A Three-Dimensional Variational Approach for Deriving the Thermodynamic Structure Using Doppler Wind Observations—An Application to a Subtropical Squall Line

YU-CHIENG LIOU, TAI-CHI CHEN WANG, AND KAO-SHEN CHUNG

Department of Atmospheric Sciences, National Central University, Chung-Li, Taiwan

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ABSTRACT

A newly designed retrieval scheme based on three-dimensional variational analysis is used to extract the thermodynamic field of a weather system from Doppler wind measurements. As compared with the traditional retrieval method, with this formulation the proposed scheme is able to find a set of optimal solutions for the pressure and buoyancy perturbations that, in the least squares sense, will simultaneously satisfy three momentum equations and a simplified thermodynamic equation. Therefore, the products of the retrieval are the complete thermodynamic fields in three dimensions. To test the performance of this method in real cases, it is applied to the analysis of a subtropical squall line. The required wind data were synthesized by two C-band Doppler radars during the 1987 Taiwan Area Mesoscale Experiment (TAMEX). The emphasis of this study is devoted to an examination of the validity of the retrieved thermodynamic structure, especially along the vertical direction. The results indicate that the distributions of the retrieved thermodynamic parameters are consistent with the kinematic structure and can be reasonably explained by the conceptual model of a squall line. Evidence is collected that strongly supports the validity of the derived thermodynamic structure. Thus, the applicability of this new retrieval scheme is demonstrated.

1. Introduction

Doppler radar can measure the radial wind with a high spatial and temporal resolution. Using the multiple-Doppler synthesis method (e.g., Armijo 1969; Ray et al. 1980), the complete three-dimensional structure of the wind field can be constructed. However, the structure of the thermodynamic variables, such as the temperature and buoyancy, as well as the pressure fluctuations, cannot be directly detected by radar. Therefore, the development of the so-called thermodynamic retrieval technique, whereby the thermodynamic perturbations can be inferred from the wind measurements, has drawn a lot of attention in the radar meteorology community over the past two decades. The pioneering work of Gal-Chen (1978, hereinafter GC78) demonstrated that the pressure and potential temperature perturbations can be extracted from the wind observations. The most important feature of GC78 is that the boundary conditions for solving a Poisson equation for the pressure perturbations are not from any artificial assumptions. Instead, they can be obtained directly from Doppler wind observations. This method turned out to be particularly useful, and it was later widely adopted by many researchers to study different weather phenomena (Hane et al. 1981; Gal-Chen and Kropfli 1984; Hane and Ray 1985; Lin et al. 1986; Parsons et al. 1987, among others).

Despite encouraging results, it should be pointed out that in GC78 only deviations (relative to the horizontal average) of the thermodynamic perturbations (with respect to a basic state) are deduced for each altitude. The absolute thermodynamic perturbations cannot be retrieved because their horizontal averages are unknown quantities. As a result, the ambiguity in the vertical structure of the thermodynamic field itself is difficult to avoid, unless one has, for the thermodynamic parameters, at least one single point of independent field measurement for each layer.

In an attempt to overcome this problem, Brandes (1984) formulated a two- and three-dimensional elliptic equation for the buoyancy and pressure perturbations, respectively. However, to solve the equations, the required lateral boundary conditions are based on an assumption that is appropriate only when the weather system is well within the retrieval domain. A major step forward was taken by Roux (1985, 1988). His approach could solve for pressure and potential temperature perturbations uniquely up to a volume-wide constant. To deduce this unknown constant, only a single point of independent pressure and temperature observations somewhere in the retrieval domain is needed. Using Roux’s results as boundary conditions, Sun and Roux

Corresponding author address: Dr. Yu-Chieng Liou, Department of Atmospheric Sciences, National Central University, Chung-Li, 320, Taiwan.
E-mail: tyliou@atm.ncu.edu.tw

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(1988) analyzed the trailing anvil clouds of squall lines over a two-dimensional vertical cross section. Hauser et al. (1988) incorporated Roux’s variational procedure into a retrieval system for deducing both the thermodynamic and microphysical variables. Roux and Sun (1990) proposed further modifications by taking into account a simplified form of the thermodynamic equation everywhere in the domain, so that the temperature gradients are constrained along both horizontal and vertical directions. Sun and Houze (1992) investigated the validity of this modification using data from a simulated two-dimensional squall line with stratiform precipitation. Additional improvements in Roux’s retrieval technique can be found in Roux et al. (1993). Most recently, Yu et al. (2001) utilized Roux’s method to study the thermodynamic structure of a subtropical, orographically influenced convective line. Cai and Waki-moto (2001) also applied Roux’s algorithm to airborne Doppler radar data to analyze the propagation of a supercell thunderstorm.

Sun and Crook (1996) demonstrated the usage of the adjoint method in thermodynamic retrieval. The major advantage of the four-dimensional variational data assimilation (4DVAR) technique over the traditional GC78 method is that the absolute temperature perturbations can be deduced without any independent measurements. Liou (2001, hereinafter L01) also formulated a retrieval technique based on three-dimensional variational analysis. This method involved implementing the horizontal and vertical equations of motion and a simplified thermodynamic equation into a single cost function. Through the variational approach, a set of optimally determined temperature and pressure fluctuations that minimizes this cost function can be deduced simultaneously. When tested against model-generated data, very satisfactory results were obtained.

In this research, the L01 method is further extended, and its applicability to a real case study is evaluated. The wind data were collected by two C-band Doppler radars operating during the 1987 Taiwan Area Meso-scale Experiment (TAMEX). The manuscript is organized as follows. In the next section, the proposed modified L01 method is briefly introduced. Section 3 describes the Doppler radar datasets needed for retrievals. Discussions of the retrieved results are presented in sections 4 and 5, followed by the conclusions in section 6.

2. Methodology

The basic equations of motion employed for this research are different from those used in L01, because moisture has to be considered. Similar to Klemp and Wilhelmson (1978), the equations may be written in a fixed frame of reference as follows:

\[
\frac{1}{\theta_{0}} \frac{\partial u}{\partial t} + \mathbf{V} \cdot \nabla u - fu + \text{turb}(u) = -\frac{\partial \pi'}{\partial y} = -G, \tag{1}
\]

and

\[
\frac{1}{\theta_{0}} \frac{\partial w}{\partial t} + \mathbf{V} \cdot \nabla w + \text{turb}(w) + \theta' = \frac{\partial \pi'}{\partial z} + g \frac{\theta'}{\theta_{0}} = -H. \tag{3}
\]

Note that subscript “0” represents a horizontally homogeneous basic state, from which the nonhydrostatic perturbations are expressed by variables with a single prime. In (1)–(3), the wind vector \( \mathbf{V} \) consists of three Cartesian components \((u, v, w)\), \( f \) refers to the Coriolis parameter, \( g \) stands for the gravity, and \( \text{turb}(\cdot) \) denotes a subgrid-scale turbulence parameterization operator. In the real case application discussed later in section 4, these turbulence terms are parameterized using a simple first-order closure scheme. A normalized pressure called the Exner function \( \pi \) is employed. It is defined as

\[
\pi = C_{p} \left( \frac{P}{P_{0}} \right)^{R/\gamma}, \tag{4}
\]

in which \( P \) is the pressure, \( P_{0} = 100 \text{ kPa} \), \( R \) is the gas constant, and \( C_{p} \) is the specific heat capacity at a constant pressure. The potential temperature, virtual potential temperature, and virtual cloud temperature perturbation (Roux 1985) are represented by \( \theta, \theta_{c}, \) and \( \theta' \), respectively. The latter two quantities are defined by

\[
\theta_{v} = \theta(1 + 0.61q_{c}), \tag{5}
\]

where \( q_{c} \) stands for the perturbation of the water vapor mixing ratio, and \( q_{c} \) is the cloud water mixing ratio. The virtual cloud temperature perturbation \( \theta' \) is treated as a retrieved parameter in the proposed formulation.

In (3), \( q_{c} \) refers to the rainwater mixing ratio and is considered to be an observable quantity. Assuming a Marshall–Palmer type of drop size distribution, \( q_{c} \) can be estimated using the radar reflectivity data by

\[
\frac{35}{2} \log(\rho q_{c}) + 95.6 = \eta, \tag{7}
\]

where \( \rho \) is the air density, and \( \eta \) is the radar reflectivity factor (dBZ). Using presquall line environmental sounding data to determine the basic state, the values of \( F, G, \) and \( H \) can be obtained once the three-dimensional air motion is known through a procedure such as multiple-Doppler radar syntheses.

In addition to the momentum equations, a simplified form of thermodynamic equation is also employed for the virtual cloud temperature perturbation \( \theta' \), to provide information on temperature gradients. This equation is expressed as

\[
\frac{\partial \theta'}{\partial x} + \frac{\partial \theta'}{\partial y} + \frac{\partial \theta'}{\partial z} + \frac{d \theta}{dz} = S. \tag{8}
\]
where $S$ stands for the total contributions from the temporal variation, the diffusion, and the source/sink of $\theta'$. According to the experiments in L01, the term $S$ can also be reproduced along with the temperature and pressure. Therefore, in this study, it is treated as a retrieved parameter. Gal-Chen (1982) suggested using the radar observed radial velocity or reflectivity fields to define a moving frame of reference on which the global temporal evolution can be minimized. Note that this concept can be adopted to reduce the errors in estimating the local time derivatives of winds or radar reflectivities, but is not applicable to (8), because in this case the temperature is not a directly observable quantity.

From Eqs. (1)–(3) and (8), a cost function can be formulated:

$$J = \frac{1}{2} \int \int \int (\alpha_i P_i^2) \, dx \, dy \, dz. \quad (9)$$

In (9), a total of 10 constraints are included. They are

$$P_1 = \left( \frac{\partial \pi'}{\partial x} - F \right),$$

$$P_2 = \left( \frac{\partial \pi'}{\partial y} - G \right),$$

$$P_3 = \left( \frac{\partial \pi'}{\partial z} - g \frac{\theta'_e}{\theta_e} - H \right),$$

$$P_4 = \left( \frac{\partial \theta'_e}{\partial x} + \frac{\partial \theta'_e}{\partial y} + \frac{\partial \theta'_e}{\partial z} + w \frac{\partial \theta'_e}{\partial z} - S \right),$$

$$P_5 = \left( \frac{\partial \theta'_e}{\partial z} \right),$$

$$P_6 = [(\pi' - \langle \pi' \rangle) - (\pi' - \langle \pi' \rangle)_{GC}],$$

$$P_7 = [(\theta'_e - \langle \theta'_e \rangle) - (\theta'_e - \langle \theta'_e \rangle)_{GC}],$$

$$P_8 = \frac{\partial^2 \pi'}{\partial x^2} + \frac{\partial^2 \pi'}{\partial y^2} + \frac{\partial^2 \pi'}{\partial z^2},$$

$$P_9 = \frac{\partial^2 \theta'_e}{\partial x^2} + \frac{\partial^2 \theta'_e}{\partial y^2} + \frac{\partial^2 \theta'_e}{\partial z^2},$$

$$P_{10} = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}. \quad (10)$$

In (9), the first two terms ($P_1^2$, $P_2^2$) measure the distance between the retrieved horizontal pressure gradients and the calculated $F$ and $G$ (from the wind observations), respectively. The third term $P_3^2$ denotes the difference in the retrieved vertical pressure gradient, the buoyancy (without the contribution from rainwater loading), and the quantity $H$. The fourth term $P_4^2$ originates from the thermodynamic equation, and offers a constraint for the temperature’s spatial distributions. Similar to Sun and Crook (1996), $P_5^2$ is imposed to restrict the growth of “spurious” thermal instability. Note that $P_6^2$ is added as a “weak” constraint, so that with adequate weighting, the “true” thermally unstable region can be kept intact (Liu 2001). Compared to L01, the newly implemented terms are $P_6$ and $P_7$. In these two constraints, the symbol $\langle \rangle$ represents an average over the horizontal domain. It is known that the deviations (from the horizontal mean) of the thermodynamic perturbations can be uniquely retrieved by GC78. Because of the use of different numerical methods to minimize the cost function, the deviations of the perturbations deduced by our method are not exactly the same as those of GC78 (i.e., the quantity with subscript GC). Therefore, constraints $P_6$ and $P_7$ are included in the formulation to minimize this difference. By explicitly adding these two constraints, the discrepancies between two deviations obtained separately by the GC78 method and our scheme can be effectively reduced. Last, constraints $P_8^2$ through $P_{10}^2$ are added as low-pass filters to suppress noise.

The weighting coefficients $\alpha_i (i = 1–10)$ in (9) need to be carefully specified. A detailed description about the estimation of these quantities can also be found in L01, except for $\alpha_6$ and $\alpha_7$. In this research, they are inversely proportional to the volume-averaged square of the buoyancy/pressure deviations inferred from GC78, or

$$\alpha_{6,7} = \left( \frac{1}{\Omega} \int \int \int [(\phi - \langle \phi \rangle)_{GC}^2] \, dx \, dy \, dz \right)^{-1}, \quad (11)$$

where $\phi$ can be either $\theta'$ or $\pi'$, and $\Omega$ is the volume of the entire retrieval domain.

The variational analysis is applied to minimize (9), which requires knowledge of the cost function gradients with respect to $\pi'$, $\theta'_e$, and $S$. Readers can refer to the appendix for details about the derivation and the minimization procedure. The resulting fields provide the structure of the virtual cloud temperature perturbations, as well as the pressure fluctuations in three dimensions.

In L01, the wind data of a collapsing dry cold pool simulated by a numerical model are used to recover the thermodynamic fields, which are then compared against their model-generated counterparts, so that the accuracy of the retrieval scheme can be evaluated. In this study, the modified variational formulation introduced in (9) and (10) is also tested, using the same datasets, before being applied to real cases. Figure 1 shows the true potential temperature perturbations, superimposed by the flow field, of a simulated cold pool after 20 min of numerical integration. The display follows a vertical X–Z cross section passing through the center of the cold pool. Figure 2 depicts the retrieved potential temperature perturbation field along the same cross section, using the modified L01 method and the wind fields. The comparison does reveal further improvements over the original L01 formulation. The three-dimensional root-mean-square (rms) error for the potential temperature perturbation is reduced from over 0.02° in L01 to 0.01°.
or 4.25% if measured by the relative rms difference (normalized by the rms magnitude of the true potential temperature perturbation). In addition, if one compares the retrieved $\theta'$ deviation (i.e., $\theta' - \langle \theta' \rangle$) for each horizontal plane, then the relative rms errors in $L_0$ are generally over 10%, while the modified formulation in this study can reduce the errors to less than 5%. Apparently, this improvement can be attributed to the implement of constraints $P_x$ and $P_\theta$.

Although the results are produced using idealized datasets, in which the observational errors can be ignored, this experiment does demonstrate the capability of our formulation to resolve the vertical structure of the thermodynamic fields. In the following, the applicability of this retrieval method to a real case study is investigated.

3. Dual-Doppler radar data of TAMEX IOP 2

The TAMEX Intensive Observation Period (IOP) 2 took place on 16–17 May 1987. This particular case represented a long-lasting subtropical squall line. During IOP 2, two C-band Doppler radars [CP-4 and Tropical Ocean and Global Atmosphere (TOGA)], provided by the National Center for Atmospheric Research (NCAR) and the National Oceanic and Atmospheric Administration (NOAA), respectively were operated simultaneously in the early morning of 17 May. Only the northern portion of the squall line was observed while it was still over the ocean. Figure 3 shows the dual-Doppler analysis domain, which covers an area of 45 km $\times$ 50 km. The coordinates of CP-4 and TOGA in Fig. 3 are (0.0, 0.0, 0.009) km and $(-14.66, -41.25, 0.206)$ km, respectively. The shaded contour lines denote the radar reflectivity at 2 km above ground level (AGL). A north–south-oriented high-reflectivity line is easily identifiable. To the east of the analysis domain the reflectivity data were not available. This is because both radars were performing intensive sector scans toward the west during the observation period. According to Wang et al. (1990), the squall line moved from 250°
at a nearly constant velocity 16.5 m s\(^{-1}\). The line AB drawn in Fig. 3, which represents the central portion of the analysis domain and passes a reflectivity maximum zone, is oriented so that it is parallel to the system’s motion. In section 4, the vertical cross section for which the validity of the retrieved thermodynamic structure is discussed will coincide with line AB.

The dual-Doppler syntheses were performed using the Custom Editing and Display of Reduced Information in Cartesian Space (CEDRIC) software developed by NCAR (Mohr et al. 1986) to generate the complete three-dimensional wind fields at 1640:00, 1643:00, and 1646:00 coordinated universal time (UTC) on 16 May. The local time is 8 h after UTC. The horizontal and vertical spatial resolutions of the wind measurements are 1.0 and 0.5 km, respectively. The Doppler wind data are then inserted into the retrieval scheme so that the thermodynamic variables can be inferred using the variational approach introduced in section 2. Figure 4 depicts the evolution of the cost function \(J\) defined in (9) during the minimization procedure. It declines to approximately 19\% of the initial value after the first 50 iterations. The resulting thermodynamic fields are subjected to a three-dimensional momentum checking \((E_r)\) to determine the level of internal consistency. The parameter, \(E_r\), is defined by

\[
E_r = \frac{\iiint \left[ \frac{\partial \pi'}{\partial x} - F \right]^2 + \left( \frac{\partial \pi'}{\partial y} - G \right)^2 + \left( \frac{\partial \pi'}{\partial z} - g \frac{\partial \theta'}{\partial \theta_0} - H \right)^2 \, dx \, dy \, dz}{\iiint \left( F^2 + G^2 + H^2 \right) \, dx \, dy \, dz}. \tag{12}
\]

In this study the value of \(E_r\) is about 0.23, which indicates a satisfactory fit between the retrieved and the kinematic fields (Hane and Ray 1985).

4. Discussion of the results

Lin et al. (1990) have investigated the thermodynamic structure of this particular case (IOP 2) in detail. However, their analysis mostly covered the horizontal domain at different altitudes. Apparently this was because they adopted the traditional GC78 method of thermodynamic retrieval in their study. An intercomparison of \(\theta' - \langle \theta' \rangle\) at each layer recovered, respectively, by the proposed modified L01 and the GC78 method is conducted, because this is the quantity that can be uniquely determined by GC78. The comparison reveals an extremely high degree of similarity, with relative rms differences ranging from 0.08\% to 2.95\% at different heights. This result demonstrates that our modified method L01 can produce unambiguous horizontal thermodynamic structures, and the quantitative accuracy is comparable to that obtained by the GC78 method. Therefore, for brevity, a description of the horizontal structure of the pressure and buoyancy field will not be repeated here. Instead, we will focus on discussing the vertical distribution of the retrieved thermodynamic parameters.

The results presented in the following are along line AB, as designated in Fig. 3. The field presented by each vertical panel is a composite result generated by averaging the retrieved variables over multiple layers. These layers are centered by line AB, and extend over a distance about 10 km horizontally. In addition, on each vertical cross section, the right portion of the domain is defined as the front, toward which the squall line is moving. Figure 5 illustrates the vertical velocity \((w)\) field, superimposed by radar reflectivity. On the right (front) side of the system \((X > 25 \text{ km})\), one finds strong updrafts/downdrafts. By contrast, the left portion of the domain \((X < 20 \text{ km})\) is the stratiform region, where the vertical motion is much weaker \((|w| < 1.0 \text{ m s}^{-1})\). Note that the convective available potential energy
Fig. 5. The vertical distribution of vertical velocity \( (w) \) along line AB (as designated in Fig. 3), superimposed by the radar reflectivity factor \( (\eta) \) field. The contour interval for \( w \) is 0.5 m s\(^{-1}\). The field of \( \eta \) is characterized into three categories: 20 \( < \eta < 30 \) dBZ (light shaded), 30 \( < \eta < 40 \) dBZ (moderate shaded), and \( \eta > 40 \) dBZ (dark shaded), respectively. The right (left) portion of the diagram is defined as the front (rear) of the squall line.

Fig. 6. Same as in Fig. 5, but for the system-relative wind field. The FTR and RTF flows are evident. The flow pattern near the location \((X = 32 \text{ km}, \ Z = 4.0 \text{ km})\) implies a horizontal vortex with vorticity pointing toward the negative Y direction, as denoted by a circle.

CAPE) of this case is 1369 m\(^2\) s\(^{-2}\) (Wang et al. 1990). Based on this value, the maximum vertical velocity \( (w_{max}) \) that can be reached by an air parcel will be approximately \( w_{max} = \sqrt{2 \times \text{CAPE}} \approx 52.3 \text{ m s}^{-1} \). On the other hand, the strongest updraft in the dual-Doppler analysis domain is about 9.7 m s\(^{-1}\) (not shown in the figure). The authors believe that this discrepancy can be attributed mainly to (a) the neglect of vertical pressure gradient and water/ice loading, as well as the entrainment of unsaturated air from the environment, while estimating the CAPE (These factors may lead to a serious overestimation of the vertical velocity); and (b) the sounding data, which represents a pre-squall-line atmosphere (The time of the sounding measurements was 4 h ahead of the radar observations.) In addition, the location where the sounding was released was outside of the dual-Doppler analysis domain.

Figure 5 also shows that the maximum reflectivity is about 44 dBZ, which corresponds to the convective region of the squall line. Near the left boundary of the domain, one finds a sharp increase \((-10 \text{ dBZ})\) of the radar reflectivity from the height of \( Z = 5.5 \) to 4.0 km, with the maximum reflectivity reaching 33 dBZ. This is believed to be the place where the bright band occurs. Figure 6 displays the storm-relative flow pattern. The features of the rear-to-front (RTF) and front-to-rear (FTR) flow are evident. Note that the lowest level of the retrieval domain starts above 1 km, therefore, the returning FTR flow very near the surface in the stratiform zone is not resolved. From the above descriptions, it is realized that the kinematic character of this squall line seems to be rather classical.

The retrieved pressure perturbation \( (\pi') \) field and buoyancy perturbation \( (B') \) field are shown in Figs. 7 and 8, respectively. Note that \( B' \) is computed using the retrieved virtual cloud temperature perturbation as well as the rainwater content \( (q_r) \) derived from radar reflectivity data [see (7)]. That is,

\[
B' = g \left( \frac{\theta'_{v}}{\theta_{h}} - q_r \right).
\] (13)

In Fig. 7, the pressures are labeled as “H” and “L” for convenience of discussion. At low levels, a high pressure region \( (H_1) \) can be identified at the front of the squall line \( (X \approx 40 \text{ km}) \). By comparing with Fig. 6, one finds that at this location, the high pressure should be induced by an airmass accumulation due to the convergence of the RTF and FTR flows, as explained by Droegemeier and Wilhelmson (1987). A second high pressure zone \( (H_2) \) is found in the stratiform region near the surface. This pressure maximum can be attributed to the cold dry RTF flow, as well as to the effect of...
precipitation cooling, which make the air in this area heavier.

Two low pressure centers (L1 and L2) are found in the convective region. The locations of L1 (X = 30.0 km, Z = 6.0 km) and L2 (X = 32.0 km, Z = 4.0 km) are within the region of positive B’ (see Fig. 8), therefore, their formation should be relevant to the hydrostatic effects (LeMone et al. 1984). However, a visual comparison of the pressure and the flow field indicates that L2 seems to be associated with a horizontal vortex center (ζ = ∂u/∂z − ∂w/∂x, see the circle drawn in Fig. 6), where the vorticity is pointed toward the negative Y direction. This phenomenon can also be illustrated by plotting a horizontal line at Z = 4.0 km, along which the variables π’, w, and B’ are displayed, as shown in Fig. 9. It can be seen from Figs. 9a and 9b that exactly at X = 32.0 km, π’ reaches a local minimum, while the vertical velocity w changes from negative at X < 32.0 km, to positive at X > 32.0 km. This is a good indication of the existence of a horizontal vortex. Moreover, the horizontal buoyancy gradient illustrated by Fig. 9c is positive (e.g., ∂B’/∂x > 0), which represents a favorable condition for the maintenance of the negative horizontal vorticity, because the simplified vorticity equation gives dζ/dt = −∂B’/∂x < 0. Thus, it is believed that dynamic effects should play an important role in causing this pressure minimum (L2).

To further clarify the causes of the pressure pattern, the divergence operator (∇·) is applied to the equations of motion (1)–(3). Assuming an anelastic-type continuity equation (∇·ρ0V = 0), one arrives at a Poisson equation for the pressure perturbation:

∇²π’ = -(T_{b1} + T_{b2} + T_{b1} + T_{b2}),

(14)

where ∇² is a three-dimensional Laplacian operator, and

\[ T_{b1} = -\frac{1}{\theta_0} \frac{\ln(\rho_0 \theta_0)}{\theta_0} \frac{\partial \pi'}{\partial z}, \]

(15)

\[ T_{b2} = \frac{1}{\theta_0} \left[ -\frac{1}{\rho_0} \frac{\partial (\rho_0 B')}{\partial z} \right]. \]

(16)
In (14), $T_{B1}$ and $T_{D2}$ are originated from the buoyancy forcing terms, while $T_{D1}$ and $T_{D2}$ represent the dynamic forcing terms. Substituting the Doppler radar–observed wind fields and the retrieved thermodynamic parameters into (14), it is realized that $T_{D1}$ is much smaller than the other terms, and, therefore, can be ignored. Assuming a wavelike solution for the pressure perturbations, (14) can be approximated by:

$$\pi' \propto -\nabla^2 \pi' = T_B + T_D,$$

where $T_B$ and $T_D$ stand for the summation of $(T_{B1}, T_{B2})$ and $(T_{D1}, T_{D2})$, respectively. The horizontal distributions of $\pi'$, $T_B$, and $T_D$ are then plotted at different heights to examine the relative importance of the buoyancy and dynamic forcing to $\pi'$. At lower layers, it is found that the behavior of $\pi'$ can be explained mostly by the buoyancy forcing term $(T_B)$ alone. For example, Fig. 10c depicts that at $Z = 2.5$ km, the curve of $T_B$ varies synchronously with $\pi'$, while $T_D$ is near zero. By contrast, at the level where $L_2$ is located (i.e., $Z = 4.0$ km), one finds that the contribution of $T_D$ to $\pi'$ is small in the stratiform region, but gradually increases toward the convective zone. Near the position of $L_2$ ($X = 32$ km) the magnitude of $T_D$ even exceeds that of $T_B$, as illustrated by Fig. 10b. Figure 10a reveals that at an even higher altitude ($Z = 6.0$ km) where $L_1$ is located, the buoyancy forcing $T_B$ again becomes the major contributor to $\pi'$, although a small contribution from $T_D$ near $X = 32$ km is still identifiable. These results clearly show the role played by the dynamic forcing in terms of determining the low pressure centers $L_1$ and $L_2$.

Figure 7 also indicates that an elongated region of low pressure $L_3$ lies in the stratiform region at a height of about $Z = 5.5$ km. Its formation should be associated with the positively buoyant FTR current aloft (see Figs. 6 and 8). In addition, Fig. 7 reveals that above (below) the 5.5-km height, the vertical pressure gradient force is downward (upward), while Fig. 8 illustrates that the buoyancy is positive (negative).
This pressure and buoyancy pattern is consistent with the observational results of Sun and Roux (1988), and model results of Fovell and Ogura (1988). Recall that in Fig. 5, a dramatic increase of radar reflectivity is identified from Z = 5.5 to 4.0 km in the stratiform region. Therefore, the height Z = 5.5 km should be the layer of 0°C isotherm below which the hydrometers start to melt and form the bright band.

Figure 11 depicts the total forcing along the vertical direction (i.e., vertical pressure gradient force + buoyancy). It can be seen that in the stratiform region the pressure gradient force basically balances the buoyancy, which indicates that the atmosphere is hydrostatic. By contrast, in the convective zone of the squall line, the conditions are favorable for the further development of cells, as indicated by the strong positive forcing located at X ~ 33 and X ~ 40 km. If we compare this with Fig. 8, it is also interesting to find that although the total forcing is positive, the buoyancy near the surface is negative. This implies that the initiation of the upward motion at lower altitudes is not caused by thermal instability. Instead, it is driven mainly by the force of upward pressure gradient produced at the interface of two opposing currents. Roux (1985) also described a similar feature in a West African squall line system. When the air reaches a higher layer (e.g., 3 km), both the force of vertical pressure gradient and the buoyancy become suitable for additional ascending.

5. Other supporting evidence

Because there was no direct observation of the thermodynamic variables within the squall line system during IOP 2, in this section we attempt to collect more evidence so that the validity of the retrievals can be verified at least indirectly. By comparing with previous squall line studies, it is found that the major features exhibited by the derived pressure and buoyancy perturbations shown in Figs. 7 and 8 can also be identified in other research. For example, the thermodynamic structure in the trailing stratiform region is very similar to that found by Sun and Roux (1988, see their Figs. 6e and 6f) in a tropical squall line. The “warm tongue,” which stretches from the middle to lower altitudes in the convective region, also resembles Fovell and Ogura’s simulation result (1988, see their Fig. 25a). Furthermore, numerical studies of this particular case (TAMEX IOP 2) were performed by Tao et al. (1991) and Chen (1991). By making a comparison with their findings, it is interesting to note that the features of two pressure minima (i.e., L1 and L2 in our retrieved pressure field) in the convective region of the squall line also existed in Tao et al. (1991, see their Fig. 6b). Chen’s (1991) model results revealed a similar pressure pattern (see his Fig. 7c) as well.

The observational studies of this case by Wang et al. (1990) and Lin et al. (1990) indicated that this was a long-lasting squall line, with old cells being successively replaced by new cells. Rotunno et al. (1988) suggested that the longevity was the result of a balance between the positive vorticity associated with the environmental low-level wind shear and the negative vorticity generated by the cold pool. According to their analysis, the former can be represented by the speed difference ∆u between the top and bottom of the shear zone, while the latter can be expressed by a parameter C:

\[ C^2 = 2 \int_0^H B' \, dz, \]  

where H is the depth of the cold pool, and C has the units of meters per second. When C is similar to ∆u, the optimal balance is reached.

To examine whether the above criterion is satisfied in our case, the values of ∆u and C are estimated. In previous studies such as Wang et al. (1990), it has been pointed out that a low-level wind shear perpendicular to the squall line, with a magnitude about 4.3×10⁻³ s⁻¹, is present from Z = 0.6 to 3.6 km. Thus, the speed difference ∆u is approximately 4.3×10⁻³ s⁻¹ × 3000 m = 12.9 m s⁻¹. In order to evaluate C, we use the retrieved thermodynamic field and compute the horizontally averaged buoyancy for each height. The average is made only inside the stratiform region (X < 20 km) where the vertical motion is rather weak (|w| < 1.0 m s⁻¹). The resulting \( \langle B' \rangle \) ranges from −4.5×10⁻² to −3.5×10⁻² m s⁻¹ within the height interval from Z = 1.0 to 3.5 km. Substituting the averaged buoyancy \( \langle B' \rangle \) into (20) yields an estimation of \( C = 14.0 \) m s⁻¹, which agrees very well with ∆u. This agreement can be considered as a piece of “indirect” evidence that demonstrates that the retrieved buoyancy is quantitatively reasonable.

The pre-squall-line sounding data reveal that if an undiluted air parcel ascends moist adiabatically from
the surface, then the maximum temperature excess should not be larger than 6°C (see Wang et al. 1990, Fig. 7a). This quantity imposes an upper limit to the retrieved temperature field. An examination of the retrieved virtual cloud temperature perturbation ($\theta_c'$) field shows that the maximum value at each altitude is indeed below 6°C.

Last, using the same Doppler wind observations, the traditional GC78 method is also applied to the recovery of the thermodynamic structure for this case. The retrieved buoyancy perturbation field (actually, the deviation of the buoyancy perturbation from a horizontal mean, or $B' - \langle B' \rangle$) is shown in Fig. 12. It can be found that in the stratiform region near the surface ($Z < 3.0$ km) where the cold pool is located, the air is now positively buoyant. This contradicts the conceptual model of a squall line and is apparently caused by the unknown horizontal average ($\langle B' \rangle$). If the retrieval domain is sufficiently large, and the weather system is well within the boundaries, one can expect to have $\langle B' \rangle \sim 0$, so that the GC78 inferred deviation ($B' - \langle B' \rangle$) approaches to the perturbation itself ($B'$). However, this case study presents an example that shows that, in real applications, the radar often covers only a portion of the weather phenomenon, and, thus, the perturbation’s horizontal average may not be negligible.

6. Conclusions and future work

In this research we propose a different thermodynamic retrieval scheme for extracting three-dimensional pressure and buoyancy field information from Doppler wind observations. The performance of this newly designed retrieval algorithm is tested using field experiment Doppler radar data collected during the 1987 TAMEX IOP 2, which represents a long-lasting subtropical squall line. The emphasis of this study is placed on examining the capability of the retrieval scheme to resolve the vertical structure of the thermodynamic parameters. The validity of the retrieved fields is confirmed (at least indirectly) by the following supporting evidence:

1) The momentum-checking parameter is about 0.23, which indicates a good internal consistence between the kinematic structures and the retrieved thermodynamic variables.
2) The configuration of the deduced pressure and buoyancy perturbations turns out to be classical and can be reasonably explained by an existing conceptual model of a squall line.
3) The major features exhibited by the retrieved thermodynamic fields over the vertical cross section agree well with previous numerical studies of a squall line.
4) The theoretical criterion explaining the longevity of a squall line is satisfied, which implies that the magnitude of the retrieved cold pool is quantitatively correct.
5) The retrieved temperature perturbations are less than the upper limit determined by pre-squall-line sounding data.
6) As compared with the reasonable results produced by our proposed scheme, the traditional GC78 method leads to an erroneous interpretation of the vertical thermodynamic structure. This is apparently because the latter is not able to recover the horizontal average of the pressure and buoyancy perturbations.

Based on the above analyses, it is believed that the applicability of this proposed method for recovering three-dimensional thermodynamic structures from wind observations to real case studies has been demonstrated.

When a weather system reaches the coast of Taiwan, its interaction with the island’s terrain (the maximum height reaches about 4.0 km) becomes significant. Therefore, future work will extend the current retrieval method to incorporate generalized terrain-following coordinates so that the orographic effects can be taken into account and studied.

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APPENDIX

The Derivation of the Gradients of the Cost Function with Respect to the Unknowns

In this appendix the gradients of the cost function with respect to the unknown variables (e.g., $\pi'$, $\theta_c'$, and $S$) are derived. Starting from (9) and (10), and neglecting the smoothing constraints ($P_s - P_{sw}$) for simplicity, the variation of the cost function $J$ can be written as

\[
\frac{\partial J}{\partial \pi'} = -2 \sum_{k} \sum_{n} w_{kn} \left( \frac{\partial \pi'}{\partial S_k} \right) \left( \frac{\partial S_k}{\partial \pi'} \right) + 2 \sum_{k} \left( \frac{\partial \pi'}{\partial S_k} \right) \left( \frac{\partial S_k}{\partial \pi'} \right) - 2 \sum_{k} \left( \frac{\partial \pi'}{\partial S_k} \right) \left( \frac{\partial S_k}{\partial \pi'} \right)
\]

\[
\frac{\partial J}{\partial \theta_c'} = -2 \sum_{k} \sum_{n} w_{kn} \left( \frac{\partial \theta_c'}{\partial S_k} \right) \left( \frac{\partial S_k}{\partial \theta_c'} \right) + 2 \sum_{k} \left( \frac{\partial \theta_c'}{\partial S_k} \right) \left( \frac{\partial S_k}{\partial \theta_c'} \right) - 2 \sum_{k} \left( \frac{\partial \theta_c'}{\partial S_k} \right) \left( \frac{\partial S_k}{\partial \theta_c'} \right)
\]

\[
\frac{\partial J}{\partial S} = -2 \sum_{k} \sum_{n} w_{kn} \left( \frac{\partial S}{\partial S_k} \right) \left( \frac{\partial S_k}{\partial S} \right) + 2 \sum_{k} \left( \frac{\partial S}{\partial S_k} \right) \left( \frac{\partial S_k}{\partial S} \right) - 2 \sum_{k} \left( \frac{\partial S}{\partial S_k} \right) \left( \frac{\partial S_k}{\partial S} \right)
\]
where \( X_{left}, X_{right}, Y_{left}, Y_{right}, Z_{bottom}, \) and \( Z_{top} \) represent the boundary surfaces along the \( x, y, \) and \( z \) directions, respectively. Since a small change of any unknown variable at any grid point contributes to the total variation of \( J, \) \( \Delta J \) can also be written by

\[
\Delta J = \iint \left( \frac{\partial J}{\partial \pi} \delta \pi' + \frac{\partial J}{\partial \theta'} \delta \theta' + \frac{\partial J}{\partial S} \delta S \right) \, dx \, dy, \tag{A2}
\]

Equation (A2) actually contains surface integration over the boundaries and a three-dimensional integration over the interior grid points. By comparing (A1) and (A2), the following expressions for the gradients can be obtained.

For interior grid points,

\[
\frac{\partial J}{\partial \pi'} = - \left[ \frac{\partial (\alpha_i P)}{\partial x} + \frac{\partial (\alpha_j P)}{\partial y} + \frac{\partial (\alpha_k P)}{\partial z} - \alpha_i P \right],
\]

\[
\frac{\partial J}{\partial \theta'} = - \frac{g \alpha_i P}{\theta_{0,\theta_{0}}} - \frac{\partial (\alpha_i P)}{\partial y} - \frac{\partial (\alpha_j P)}{\partial y} - \frac{\partial (\alpha_k P)}{\partial z} + \alpha_i P \right), \quad \text{and}
\]

\[
\frac{\partial J}{\partial S} = - \alpha_i P. \tag{A3}
\]

For grid points at the boundaries along the \( x \) direction,

\[
\frac{\partial J}{\partial \pi'} = \begin{cases} - \alpha_i P_1 & \text{at } x = X_{left}, \\
\alpha_i P_1 & \text{at } x = X_{right}, \end{cases}
\]

\[
\frac{\partial J}{\partial \theta'} = \begin{cases} - u \alpha_i P_4 & \text{at } x = X_{left}, \\
u \alpha_i P_4 & \text{at } x = X_{right}, \end{cases}
\]

\[
\frac{\partial J}{\partial S} = 0. \tag{A4}
\]

For grid points at the boundaries along the \( y \) direction,

\[
\frac{\partial J}{\partial \pi'} = \begin{cases} - \alpha_i P_2 & \text{at } y = Y_{left}, \\
\alpha_i P_2 & \text{at } y = Y_{right}, \end{cases}
\]

\[
\frac{\partial J}{\partial \theta'} = \begin{cases} - u \alpha_i P_4 & \text{at } y = Y_{left}, \\
u \alpha_i P_4 & \text{at } y = Y_{right}, \end{cases}
\]

\[
\frac{\partial J}{\partial S} = 0. \tag{A5}
\]

For grid points at the boundaries along the \( z \) direction,

\[
\frac{\partial J}{\partial \pi'} = \begin{cases} - \alpha_i P_3 & \text{at } z = Z_{bottom}, \\
\alpha_i P_3 & \text{at } z = Z_{top}, \end{cases}
\]

\[
\frac{\partial J}{\partial \theta'} = \begin{cases} - u \alpha_i P_4 - \alpha_i P_5 & \text{at } z = Z_{bottom}, \\
u \alpha_i P_4 + \alpha_i P_5 & \text{at } z = Z_{top}, \end{cases}
\]

\[
\frac{\partial J}{\partial S} = 0. \tag{A6}
\]

Using the same procedure, the contributions of the smoothing constraints \( (P_1 - P_{10}) \) to the cost function gradients can be added. For example, for the interior points, we have

\[
\frac{\partial J}{\partial A} = \sum_{i=1}^{3} \left[ \frac{\partial^2 A}{\partial x^i} \right] \frac{\alpha \partial^2 A}{\partial x^i} \tag{A7}
\]

where \( x_i \) \((i = 1\text{--}3)\) stands for \( x, y, \) and \( z, \) respectively. Variable \( A \) can be any of the unknown parameters (i.e., \( \pi', \theta', \) and \( S \)), and their corresponding weighting coefficients are represented by \( \alpha. \)

Starting from an initial guess field and using the gradients computed from (A3)--(A7), a new estimate of the solution is achieved. This process is iterated until an optimal solution is obtained. This solution is called optimal in the sense that it can make the cost function \( J \) reach a nearly steady state near the minimum.
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