A METHOD FOR THE INTEGRATION OF THE RADIATIVE-TRANSFER EQUATION

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ABSTRACT

The equations for radiative transfer are integrated exactly for a band of spectral lines which do not overlap and for an Elsasser band. The result of the two-fold integration of the absorption over frequency and over the atmospheric path can be expressed in terms of the Legendre functions. Here it is assumed that the mixing ratio is constant, that the line intensity is independent of temperature, and that the Lorentz line shape is valid. Asymptotic forms of the Legendre functions are used to obtain the solutions to the following problems from these exact results. The regions of validity of the single-line and strong-line approximations are precisely stated. It is shown that the strongest line in the band absorbs more radiation than any other line in an atmospheric layer, when the overlap of the lines can be neglected. For an Elsasser band, an expression is derived for the line strength that gives the maximum absorption in an atmospheric layer for radiation emitted either by a black body at another level or by another atmospheric layer.

1. Introduction

The accurate evaluation of the radiation flux in the atmosphere requires a reasonably precise knowledge of the transmission function and its derivatives. In recent years, a number of papers[2] [1 to 13] have presented various expressions for the transmission function for a variety of atmospheric models. The majority of these papers have considered in detail the following atmospheric model: (A) the mixing ratio of the radiating gas is constant with height; (B) the temperature dependence of the line intensities and of their half-widths is neglected; and (C) the spectral lines have the Lorentz line shape with a half-width proportional to the pressure. For CO_2, this should be a reasonably good model, especially when the temperature dependence of the line intensities is taken into account in the calculations in some other fashion. This correction can be made in quite a simple manner. The writers plan to describe this procedure in a paper that will analyze the experimental measurements for CO_2. However, for H_2O and O_3, the three assumptions (A, B and C) can apply only over relatively short path lengths, because of the rapid variation of their mixing ratios with height.

The popularity of the constant mixing ratio, constant line intensity atmospheric model stems from the fact that the equations of radiative transfer then have the least complicated, non-trivial form. It is important to have the exact results that can be obtained from the integration of these equations, since they can be used with confidence to investigate a wide variety of radiation problems. The results of these investigations in turn show which approximations are useful, together with their range of validity. These proven methods can then be used for H_2O and O_3, where approximate methods must necessarily be used.

In this article there is presented a new method for the integration of the equations of radiative transfer. The result is the simplest expression that has yet been found for the transmission function for either a single line or an Elsasser band. For example, (31) shows that the derivative of the transmission function for an Elsasser band can be expressed as a Legendre function with a relatively simple argument. This is an exact result for the model used here. The proof is valid for any value (not necessarily integral) of the dimensionless parameter, γ, that occurs in all radiation problems where conditions along the path are not uniform. The simplicity of these results should make them useful in a number of investigations, particularly since the Legendre functions are well known and well tabulated.

In this article, these exact results are used to obtain series expansions that determine the precise ranges where certain approximate expressions for the integrated absorption may be used. Together with the asymptotic expansion for the Legendre functions, these exact results are used to determine the line strength that gives the maximum absorption of radiation from various sources by an atmospheric layer.

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[2] The paper by Pedersen [1] should have been included in the lists of references in our earlier papers. We especially regret its omission because it contains the first clear statement that we have seen in the literature of the importance of the pressure effect for radiation calculations. We trust that the general unavailability of the journal in this country will not prevent others from reading it for as many years as we have overlooked it.

2. The atmospheric model

The fundamental problem in the calculation of the infrared radiation flux is the evaluation of the integrated absorption, $\Lambda(u_0, u_1)$, defined as

$$\Lambda(u_0, u_1) = \int_{\Delta\nu} \left\{ 1 - \exp\left[-\sec\theta \int_{u_0}^{u_1} k(\nu, u)\, du \right] \right\} d\nu, \quad (1)$$

where $\Delta\nu$ is the frequency interval, θ the angle that the radiation makes with the vertical, and $k(\nu, u)$ is the absorption coefficient. The mass per unit area of radiating gas measured from the top of the atmosphere, u, is given by

$$u = \int_z^\infty c\rho\, dz = \int_z^\infty \rho_r\, dz, \quad (2)$$

where the fractional concentration (mixing ratio) c is the ratio of the density of the radiating gas, ρ_r, to the total density, ρ. The integrated absorption between two layers in the atmosphere is specified by the mass per unit area, u_0 and u_1, above these levels.

Throughout this article, the three assumptions as given in the introduction for an atmosphere with a constant mixing ratio and constant line intensity are made. From assumption (C), the absorption coefficient is

$$k(\nu) = (S\alpha/\pi)[(\nu - \nu_0)^2 + \alpha^2]^{-1}, \quad (3)$$

where S is the line intensity, and α is the half-width of the lines which is proportional to the pressure, p, at constant temperature. Thus,

$$\alpha = (\alpha_s/p_s)p, \quad (4)$$

where α_s is the half-width at some standard pressure p_s.

The integrated absorption, Λ, and its derivatives have a simple physical meaning. In the remainder of the article, Λ is considered as a function of the half-width of the spectral lines at the appropriate atmospheric level, and α is considered as the independent variable. In an isothermal atmosphere, $\Lambda(0, \alpha_1)$ is proportional to the total flux of radiation from the entire atmosphere that is radiated to space. $\partial[\Lambda(\alpha_0, \alpha_1)]/\partial\alpha_0$ is proportional to the radiation from a black surface (at a height that corresponds to α_1) that is absorbed by a higher atmospheric layer (at a height that corresponds to α_0) with a thickness such that α changes by $d\alpha_0$ in the layer. $\partial[\Lambda(\alpha_0, \alpha_1)]/\partial\alpha_1$ is proportional to the radiation from a black surface (at α_0) that is absorbed by a lower atmospheric layer of thickness $d\alpha_1$ (at a height that corresponds to α_1). $\partial^2[\Lambda(\alpha_0, \alpha_1)]/\partial\alpha_0\,\partial\alpha_1$ is proportional to the radiation from an atmospheric layer of thickness $d\alpha_0$ (at α_0) that is absorbed by another atmospheric layer of thickness $d\alpha_1$ (at α_1). Λ and its derivatives are calculated and discussed in the remainder of this article.

It should be noted that (B) still permits an arbitrary pressure variation with height; in particular, the exponential decrease of pressure with height in an isothermal atmosphere need not be assumed. There is now considerable theoretical and experimental evidence that supports the validity of (3) and (4). For frequencies very far from the line center, the Lorentz shape is no longer valid. However, this is only important for the strongest spectral lines and very long path lengths [3]. Further, the Doppler width may be neglected below 50 km [10].

The principal reason for the use of assumptions (A) and (B) is that this is one of the few cases where the equations of radiative transfer can be integrated exactly. These exact solutions can provide an important insight into more complicated situations, where the radiation flux can only be obtained by approximate methods. Thus our atmospheric model assumes that the radiating gas has a constant mixing ratio with height, no temperature variation of the line-strengths and half-widths, and spectral lines with the Lorentz line shape.

3. Integrated absorption for a single line

In this section, the integrated absorption for a single line is expressed exactly in terms of Legendre functions. When (3) and (4) are substituted into (1), and the relation

$$g\, du = c\, dp = (cp_s/\alpha_s)\, d\alpha \quad (5)$$

is used, g being the gravitational acceleration, the integration over u in (1) can be performed immediately. The result is [2]

$$\Lambda(\alpha_0, \alpha_1) = 2\alpha_1 \int_0^\infty \left\{ 1 - \left[\frac{x^2 + (\alpha_0/\alpha_1)^2}{x^2 + 1} \right]^\gamma \right\} dx, \quad (6)$$

where α_0 and α_1 are the values of the half-width at the levels u_0 and u_1, and where γ is the dimensionless constant that occurs in all radiation problems when the pressure varies along the path [2] and is defined as

$$\gamma = (Scp_s \sec\theta)/(2\pi\alpha_s g). \quad (7)$$

The evaluation of the integral in (6) is complicated by the fact that the integrand is the difference of two terms. Each of these terms evaluated separately gives an infinite answer, and it is difficult to obtain their difference (which, of course, is finite). A more suitable form can be obtained from a transformation that combines these two parts of the integrand into a single term. First write (6) as a double integral and then reverse the order of integration (this process can be shown to be valid here), to obtain

$$\Lambda(\alpha_0, \alpha_1) = 2\alpha_1 \int_0^\infty dx \int_0^{\gamma \ln[(x^2+1)/(x^2+p^2)]} \times \exp(-y)\, dy$$

$$= 2\alpha_1 \int_0^{\ln p^{-2\gamma}} dy \exp(-y) \times \int_0^{\left[\frac{\exp(-y/\gamma)-p^2}{1-\exp(-y/\gamma)}\right]^{\frac{1}{2}}} dx$$

$$= 2\alpha_1 \int_0^{\ln p^{-2\gamma}} \exp(-y) \times \left[\frac{\exp(-y/\gamma) - p^2}{1 - \exp(-y/\gamma)}\right]^{\frac{1}{2}} dy, \quad (8)$$

where $p = \alpha_0/\alpha_1$.

Finally, from the substitution $y = -\gamma \ln[1 - (1 - p^2)t]$, we obtain

$$\Lambda(\alpha_0, \alpha_1) = 2\alpha_1\gamma(1 - p^2) \times \int_0^1 [1 - (1 - p^2)t]^{\gamma-1}(1 - t)^{\frac{1}{2}} t^{-\frac{1}{2}}\, dt. \quad (9)$$

This integral can be evaluated in terms of the hypergeometric function from the relation, given by Bateman [14, Vol. I, p. 114],

$$F(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \times \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a}\, dt, \quad (10)$$

where $\Gamma(c)$ is the gamma function. This equation is valid for all real, positive values of b and c.

If we let $a = 1 - \gamma$, $b = \frac{1}{2}$, and $c = 2$, a comparison of (9) and (10) shows that

$$\Lambda(\alpha_0, \alpha_1) = \pi\alpha_1\gamma(1 - p^2)\, F(1 - \gamma, \tfrac{1}{2}, 2, 1-p^2). \quad (11)$$

Now, from Bateman [14, Vol. I, p. 103], we may write

$$(c - a)\,(c - b)z\, F(a, b, c + 1, z) = ac(1 - z)\, F(a + 1, b, c, z) - c[a - (c - b)z]\, F(a, b, c, z). \quad (12)$$

Define

$$\eta = -\ln p = \ln(\alpha_1/\alpha_0). \quad (13)$$

From (11) and (12) with $a = 1 - \gamma$, $b = \frac{1}{2}$, $c = 1$, and $z = 1 - p^2$, we obtain

$$\Lambda(\alpha_0, \alpha_1) = 2\pi\alpha_1\{(1 - \gamma)e^{-2\eta}\, F(2 - \gamma, \tfrac{1}{2}, 1, 1 - e^{-2\eta}) - (1 - \gamma - e^{-\eta}\sinh\eta) \times F(1 - \gamma, \tfrac{1}{2}, 1, 1 - e^{-2\eta})\}. \quad (14)$$

These hypergeometric functions can be simply expressed in terms of the Legendre functions. From Bateman [14, Vol. I, p. 173], we have

$$P_{\nu-\frac{1}{2}}(\cosh\eta) = e^{-(\nu+\frac{1}{2})\eta}\, F(\tfrac{1}{2} + \nu, \tfrac{1}{2}, 1, 1 - e^{-2\eta}). \quad (15)$$

Also, since $F(a, b, c, z) = F(b, a, c, z)$, and $P_\nu(z) = P_{-\nu-1}(z)$, the integrated absorption can be written as

$$\Lambda(\alpha_0, \alpha_1) = 2\pi\alpha_1\{(1 - \gamma)e^{-\gamma\eta} \times P_{\gamma-2}(\cosh\eta) - [(1 - \gamma)e^{\eta} - \sinh\eta]e^{-\gamma\eta}\, P_{\gamma-1}(\cosh\eta)\}. \quad (16)$$

A more convenient form for this equation can be obtained with the use of the recursion relation for Legendre functions [14, Vol. I, p. 160],

$$(2\nu + 1)z\, P_\nu(z) = (\nu + 1)\, P_{\nu+1}(z) + \nu\, P_{\nu-1}(z). \quad (17)$$

Let $\nu = \gamma - 1$ in (17), and substitute for $P_{\nu-2}$ in (16). The result is obtained that the integrated absorption is

$$\Lambda(\alpha_0, \alpha_1) = 2\pi\alpha_1\gamma e^{-\gamma\eta}[P_\gamma(\cosh\eta) - e^{-\eta}\, P_{\gamma-1}(\cosh\eta)], \quad (18)$$

where η is defined by (13). The results are expressed as a function of η here, to facilitate comparison with similar results that are given in section 4 for the Elsasser band.

With the use of recursion formulae for the Legendre functions and their derivates, the derivatives of the integrated absorption can be written as

$$\partial[\Lambda(\alpha_0, \alpha_1)]/\partial\alpha_0 = -2\pi\gamma e^{-\gamma\eta}\, P_{\gamma-1}(\cosh\eta), \quad (19)$$

$$\partial[\Lambda(\alpha_0, \alpha_1)]/\partial\alpha_1 = 2\pi\gamma e^{-\gamma\eta}\, P_\gamma(\cosh\eta), \quad (20)$$

and

$$\frac{\partial^2\Lambda(\alpha_0, \alpha_1)}{\partial\alpha_0\partial\alpha_1} = \frac{4\pi\gamma^2}{\alpha_1}\,\frac{e^{-\gamma\eta}}{1 - e^{-2\eta}} \times [P_{\gamma-1}(\cosh\eta) - e^{-\eta}\, P_\gamma(\cosh\eta)]. \quad (21)$$

The principal results of this section are (18) to (21). These are exact results for the atmospheric model described in the introduction. In particular, (18) to (21) are valid for any value of the parameter γ, whether or not it is an integer. King [7; 9] has previously shown that the derivatives of the integrated absorption can be expressed in terms of the hypergeometric function for integral values of γ.

The Legendre functions are well-known polynomials when the order, γ, is an integer. In this case, the integrated absorption can be written as a sum of terms. The number of terms in the sum is equal to the integral value of γ. For example, the first few such terms are:

$$\begin{aligned}
\gamma = 1 &: \Lambda(\alpha_0, \alpha_1) = \pi\alpha_1\delta^2, \\
\gamma = 2 &: \Lambda(\alpha_0, \alpha_1) = 2\pi\alpha_1\delta^2(1 - \tfrac{1}{4}\delta^2), \\
\gamma = 3 &: \Lambda(\alpha_0, \alpha_1) = 3\pi\alpha_1\delta^2(1 - \tfrac{1}{2}\delta^2 + \tfrac{1}{8}\delta^4), \\
\gamma = 4 &: \Lambda(\alpha_0, \alpha_1) = 4\pi\alpha_1\delta^2(1 - \tfrac{3}{4}\delta^2 + \tfrac{3}{8}\delta^4 - \tfrac{5}{64}\delta^6), \\
\gamma = 5 &: \Lambda(\alpha_0, \alpha_1) = 5\pi\alpha_1\delta^2(1 - \delta^2 + \tfrac{3}{4}\delta^4 - \tfrac{5}{16}\delta^6 + \tfrac{7}{128}\delta^8), \quad (22) \\
&\cdot \quad \cdot \quad \cdot \\
&\cdot \quad \cdot \quad \cdot \\
&\cdot \quad \cdot \quad \cdot
\end{aligned}$$

where $\delta^2 = 1 - p^2 = 1 - e^{-2\eta} = 1 - (\alpha_0^2/\alpha_1^2)$. These expressions can be very useful, since it is often difficult to obtain an adequate approximation for the integrated absorption when γ has a value near unity.

For half-integral values of γ, the integrated absorption can be written in terms of the complete elliptic integrals of the first and second kind,

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-\frac{1}{2}} d\theta,$$

and (23)

$$E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{\frac{1}{2}} d\theta,$$

respectively. From Bateman [14, Vol. I, p. 173] we have that

$$P_{-\frac{1}{2}}(\cosh x) = \frac{2}{\pi \cosh \frac{1}{2}x} K(\tanh \tfrac{1}{2}x),$$

and (24)

$$P_{\frac{1}{2}}(\cosh x) = \frac{2}{\pi} e^{\frac{1}{2}x} E[(1 - e^{-2x})^{\frac{1}{2}}].$$

From the recursion relation (17), it follows that any Legendre function of half-integral order can be expressed in terms of the complete elliptic integrals of the first and second kind. Since these are well-tabulated functions, this is often a useful form for the result. This result is particularly simple for the derivatives of the integrated absorption, as given by (19) and (20).

Before we consider some applications of the above results, we derive by a somewhat different method an exact expression for the derivatives of the integrated absorption for an Elsasser band.

4. Integrated absorption for an Elsasser band

The Elsasser model of a band assumes that all the lines have the same intensity, spacing and half-width. Although no actual band satisfies these requirements over a large frequency interval, they may be fairly well satisfied over a restricted frequency range in some bands. Further, the results obtained from the Elsasser model of a band enable an estimate to be made of the importance of such phenomena as the overlapping of the lines in a band for the calculation of the integrated absorption.

The integrated absorption for the Elsasser band may be written [3; 7; 9; 15] as

$$\Lambda(\alpha_0, \alpha_1) = \int_{-\frac{1}{2}d}^{\frac{1}{2}d} \times \left\{ 1 - \exp\left[-\gamma \int_{\beta_0}^{\beta_1} \frac{\sinh \beta \, d\beta}{\cosh \beta - \cos \theta} \right] \right\} d\nu$$
$$= d - \frac{d}{2\pi} \int_{-\pi}^{\pi} \left[\frac{\cosh \beta_0 - \cos \theta}{\cosh \beta_1 - \cos \theta} \right]^\gamma d\theta, \qquad (25)$$

where $\beta = 2\pi\alpha/d$ and $\theta = 2\pi\nu/d$.

Now, Whittaker and Watson [16, pp. 326–329] show that, for *any* value of γ,

$$P_{\gamma-1}(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{[x + (x^2-1)^{\frac{1}{2}} \cos(\omega - \phi)]^{\gamma-1}}{[y + (y^2-1)^{\frac{1}{2}} \cos \phi]^\gamma} d\phi, \qquad (26)$$

where $z = xy - (x^2 - 1)^{\frac{1}{2}} (y^2 - 1)^{\frac{1}{2}} \cos \omega$, and $P_{\gamma-1}(z)$ is the Legendre function.

In (26), let $x = \coth \beta_0$, $y = \coth \beta_1$, $\omega = 0$, and $\theta = \phi + \pi$, to obtain

$$\int_{-\pi}^{\pi} \frac{[\cosh \beta_0 - \cos \theta]^{\gamma-1}}{[\cosh \beta_1 - \cos \theta]^\gamma} d\theta = 2\pi \frac{\sinh^{\gamma-1} \beta_0}{\sinh^\gamma \beta_1} P_{\gamma-1}\left(\frac{\cosh \beta_0 \cosh \beta_1 - 1}{\sinh \beta_0 \sinh \beta_1} \right), \qquad (27)$$

where the periodic character of the integrand allows the limits to be taken from $-\pi$ to π.

From (25), we obtain by differentiation that

$$\frac{\partial \Lambda(\alpha_0, \alpha_1)}{\partial \alpha_0} = -\gamma \sinh \beta_0 \times \int_{-\pi}^{\pi} \frac{[\cosh \beta_0 - \cos \theta]^{\gamma-1}}{[\cosh \beta_1 - \cos \theta]^\gamma} d\theta. \qquad (28)$$

From substitution of (27) in (28), we obtain

$$\frac{\partial \Lambda(\alpha_0, \alpha_1)}{\partial \alpha_0} = -2\pi\gamma \frac{\sinh^\gamma \beta_0}{\sinh^\gamma \beta_1} \times P_{\gamma-1}\left(\frac{\cosh \beta_0 \cosh \beta_1 - 1}{\sinh \beta_0 \sinh \beta_1} \right). \qquad (29)$$

This is the desired exact evaluation of the derivative of the integrated absorption. By a substitution of variables, (29) can be written in a form that is more suitable for many purposes and that formally simplifies the argument of the Legendre function. Let

$$\sinh \xi_0 = (\sinh \beta_0)^{-1},$$

or

$$\cosh \xi_0 = \coth \beta_0, \qquad (30)$$
$$\exp(-\xi_0) = \tanh \tfrac{1}{2}\beta_0.$$

ξ_1 is defined by similar equations in terms of β_1.

The derivative of the integrated absorption can then be written in terms of the variables ξ_0 and ξ_1, as

$$\frac{\partial \Lambda(\alpha_0, \alpha_1)}{\partial \alpha_0} = -2\pi\gamma \frac{\sinh^\gamma \xi_1}{\sinh^\gamma \xi_0} P_{\gamma-1}[\cosh(\xi_0 - \xi_1)]. \qquad (31)$$

From the derivative of (25) with respect to α_1, it follows that

$$\frac{\partial \Lambda_\gamma}{\partial \alpha_1} = -\frac{\gamma}{\gamma + 1} \frac{\sinh \xi_0}{\sinh \xi_1} \frac{\partial \Lambda_{\gamma+1}}{\partial \alpha_0}, \qquad (32)$$

where Λ_γ indicates the integrated absorption evaluated for the value γ. Thus, we obtain

$$\frac{\partial\Lambda(\alpha_0,\alpha_1)}{\partial\alpha_1}=2\pi\gamma\frac{\sinh^\gamma\beta_0}{\sinh^\gamma\beta_1}P_\gamma\left(\frac{\cosh\beta_0\cosh\beta_1-1}{\sinh\beta_0\sinh\beta_1}\right)$$
$$=2\pi\gamma\frac{\sinh^\gamma\xi_1}{\sinh^\gamma\xi_0}P_\gamma[\cosh(\xi_0-\xi_1)]. \qquad (33)$$

With the aid of the recursion formulae for the derivative of the Legendre function, and from either (31) or (33), the second derivative of the integrated absorption can be written as

$$\frac{\partial^2\Lambda(\alpha_0,\alpha_1)}{\partial\alpha_0\,\partial\alpha_1}=\frac{4\pi^2\gamma^2}{d}\frac{\sinh^{\gamma+1}\xi_1}{\sinh^\gamma\xi_0\sinh(\xi_0-\xi_1)}$$
$$\times\{\sinh\xi_0\,P_{\gamma-1}[\cosh(\xi_0-\xi_1)]$$
$$-\sinh\xi_1\,P_\gamma[\cosh(\xi_0-\xi_1)]\}. \qquad (34)$$

The Legendre functions in the above expressions become Legendre polynomials when γ is an integer. The exact expressions are particularly simple for the first few integers and are very useful, since it is difficult to obtain adequate approximations for this range of values of γ. For example, from (31),

$$\gamma=1:\ \frac{\partial\Lambda(\alpha_0,\alpha_1)}{\partial\alpha_0}=-2\pi\frac{\sinh\xi_1}{\sinh\xi_0},$$
$$\gamma=2:\ \frac{\partial\Lambda(\alpha_0,\alpha_1)}{\partial\alpha_0}=-2\pi\left(\frac{\sinh\xi_1}{\sinh\xi_0}\right)^2(2)\Xi,$$
$$\gamma=3:\ \frac{\partial\Lambda(\alpha_0,\alpha_1)}{\partial\alpha_0}=-2\pi\left(\frac{\sinh\xi_1}{\sinh\xi_0}\right)^3(\tfrac{3}{2})(3\Xi^2-1), \qquad (35)$$
$$\gamma=4:\ \frac{\partial\Lambda(\alpha_0,\alpha_1)}{\partial\alpha_0}=-2\pi\left(\frac{\sinh\xi_1}{\sinh\xi_0}\right)^4(2)(5\Xi^3-3\Xi),$$
$$\vdots$$

where $\Xi=\cosh(\xi_0-\xi_1)$.

To provide additional interpolation when γ is near unity, it is also useful to have available an expression for Λ in terms of tabulated functions for half-integral values of γ. From (23), (24) and (31), it follows that

$$\gamma=\tfrac{1}{2}:\ \frac{\partial\Lambda(\alpha_0,\alpha_1)}{\partial\alpha_0}=\frac{2}{\cosh[\frac{1}{2}(\xi_0-\xi_1)]}\times\left(\frac{\sinh\xi_1}{\sinh\xi_0}\right)^{\frac{1}{2}}K\left[\tanh\left(\frac{\xi_0-\xi_1}{2}\right)\right],$$
$$\gamma=\tfrac{3}{2}:\ \frac{\partial\Lambda(\alpha_0,\alpha_1)}{\partial\alpha_0}=6\exp\left(\frac{\xi_0-\xi_1}{2}\right)\times\left(\frac{\sinh\xi_1}{\sinh\xi_0}\right)^{\frac{3}{2}}E[(1-e^{-2(\xi_0-\xi_1)})^{\frac{1}{2}}], \qquad (36)$$

where K and E are the elliptic integrals of the first and second kind. The derivative of the integrated absorption can be obtained in terms of the elliptic integrals for higher half-integral values of γ from the recursion formulae for the Legendre functions.

When the half-width is small compared with the line spacing, (29) to (31), (33), and (34) reduce to (19), (20), and (21), respectively. The writers have been unable to obtain a general expression for the integrated absorption of an Elsasser band that corresponds to (18) for a single line. The difficulty in certain limits can be reduced to the impossibility of writing a general expression for the integral of $\sinh^\gamma\beta_0$, whereas the corresponding integrand for the single line case, $\beta_0{}^\gamma$, can be evaluated immediately for *any* value of γ. Therefore, it is not believed that such a general expression can be obtained for the Elsasser band.

However, if γ is restricted to *integral* values, it is easily shown that (29) and (33) always may be integrated in terms of known functions. For example,

$$\gamma=1:\ \Lambda(\alpha_0,\alpha_1)=\frac{d}{\sinh\beta_1}[\cosh\beta_1-\cosh\beta_0],$$
$$\gamma=2:\ \Lambda(\alpha_0,\alpha_1)=\frac{d}{\sinh^3\beta_1}[2(\cosh\beta_0-\cosh\beta_1)+\cosh\beta_1(\cosh^2\beta_1-\cosh^2\beta_0)]. \qquad (37)$$

All the equations through (37) in this article are exact for the atmospheric model discussed in section 1. As such, they are useful in a variety of problems. For example, they can be used to test the regions of validity of the various commonly used approximations, and to determine the line strengths that give maximum radiation transfer under various conditions. This is done in the following sections.

5. Regions of validity of various approximations

In the next two sections, we shall have occasion to use the following approximate expression for the Legendre functions, valid when $x^2>\frac{1}{4}\gamma$:

$$P_\gamma(x)=\frac{2^\gamma\,\Gamma(\gamma+\frac{1}{2})\,x^\gamma}{\pi^{\frac{1}{2}}\,\Gamma(\gamma+1)},\qquad \gamma>-\tfrac{1}{2};$$
$$P_\gamma(x)=\frac{2^{-\gamma-1}\,\Gamma(-\gamma-\frac{1}{2})}{\pi^{\frac{1}{2}}\,\Gamma(-\gamma)\,x^{\gamma+1}},\qquad \gamma<-\tfrac{1}{2}; \qquad (38)$$

These expressions are given by Bateman [14, Vol. I, p. 164]. The condition for the validity of (38), that $x^2>\frac{1}{4}\gamma$, may be derived in various ways. For integral values of γ it follows immediately from the series expression for the Legendre polynomials.

To determine the regions of validity of the customary approximations, we use the same method described in detail by Plass [12]. Briefly, the procedure is to expand the exact expression for the quantity of interest in terms of an appropriate parameter, so that the approximate equation is the first term of this series. If the resulting series is rapidly convergent, the error made by the use of only the first term in the series may be easily estimated from the remaining terms. This may be done rigorously if the terms in the series alternate in sign, as is the case with most of the series that occur in radiation problems.

First, we determine the region of validity of the single-line equations with the neglect of the overlap of various spectral lines when the upper layer is taken near the top of the atmosphere. As α_0 or β_0 becomes very small, the argument of the Legendre functions becomes large, and the approximate equations (38) may be used. With the use of (38) to evaluate the ratios of the corresponding Elsasser band and single-line equations, (18) and (37) (see also [12]), (19) and (29), (20) and (33), and (21) and (34), it is found that the single-line equations have an error of less than p per cent due to the neglect of the overlap of the lines when, for (18),

$$\gamma \leq 0.12 p \beta_1^{-2}, \tag{39}$$

and when, for (19), (20) and (21),

$$\gamma \leq 0.04 p \beta_1^{-2}. \tag{40}$$

For example, for 5 per cent accuracy and $\beta_1 = 0.1$, (18) is valid if $\gamma \leq 60$, and (19), (20) and (21) are valid if $\gamma \leq 20$. From these results, the line strength can immediately be determined at which the overlap of different lines begins to be important for a given optical path.

Another important approximation is the strong-line approximation, which is the first term of

$$\Lambda(\alpha_0, \alpha_1) = 2\pi^{\frac{1}{2}}\gamma^{\frac{1}{2}}(\alpha_1^2 - \alpha_0^2)^{\frac{1}{2}} \times \left\{1 - \frac{\alpha_1^2 + \alpha_0^2}{8\gamma(\alpha_1^2 - \alpha_0^2)} + \cdots\right\}, \tag{41}$$

and can be obtained by the expansion of (18). From this result, it is found that the strong-line approximation [defined here as the first term of (41)] is valid within p per cent when the following conditions are satisfied for the stated equation:

$$\begin{aligned}
&\text{for (18)}: \gamma \geq 12.5\, p^{-1}(1 + e^{-2\eta})\,(1 - e^{-2\eta})^{-1},\\
&\text{for (19)}: \gamma \geq 37.5\, p^{-1}(1 - \tfrac{1}{3}e^{-2\eta})\,(1 - e^{-2\eta})^{-1},\\
&\text{for (20)}: \gamma \geq 12.5\, p^{-1}|\,(1 - 3e^{-2\eta})\,|\,(1 - e^{-2\eta})^{-1},\\
&\text{for (21)}: \gamma \geq 37.5\, p^{-1}(1 + e^{-2\eta})\,(1 - e^{-2\eta})^{-1}.
\end{aligned} \tag{42}$$

The strong-line equations can be derived alternatively from the assumption that the absorption is total for frequencies near the line center and that only frequencies more than several half-widths from the line center are important. Thus, (42) shows clearly that the radiation transfer occurs predominately in the wings when γ is sufficiently large and when the two layers are sufficiently far apart, so that $\exp(-2\eta) = \alpha_0^2/\alpha_1^2$ is small. However, as the layers are allowed to come closer together, for a line of *given* strength there is some critical spacing of the layers at which the frequencies near the line center become important. This critical spacing is given by (42) with an equality sign instead of a "greater than or equal to" symbol. For layers that are very close together, the weak-line approximation can be used when the absorption at the frequency of the line center is reduced to about one-half.

An important consequence of (39), (40) and (42) is that one or two differentiations of the integrated absorption do not increase the per cent error of the original Λ by more than a factor of three. Alternatively, it can be stated that the smallest and largest allowed values of γ for the validity of a given approximation are not changed by more than a factor of three by one or two differentiations of Λ.

6. Energy transfer by single lines

In this section we determine whether strong or weak spectral lines make a larger contribution to the integrated absorption and to the radiation transfer between atmospheric layers, when the overlap of the spectral lines is neglected. Although this model has a very limited range of validity at atmospheric pressure, it can greatly simplify calculations made in regions that have lower pressures. For example, if $\beta \approx 0.3$ at atmospheric pressure as now seems indicated, the single-line model gives results that are accurate to within 5 per cent when $\gamma < 125$, for any path in the stratosphere. Thus, for all except a few of the strongest lines in the band, the overlap of the lines can be neglected in the stratosphere.

Here the asymptotic expansion is needed for the Legendre function in inverse powers of the order, γ, for any *fixed* value of the argument that is greater than unity. This result can be obtained by the method of steepest descents and has been given by Watson [17, p. 291]. The result is that

$$P_\gamma(\cosh \eta) = \frac{e^{(\gamma+\frac{1}{2})\eta}}{(2\pi\gamma \sinh \eta)^{\frac{1}{2}}}\left(1 - \frac{3}{8\gamma} + \cdots\right), \tag{43}$$

valid when $\gamma \gg 1$ and for a fixed $\eta > 0$. If $\gamma > 10$, the error made by the use of just the first term of (43) is less than 4 per cent, unless η is very close to zero. The asymptotic expansion (43) should not be confused with (38), which is valid for a *fixed* value of the order γ, as the argument becomes large. An unusual result that can be obtained from (43) is that, for $\eta > 1$ and γ an integer such that $\gamma \gg \cosh^2 \eta$, the sum of all the terms in the alternating series that represents the Legendre function is equal to the term that contains the highest power of $\cosh \eta$, even though this term is much smaller than later terms in the alternating series. Those readers who are mathematically inclined will find that a further investigation of this point is rewarding.

The substitution of (43) in (18), (19), (20) and (21) shows that, for a fixed $\eta = \ln(\alpha_1/\alpha_0)$ and as γ becomes large, the asymptotic forms of these equations are

$$\Lambda(\alpha_0,\alpha_1) = 2\pi^{\frac{1}{2}}\gamma^{\frac{1}{2}}\alpha_1(1-e^{-2\eta})^{\frac{1}{2}}, \quad (18a)$$

$$\partial[\Lambda(\alpha_0,\alpha_1)]/\partial\alpha_0 = -2\pi^{\frac{1}{2}}\gamma^{\frac{1}{2}}e^{-\eta}(1-e^{-2\eta})^{-\frac{1}{2}}, \quad (19a)$$

$$\partial[\Lambda(\alpha_0,\alpha_1)]/\partial\alpha_1 = 2\pi^{\frac{1}{2}}\gamma^{\frac{1}{2}}(1-e^{-2\eta})^{-\frac{1}{2}}, \quad (20a)$$

and

$$\partial^2[\Lambda(\alpha_0,\alpha_1)]/\partial\alpha_0\,\partial\alpha_1 = 2\pi^{\frac{1}{2}}\gamma^{\frac{1}{2}}\alpha_1^{-1}e^{-\eta}(1-e^{-2\eta})^{-\frac{3}{2}}. \quad (21a)$$

Between any two fixed layers, as γ increases, these equations show that the magnitude of Λ and its derivatives eventually increases as $\gamma^{\frac{1}{2}}$. Thus, for a band of spectral lines that do not overlap, the integrated absorption between two atmospheric layers is a maximum for the strongest line in the band. The same result is true for the absorption of radiation by an atmospheric layer from a black body, above or below the given layer, or from another atmospheric layer. The physical explanation of these results is that frequencies farther and farther from the line center contribute to the radiation transfer as the line strength increases. Of course, for an actual band at a given pressure, the overlapping of the lines would have to be taken into account after γ had increased beyond a certain value. This is done in the next section.

7. Energy transfer by an Elsasser band

In an actual band, the radiation transfer does not indefinitely increase as $\gamma^{\frac{1}{2}}$, as is the case for a band of lines that do not overlap. This is because, as γ increases, the frequencies at which the radiation transfer occurs are farther and farther from the line center. Eventually, as γ continues to increase, the centers of the neighboring lines absorb strongly at these frequencies and the radiation transfer must decrease. Thus, when the lines overlap, there is some value of γ that makes the radiation transfer a maximum. We investigate in detail the case of the absorption of radiation from a black surface at a height corresponding to $\beta_0 = 2\pi\alpha_0/d$, by an atmospheric layer of thickness such that α changes by $d\alpha_1$ in the layer and at a height corresponding to $\beta_1 = 2\pi\alpha_1/d$. As discussed in section 2, this is proportional to $\partial[\Lambda(\alpha_0,\alpha_1)]/\partial\alpha_1$, which is given exactly by (33).

If the black surface is at the top of the atmosphere, $\beta_0 = 0$, and the argument of the Legendre function approaches infinity. The substitution of (38) into (33) gives

$$\frac{\partial\Lambda(0,\alpha_1)}{\partial\alpha_1} = \frac{2\pi^{\frac{1}{2}}\gamma\,\Gamma(\gamma+\frac{1}{2})}{\Gamma(\gamma+1)\cosh^{2\gamma}\frac{1}{2}\beta_1}. \quad (44)$$

Thus, if γ increases by unity from the value γ to $\gamma+1$, it follows that the above expression is multiplied by $(\gamma+\frac{1}{2})/(\gamma\cosh^2\frac{1}{2}\beta_1)$. From this, it follows that the value of γ that makes (44) a maximum is

$$\frac{1}{2\sinh^2\frac{1}{2}\beta_1} \leq \gamma_m \leq 1 + \frac{1}{2\sinh^2\frac{1}{2}\beta_1}. \quad (45)$$

Thus, the value of γ_m is determined within an integer by (45) when $\beta_0 = 0$.

If the black surface is at some arbitrary height corresponding to β_0, (33) may be evaluated from (43), when $\gamma_m > 10$. The result is that

$$\frac{\partial\Lambda(\alpha_0,\alpha_1)}{\partial\alpha_1} = \frac{2\pi^{\frac{1}{2}}\gamma^{\frac{1}{2}}}{[1-e^{-2(\xi_0-\xi_1)}]^{\frac{1}{2}}}\left[\frac{1-e^{-2\xi_1}}{1-e^{-2\xi_0}}\right]^{\gamma}, \quad \gamma > 10. \quad (46)$$

The maximum value of (46) is given by

$$\gamma_m = \left[4\ln\frac{\cosh\frac{1}{2}\beta_1}{\cosh\frac{1}{2}\beta_0}\right]^{-1}, \quad (47)$$

valid when $\gamma_m > 10$.

In fig. 1, $\partial\Lambda/\partial\alpha_1$ is plotted as a function of $\log\gamma$ for the two cases, $\beta_1 = 0.1$ and $\beta_0 = 0.01$, and $\beta_1 = 1.0$ and $\beta_0 = 0.1$. The radiation absorbed in these cases is thus compared for a number of different spectral regions, each of which can be represented by Elsasser bands composed of lines of a given strength. The curves were obtained by the use of exact values for the Legendre function when $\gamma \leq 10$ and (46) when $\gamma > 10$. From the latest measurements of the half-widths, it appears that $\beta < 0.3$ at atmospheric pressure for the principal atmospheric bands, if it is assumed that only the stronger, regular lines in the band have an appreciable effect on the absorption. Thus, for $\beta_1 < 0.3$ and any value of β_0, it follows from (47) that $\gamma_m > 20$. Thus, more radiation is absorbed by a region of the band that contains strong lines than

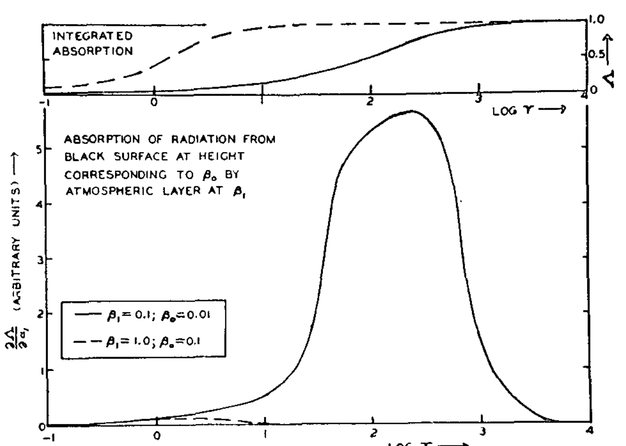

FIG. 1. Ordinate, $\partial\Lambda/\partial\alpha_1$, is proportional to absorption of radiation from black surface at height corresponding to $\beta_0 = 2\pi\alpha_0/d$ by atmospheric layer below at β_1. Thickness of layer is such that half-width, α, changes by $d\alpha_1$ in layer. This is plotted as function of dimensionless parameter γ [see (7)]. Upper curves give absorption integrated over frequency and along atmospheric path, Λ, as function of γ. Elsasser band is assumed with constant mixing ratio and constant line intensity.

by another equal frequency interval that contains the *same* number of weak lines.

Also in fig. 1, the integrated absorption, Λ, is plotted for the two sets of values of β_0 and β_1 given there. It is interesting to note that the maximum absorption of radiation occurs when the integrated absorption is slightly greater than one-half. If Λ is much greater than one-half, $\partial\Lambda/\partial\alpha_1$ rapidly falls to zero, since the absorption between the layers is too large. If Λ is much smaller than one-half, the radiation is not appreciably absorbed between the two layers. In this case, however, the absorption coefficient is too small for the radiation to be absorbed appreciably in the given atmospheric layer. The maximum effect occurs when the radiation is not absorbed a great deal in its passage between the two layers at frequencies midway between two adjacent spectral lines, but can still be absorbed at the final level at those same frequencies due to the overlap of the spectral lines. It is interesting to note that $\partial\Lambda/\partial\alpha_1$ begins to rise in fig. 1 as $\gamma^{\frac{1}{2}}$, just as happens for a single line, until finally the overlap of the lines [the factors in (46) that depend on ξ_0 and ξ_1] causes $\partial\Lambda/\partial\alpha_1$ to reach a maximum value and then rapidly to decrease to zero as γ increases still further.

The Elsasser model for a band is used here in the same manner in which it has been used many times before in the literature. Whether or not it represents the behavior of an actual band depends on the band and frequency considered and the width of the frequency interval. If the number of lines with intensity in a given range is roughly equal for equal intervals of $\ln\gamma$ for CO_2, as stated by Kaplan[3] [3; 13], more radiation is transferred for those values of γ less than γ_m than for those greater than γ_m for the entire band considered as a unit. This occurs merely because the weaker lines are so much more numerous in this assumed intensity distribution. However, even in this case, a simple calculation from fig. 1 shows that 80 per cent of the energy transfer occurs for lines with $\gamma > 10$ if $\beta_1 \leq 0.1$, *i.e.*, for pressures roughly one-third of atmospheric pressure or less. At atmospheric pressure, the maximum radiation transfer occurs for small values of γ, if the weak lines are actually this numerous. Because of the uncertainties that still exist in the determination of line strengths for the absorbing gases in the atmosphere, we have preferred in this article to compare frequency intervals that contain the same number of lines of a given strength in each interval. The lines of a band can always be grouped in this fashion, provided that there are no weak lines next to a strong line that make an appreciable contribution to the absorption. A study of the experimentally measured absorption seems to indicate that the weak lines seldom have an appreciable influence of this type.

[3] Those readers interested in a comparison of the results of this article with those of Kaplan [13] should note that the overlap of the spectral lines is *not* taken into account in [13], although this does not seem to be stated in the text. This can be seen from the fourth equation from the bottom of p. 17.

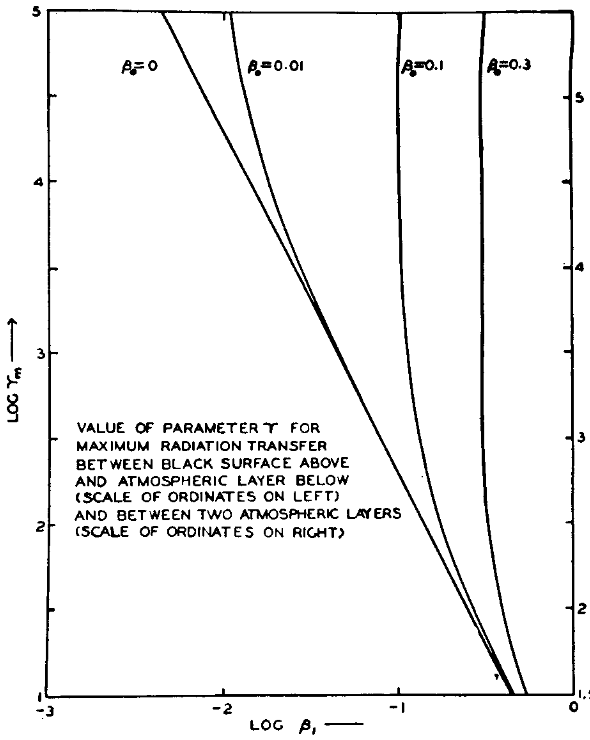

FIG. 2. Scale of ordinates on left gives value of parameter, γ, for which there is maximum absorption of radiation by atmospheric layer at β_1 below black surface at β_0. Different Elsasser bands are compared that have same number of spectral lines, but different line intensities. Scale of ordinates on left gives same quantity when radiation is emitted by atmospheric layer at β_0 instead of by black surface. Curves are shown for a number of values of β_0 and β_1.

In fig. 2, the value of the parameter, γ, for which there is the maximum absorption of radiation by an atmospheric layer at β_1 below a black surface at β_0 is plotted as a function of β_1 for several different values of β_0. These results were calculated from (45) and (47) and were checked with exact calculations from (33). Unless the two layers are quite close

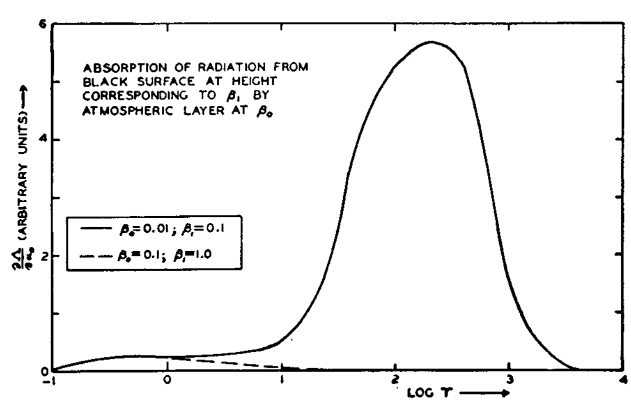

FIG. 3. Ordinate, $\partial\Lambda/\partial\alpha_0$, is proportional to absorption of radiation from black surface at height β_1 by atmospheric layer above at β_0. This is plotted as function of parameter γ.

together, the value of γ_m for a given value of β_1 does not depend strongly on β_0.

The results for $\partial\Lambda/\partial\alpha_0$ and $\partial^2\Lambda/\partial\alpha_0\,\partial\alpha_1$ are obtained in exactly the same manner and will only be summarized here. In both these cases, no useful result is obtained when β_0 approaches zero, since the radiation transfer then becomes zero. However, when $\gamma > 10$, the equation that corresponds to (46) is

$$\frac{\partial\Lambda(\alpha_0,\alpha_1)}{\partial\alpha_0} = \frac{2\pi^{\frac{1}{2}}\gamma^{\frac{1}{2}}}{(1-\gamma^{-1})^{\frac{1}{2}}[e^{2(\xi_0-\xi_1)}-1]^{\frac{1}{2}}}\left[\frac{1-e^{-2\xi_1}}{1-e^{-2\xi_0}}\right]^{\gamma}. \qquad (48)$$

From this it follows that γ_m is given by the same expression as (47) for $\partial\Lambda/\partial\alpha_0$.

In fig. 3, the absorption of radiation by an atmospheric layer above a black surface is plotted for the same values of β_0 and β_1 given in fig. 1. The exact expression for the Legendre function was used when $\gamma \leq 10$, and (48) when $\gamma > 10$. The only qualitative difference between figs. 1 and 3 is that a small subsidiary maximum occurs for small values of γ. This maximum has already been noticed by Kaplan [13]. It occurs when there is only a small absorption between the two layers at the frequency of the line center. When this radiation reaches the upper layer, some appreciable fraction of it is absorbed by the layer. The absorption at the center of the line is inversely proportional to the pressure and increases with height.

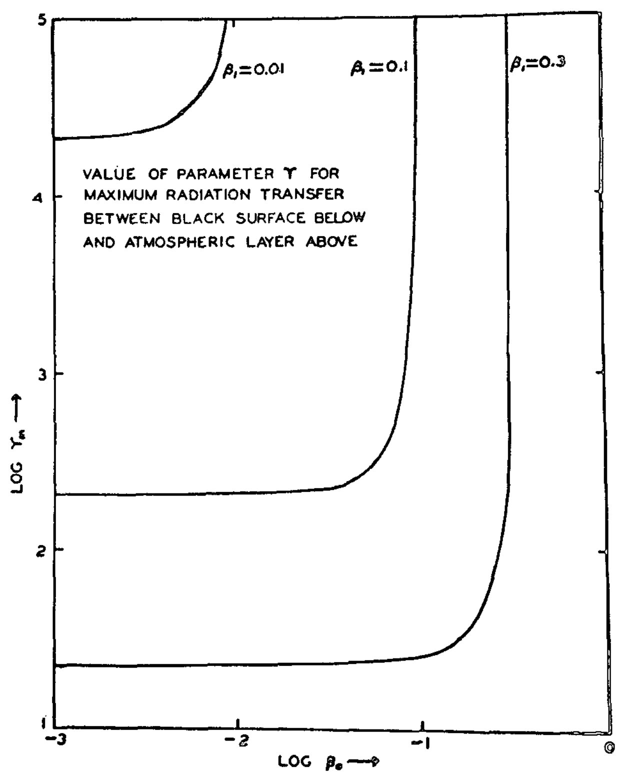

FIG. 4. Ordinate gives value of parameter, γ, for which there is maximum absorption of radiation by atmospheric layer at β_0 above black surface at β_1. Comparison is between different Elsasser bands that have same number of spectral lines, but different line intensities.

However, in a practical case, this relative maximum in the curve is small compared to the much larger maximum that occurs for large values of γ if $\beta_1 < 0.1$, *i.e.*, at pressures of roughly one-third atmospheric or less. For $\beta_1 \approx 0.3$, the two maxima of the curve are of the same order of magnitude, with the maximum at the larger γ value being somewhat larger than that at the smaller γ value.

In fig. 4, the value of the parameter, γ, for which there is the maximum absorption of radiation by an atmospheric layer at β_0 above a black surface at β_1 is plotted as a function of β_0 for several different values of β_1.

As mentioned in section 2, if the radiation is emitted by another atmospheric layer instead of by a black body, the absorption of radiation by a given layer is proportional to $\partial^2\Lambda/\partial\alpha_0\,\partial\alpha_1$. The results are entirely similar to the preceding cases, except that the equation for the maximum value of γ is

$$\gamma_m = \left[\frac{4}{3}\ln\frac{\cosh\frac{1}{2}\beta_1}{\cosh\frac{1}{2}\beta_0}\right]^{-1}, \qquad (49)$$

valid when $\gamma_m > 10$. The values of γ_m for this case are plotted in fig. 2, where the scale of ordinates on the right should be used. In this case, the values of γ_m are about three times larger for given β_0 and β_1 than those found for the other two cases.

8. Conclusion

An exact expression has been obtained for the absorption integrated over frequency and over the variable conditions along the path for a single line. The derivatives of this absorption have been obtained exactly for an Elsasser band. A constant mixing ratio, constant line intensity and the Lorentz line shape have been assumed. These results are used to find the regions of validity of certain useful approximations to the integrated absorption, Λ. The conditions when the strong-line approximation may be used with neglect of the overlap of the spectral lines are precisely stated for both Λ and its derivative. In general, the limits of the region of validity of the strong-line approximation for the derivatives of Λ are not reduced by more than a factor of three from the corresponding limits for Λ. This indicates that the strong-line approximation can be very useful at pressures somewhat less than atmospheric. It is also shown that the radiation absorbed in an atmospheric layer by a band of lines that do not overlap increases indefinitely as $\gamma^{\frac{1}{2}}$. When the overlap of the lines is taken into account through the Elsasser band model, it is found that the maximum radiation transfer always takes place in the wings of strong lines ($\gamma > 20$), provided $\beta = 2\pi\alpha/d < 0.3$, as appears to be the case for the bands of interest in atmospheric problems.

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REFERENCES

1. Pedersen, F., 1942: On the temperature-pressure effect on absorption of long-wave radiation by water vapour. *Meteor. Annaler*, 1, 115–135.
2. Strong, J., and G. N. Plass, 1950: The effect of pressure broadening of spectral lines on atmospheric temperature. *Astrophys. J.*, **112**, 365–379.
3. Kaplan, L. D., 1952: On the pressure dependence of radiative heat transfer in the atmosphere. *J. Meteor.*, 9, 1–12.
4. Plass, G. N., and D. Warner, 1952: Influence of line shift and asymmetry of spectral lines on atmospheric heat-transfer. *J. Meteor.*, 9, 333–339.
5. Plass, G. N., 1952: Parallel-beam and diffuse radiation in the atmosphere. *J. Meteor.*, 9, 429–436.
6. ——, 1952: A method for the determination of atmospheric transmission functions from laboratory absorption measurements. *J. opt. Soc. Amer.*, **42**, 677–683.
7. King, J. I., 1952: Line absorption and radiative equilibrium. *J. Meteor.*, 9, 311–321.
8. Yamamoto, G., and G. Onishi, 1952: Absorption of solar radiation by water vapor in the atmosphere. *J. Meteor.*, 9, 415–421.
9. King, J. I., 1952: Transfer theory for purely pressure-broadened band spectra. *Astrophys. J.*, **116**, 491–497.
10. Plass, G. N., and D. I. Fivel, 1953: Influence of Doppler effect and damping on line-absorption coefficient and atmospheric radiation transfer. *Astrophys. J.*, **117**, 225–233.
11. Goody, R. M., 1952: A statistical model for water-vapour absorption. *Quart. J. r. meteor. Soc.*, **78**, 165–169.
12. Plass, G. N., 1954: Regions of validity of various absorption-coefficient approximations. *J. Meteor.*, **11**, 163–165.
13. Kaplan, L. D., 1954: Energy transfer by infrared bands. *J. Meteor.*, 11, 16–19.
14. Bateman, H., *et al*, 1953: *Higher transcendental functions*. New York, McGraw-Hill, Vol. I, 302 pp.; Vol. II, 396 pp.
15. Elsasser, W. M., 1942: Heat transfer by infrared radiation in the atmosphere. *Harvard meteor. Stud.*, No. 6, 107 pp.
16. Whittaker, E. T., and G. N. Watson, 1927: *A course of modern analysis*. London, Cambridge Univ. Press, 608 pp.
17. Watson, G. N., 1918: Asymptotic expansions of hypergeometric functions. *Trans. Camb. phil. Soc.*, **22**, 277–308.