation}
P_i + K_i = P_f + K_f 
\end{equation}

(10)
the maximum of $\Delta P$ is obtained if all kinetic energy at time $t = 0$ is transformed into potential energy at $t = \Delta t$, so that

$$K_i = \Delta P, \quad K_f = 0.$$  

(11)

$K_i$ can be computed, using the mean value theorem as:

$$K_i = \int_0^{z_i} z_i \rho w^2 dz = \frac{1}{2} \rho_i \int_0^h w^2 dz = \frac{1}{2} \rho_i \bar{w}^2 h$$  

(12)

where it is assumed that $K_i$ was only present in the lower layer, associated only with the vertical velocity component, and that the integral can be approximated by an average. Clearly, $K_i$ can be associated with a thermal plume rooted at the surface. In the atmosphere, plumes are highly localized and represent in addition to kinetic energy a certain amount of thermal and potential energy which has to be distributed through the whole layer in passing from the initial to the final state. The balance (11) implies the constraint that these associated energies are small and therefore do not alter the initial value of $P_i$. We now use (9), (11), and (12) to get:

$$\frac{1}{16} \Delta \rho g \Delta h^2 = \frac{1}{2} \rho_i \bar{w}^2 h$$  

(13)

Solving for $\frac{\Delta h}{h}$ we get

$$\frac{\Delta h}{h} = \left( \frac{\rho_i}{\Delta \rho} \frac{\bar{w}^2}{gh} \right)^{1/2}$$  

(14)

In Eq. (14), $\bar{w}^2$ is not equivalent to the vertical component of the turbulence intensity. It is an average of the vertical velocity in an updraft. It is not unrealistic to assume that $w^2 \sim (w^*)^2$. Using the relation $\Delta \rho = \rho \beta \Delta T$, where $\beta$ is the coefficient of thermal expansion ($\frac{1}{T}$ in the atmosphere), then

$$\frac{\Delta h}{h} = \alpha \text{Ri}^{-1(1/2)}.$$  

(15)

This is an appealing result. Once again $\Delta h/h$ is parameterized in terms of a Richardson number, but in a much weaker form than found earlier. A more complete and more realistic situation is provided by assuming compressibility.

**b. Compressible atmosphere**

We consider an atmosphere at rest in hydrostatic equilibrium consisting of two adiabatic layers with top at height, pressure ($z_i$, $p_i$) separated by an interface with density jump $\Delta \rho$ (potential temperature jump $\Delta \theta$) and $z_i$ is the total depth ($z_i = h + d$). It should be pointed out that locally the atmosphere may not be in hydrostatic equilibrium as hot air is carried upward in plumes. However, averaged over many plumes and compensating downdrafts, it is likely to be a good approximation.

This atmosphere represents a certain amount of thermal, potential, and kinetic energy identified respectively as $U$, $P$, $K$ with

$$U = \frac{c_v}{g} \int_0^{z_i} T \rho dz = \frac{c_v}{g} \int_{p_i}^{p_0} Tdp$$  

(16)

$$P = g \int_0^{z_i} \rho z dz = \int_{p_i}^{p_0} z dp$$  

(17)

$$K = \frac{1}{2} \int_0^{z_i} \rho w^2 dz = \frac{1}{2} \int_{p_i}^{p_0} \frac{w^2}{g} dp$$  

(18)

where $T$ is the temperature, $c_v$ is the specific heat of dry air at constant volume, and the subscript 0 refers to the surface. The effects of water vapor and kinetic energy in horizontal velocity components are ignored in our computation.

We now consider the situation as in Fig. 1, where between time $t = 0$ and time $t = \Delta t$ an initially flat interface is convoluted. This represents an exchange of energy. Assuming for the moment no outside energy sources, nor entrainment between the two layers we have:

$$U_i + P_i + K_i = U_f + P_f + K_f$$  

(19)

where the subscript $i$ refers to the state of the atmosphere at the initial time $t = 0$, and the subscript $f$ to the state at $t = \Delta t$. Although we can carry out the integration for two layers, we let $z_i \rightarrow \infty$ because the equations then have a particularly simple form.

Using hydrostatic equilibrium and the ideal gas law, (17) can be rewritten as

$$P = g \int_0^{z} \rho z dz = R \int_0^{z} \rho T dz = \frac{R}{c_v} U$$  

(20)

where $R$ is the molar constant for dry air, and $p_i$ the pressure at the surface. This states that for an atmospheric volume in hydrostatic equilibrium the potential and thermal energy are proportional to each other (Iribarne and Godson 1973). In an adiabatic atmosphere

$$\frac{T}{T_0} = \left( \frac{p}{p_0} \right)^{\kappa}.$$  

(21)

Using (21) in (16) and integrating from the surface to the top of the atmosphere we have at the initial time $t = 0$

$$U_i = \frac{c_v}{g} \int_0^{p_0} Tdp = \frac{c_v}{g} \int_{p_i}^{p_0} Tdp + \frac{c_v}{g} \int_{p_i}^{p_0} Tdp$$

$$= \frac{c_v}{g(1 + \kappa)} (p_0 T_0 + p_i \Delta \theta).$$  

(22)
In an adiabatic atmosphere, we also have

$$T = T_0 \left( 1 - \frac{\Gamma d}{T_0} \right)$$  \hspace{1cm} (23)

where

$$\Gamma d = \frac{g}{c_p}.$$  \hspace{1cm} (24)

Combining (23) with (21) we get

$$p_h = p_0 \left( 1 - \frac{\Gamma d h}{T_0} \right)^{1/\kappa}.$$  \hspace{1cm} (25)

Substituting (25) in (22) we arrive at

$$U_i = \frac{c_v}{g(1 + \kappa)} p_0 T_0 \left( 1 + \left( 1 - \frac{\Gamma d h_i}{T_0} \right)^{1/\kappa} \frac{\Delta \theta}{T_0} \right).$$  \hspace{1cm} (26)

Using (20):

$$E_i = U_i + P_i = \frac{c_p}{g(1 + \kappa)} p_0 T_0 \left( 1 + \left( 1 - \frac{\Gamma d h_i}{T_0} \right)^{1/\kappa} \frac{\Delta \theta}{T_0} \right).$$  \hspace{1cm} (27)

In (25) through (27) $h_i$ is defined as the height of the lower adiabatic layer at the initial time $t = 0$. At the final time $t = \Delta t$, $h = h_f$ we can now compute $\Delta E = E_f - E_i$:

$$\Delta E = \frac{c_p}{g(1 + \kappa)} p_0 T_0 \left( \left( 1 + \left( 1 - \frac{\Gamma d h_f}{T_0} \right)^{1/\kappa} \frac{\Delta \theta}{T_0} \right) \right.$$  

$$\left. - \left( 1 + \left( 1 - \frac{\Gamma d h_i}{T_0} \right)^{1/\kappa} \frac{\Delta \theta}{T_0} \right) \right).$$  \hspace{1cm} (28)

The potential temperature jump $\Delta \theta$ is not changed under adiabatic lifting or lowering of the layer. Using the fact that $h_f = h_i + \delta h$, and the expansion to second order:

$$(1 + x)^n = 1 + \alpha x + \frac{\alpha - 1}{2} x^2$$  \hspace{1cm} (29)

with $x = -\frac{\Gamma d h}{T_0}$, $\alpha = \frac{1}{\kappa}$; we get

$$\Delta E = \frac{c_p}{g(1 + \kappa)} p_0 T_0 \left[ \left( 1 - \frac{\Gamma d}{\kappa T_0} \right) \delta h \right.$$  

$$\left. + \frac{1}{2} \left( \frac{1}{\kappa} - 1 \right) \frac{\Gamma d^2}{T_0^2} (2h \delta h + \delta h^2) \right] \frac{\Delta \theta}{T_0}.$$  \hspace{1cm} (30)

As in Fig. 1, $h_f$ is a function of distance $x$, so $\delta h = \delta h(x)$.

Consider as a simple functional representation of $\delta h(x)$ the sine as in equation (8). Integrating over $x$ and averaging over $\Delta x$ we get

$$\Delta E = \frac{1}{\Delta x} \int \limits_0^{\Delta x} \Delta Edx$$  

$$= \frac{c_p}{g(1 + \kappa)} p_0 T_0 \left[ \left( 1 - \frac{\Gamma d}{\kappa T_0} \right) \frac{\Delta \theta}{T_0} \right].$$  \hspace{1cm} (31)

The maximum value of $\Delta E$ is obtained if all kinetic energy in the initial state is transformed into thermal and potential energy in the final state, or

$$\dot{K}_i = \frac{1}{\Delta x} \int \limits_0^{\Delta x} K_i dx = \Delta E, \quad K_f = 0.$$  \hspace{1cm} (32)

We can then approximate $K_i$ by

$$\dot{K}_i = \frac{1}{\Delta x} \left[ \int \limits_0^{\Delta x} \frac{w^2}{g} dx \right] \approx \frac{1}{2} \int \limits_0^{\Delta x} \frac{w^2}{g} \left( \frac{p_0 - p_h}{g} \right).$$  \hspace{1cm} (33)

There we have assumed that $w^2$ can be represented by some average. We now use (30) through (33) and a first order approximation of the power in (25) to arrive at the following balance:

$$\left( \frac{\Delta h}{h} \right)^2 = \frac{1}{2} \left( \frac{\Delta h}{h} \right) \frac{T_0 w^2}{g \Delta h h}.$$  \hspace{1cm} (34)

For purely convective boundary layers it can be assumed that $w^2 = a_1 w^* \frac{w^*}{d}$ with $a_1$ of order one; hence, we arrive once again at Eq. (15).

### c. Measurements

We replotted the laboratory and atmospheric lidar data of Deardorff et al. (1980) and Boers and Eloranta (1986) on Fig. 2 to present the Ri $^{-1/2}$ dependence. Based on all data, we find $\alpha = 1.23$. The individual datasets show different slopes. The lidar data has a best fit of $\Delta h/h = 1.38 \text{ Ri}^{-0.53}$, which is very close to the expected $-\frac{1}{2}$ fit. The laboratory data has as a best fit $\Delta h/h = 0.54 \text{ Ri}^{-0.24}$. In considering the significance of this fit, an important difference between the two data sources should be considered. We define a relative stratification parameter $S$ as

$$S = \Gamma \Delta h / \Delta \theta,$$  \hspace{1cm} (35)

where $\Gamma$ is the lapse rate of temperature in the stable fluid above the mixed layer. A comparison of the two data sources shows that $S$ is much larger for the lab data than for the lidar data. Using $S = 1$ as the criterion for large ($S > 1$) versus small ($S < 1$) relative stratification, 29 of 53 laboratory data points can be considered taken with large $S$. Only 3 of 50 lidar data points fall in this category. Although the temperature jump within the entrainment zone is dominating in restricting the overshoot of upward moving convective cells, the most vigorous parcels will encounter the damping buoyancy force of the upper stratified layer. Such restriction would limit the effective depth of the entrain-
ment zone. Refitting the laboratory data using only points with \( S < 1 \) does improve the fit (\( \sim \text{Ri}^{-0.30} \)) but not enough to account for the difference. Although the significance of the fit through such a small number of points is reduced, the results suggest that \( \Gamma \) might be a significant parameter in developing more sophisticated parameterization of the entrainment zone depth.

Because the lidar data was taken under conditions of widely varying lapse rates and mixed layer depths, and \( S \) remained small, the success of this theory in explaining the atmospheric data suggests that entrainment, which is neglected in our theory, is of secondary importance in determining the depth of the entrainment zone.

4. Conclusion

We have shown that the normalized entrainment zone depth depends on the inverse Richardson number as its square root. This parameterization, derived from total energy conservation, results in a \( \text{Ri}^{-1/2} \) law. This theory explains the Ri dependence of the lidar data completely, but still does not adequately predict the lab data, although it improves considerably over parcel theory.

An important difference between the two data sources, namely, the much larger stratification parameter \( S \) for the lab data was shown to influence the Ri dependence but not enough to account for the difference with the theory. The excellent agreement of the theory with the atmospheric data suggests that potentially complicating effects of entrainment may not be significant. However, more data is needed to verify the results under a wide variety of atmospheric conditions.

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**REFERENCES**


