EQUATIONS GOVERNING THE ENERGETICS OF THE LARGER SCALES OF ATMOSPHERIC TURBULENCE IN THE DOMAIN OF WAVE NUMBER

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(Original manuscript received 3 August 1956; revised manuscript received 22 May 1957)

ABSTRACT

By considering the Fourier analysis of the planetary field of motion in the atmosphere, it is possible to define "scales" of motion and to write equations which govern the behavior of these separate scales of motion. Specifically, equations for the rate of change of the kinetic and available potential energy of a disturbance of a given wave number are presented. Such equations, which include the effects of the generation and release of potential energy, friction, and the transfer of energy among the various scales of eddies and the mean flow, can serve as a basis for studying the day-to-day variations of the spectral distribution of kinetic energy and for computing the "steady-state" atmospheric energy cycle in the domain of wave number, with the use of daily hemispheric data.

1. Introduction

It is conventional in considering the planetary atmospheric flow to think in terms of a mean motion obtained by averaging the flow along latitude circles, and an irregularly-varying eddy or disturbed motion which represents a turbulent departure from this mean condition. In the atmosphere, the turbulence elements which comprise this eddy motion vary in scale over a wide spectrum, ranging from the minute fluctuations recorded by sensitive micrometeorological instruments to the very large-scale irregularities in the flow observed on hemispheric synoptic charts. The propriety of considering the larger-scale eddies as turbulence elements was first recognized by Defant (1921).

By considering the physical processes which bring about the eddy motion, it is possible to distinguish between two basic regimes of turbulence which occur in the atmosphere. In particular, we may distinguish between a DIRECT turbulence in which kinetic energy of the eddy motion is maintained primarily by a direct conversion from other forms of energy, and an INDIRECT turbulence in which the eddy motion arises as a result of a transfer of kinetic energy from the laminar-type motions associated with a larger scale of flow. As an example of direct turbulence, we may cite the cyclone-scale disturbances in mid-latitudes which grow largely at the expense of potential energy due to baroclinic instability. The frictionally-induced, gusty wind variations near the ground surface, which represent the degradation of the energy of the larger-scale global motions, are an example of the indirect type.

Most of the classical treatments of turbulence [see Sutton (1953)] have been concerned primarily with the latter case, and consequently are not generally applicable to the larger-scale meteorological turbulence which tends to be of the direct type. Past attempts to make such an application have, accordingly, resulted in many paradoxical inconsistencies, exemplified by the requirement for an imaginary mixing-length. In recent years, however, the essential energetic differences between the two regimes of turbulence have been recognized more widely, and several writers have suggested a more general viewpoint in which the existence of both types is considered (e.g., Blackadar, 1950; Kuo, 1951; Starr, 1953; Lettau, 1954; and Hutchings, 1955). The purpose of this article is to present equations for the study of the statistical and dynamical properties of the larger-scale atmospheric turbulence from this broader viewpoint, with special regard for the energy-flow characteristics in the domain of wave number.

A customary approach to the study of turbulent fluid motion is to consider the stability of the disturbances in the flow; accordingly, much attention has been given recently to the examination of the stability characteristics of the larger-scale atmospheric disturbances. These studies have depended largely on idealizations of the flow, usually based on the method of small perturbations proposed originally by Helmholtz (1888). In this connection we may cite, as examples, studies by Charney (1947), Eady (1949), Fjörtoft (1950), Kuo (1949; 1951; 1952; 1953), Phillips (1951), and Thompson (1953). The "stability" of flows initially containing finite disturbances has been treated by Fjörtoft (1950; 1951), Starr (1950), Platz-

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1 This research has been supported by the Geophysics Research Directorate, Air Force Cambridge Research Center, Air Research and Development Command, under Contract AF19(604)-1000.

2 The term "stability" is used here in its broadest sense, being applied to disturbances of large as well as small amplitude.
man (1952), Kuo (1953), and Lorenz (1953). The flows dealt with in these latter studies were
constrained by simple boundary and initial conditions and simplified dynamical relationships such as the
barotropic vorticity equation.

In the well-known formulation of the problem from
the energy standpoint by Reynolds (1894), the basis
is provided for studying, in complete generality, the
stability properties of the large, finite disturbances
which are observed in any turbulent fluid. A serious
drawback of this approach, however, is the fact that
no information is given as to the behavior of the
separate scales of eddies, as distinct from the growth
and decay of total eddy kinetic energy.

In this article, methods similar to those used in the
modern statistical theory of turbulence based on
Fourier analysis [see, for example, Batchelor (1953)]
are used to extend the Reynolds formulation of the
turbulence problem into the domain of scale. Such an
approach makes it possible to investigate the ener-
geties of the atmospheric disturbances directly from
daily observational data. Whereas, for reasons of
mathematical expediency, most past studies have
treated either the "barotropic problem" in which the
flow has a horizontal shear or the "baroclinic problem"
in which the flow has a vertical shear, the present
formulation includes both effects simultaneously. [Re-
cently Peciniki (1955) and Phillips (1956) have made
first attempts to unify the barotropic and baroclinic
problems, the former with the framework of the per-
turbation method and the latter using numerical
integration techniques.]

Specifically, it is possible to derive from the funda-
mental equations a relation for the time rate of change
of the kinetic energy of a disturbance of any given
wave number as a function of the latitudinal spectra
of several meteorological quantities. This equation
contains terms representing the transfer of energy
among the various-wavelength disturbances and the
mean flow, and terms representing the conversions
from potential and internal energy. In accordance
with the common usage, one would speak of a given-
wave length disturbance as being barotropically un-
stable if its kinetic energy is increasing at the expense
of the kinetic energy of the mean flow and of other
disturbances, and baroclinically unstable if it tends to
grow at the expense of potential and internal energy.
The evaluation of such an equation over a sufficiently
long period of time provides a basis for determining
the "steady-state" cycle of energy conversion and
transfer in the domain of scale.

The development of this equation is presented in
section 5. As a preliminary, the basic equations to be
used in this study, a review of the more conventional
equations for zonal and total eddy kinetic energy, and
a review of the basic concepts of Fourier analysis are
presented in the following three sections.

2. Fundamental equations

In this article we shall make use of the fact that, to
a high degree of accuracy, the atmosphere is in a state
of hydrostatic equilibrium, so that we may take the
pressure, \( p \), as the vertical coordinate. Thus, if we
neglect the Coriolis-force terms involving the vertical
component of the wind, we may express the equations
of motion in spherical coordinates as follows:

\[
\frac{\partial u}{\partial t} + V \cdot \nabla u + \omega \frac{\partial u}{\partial \rho} = v \left( f + \frac{u \tan \phi}{a} \right) - \frac{g}{a \cos \phi} \frac{\partial z}{\partial \lambda} - X, \tag{1}
\]

\[
\frac{\partial v}{\partial t} + V \cdot \nabla v + \omega \frac{\partial v}{\partial \rho} = -u \left( f + \frac{u \tan \phi}{a} \right) - \frac{g}{a \cos \phi} \frac{\partial z}{\partial \phi} - Y, \tag{2}
\]

and

\[
0 = -\frac{g}{a} \frac{\partial z}{\partial \rho} - \alpha. \tag{3}
\]

In these equations, \( \lambda \) is longitude; \( \phi \) latitude; \( u \) and \( v \) are the eastward and northward components of the wind, respectively; \( V = u i + v j \) (\( i \) and \( j \) are the unit vectors in the eastward and northward directions, respectively) is the two-dimensional vector wind in a pressure
surface; \( \omega = \partial p/\partial t \); \( \nabla = i (a \cos \phi)^{-1} \partial / \partial \lambda + j a^{-1} \partial / \partial \phi \); \( a \) is the radius of the earth; \( z \) the height of an isobaric surface; \( a = 1/\rho \) is specific volume; \( X \) and \( Y \) are the eastward and northward components,
respectively, of the frictional force per unit mass;
\( f = 20 \sin \phi \) is the Coriolis parameter; and \( t \) is time.

In this \((\lambda, \phi, \rho, t)\) coordinate system the continuity
equation takes the following simple form:

\[
\frac{\partial \omega}{\partial \rho} = -\nabla \cdot V
\]

\[
= - \left( \frac{1}{a \cos \phi} \frac{\partial u}{\partial \lambda} + \frac{1}{a} \frac{\partial v}{\partial \phi} - \frac{v \tan \phi}{a} \right). \tag{4}
\]

The thermodynamical energy equation may be written
in the form:

\[
h = C_p dT/dt - \omega \alpha, \tag{5}
\]

where \( h \) is the rate of heat addition per unit mass
(including the generation of heat by friction), \( C_p \) the
specific heat at constant pressure, and \( T \) is temperature.

For completeness, we have also the equation of state,

\[
\alpha = RT/p, \tag{6}
\]

in which \( R \) is the gas constant.

The relationships (1) to (6) constitute the system
of equations which we shall use throughout this
article. In addition, we shall hereafter consider
the earth's surface to be perfectly spherical. As a result
of this simplification of the lower boundary, it will, of
course, be impossible to describe the direct effects of
orography on the energetics of the atmosphere.
3. Conventional equations for total, mean and eddy kinetic energy

If we multiply (1) and (2) by \( u \) and \( v \), respectively, we obtain the mechanical energy equations for the "horizontal" wind components in the form

\[
\frac{\partial}{\partial t} \left( \frac{u^2}{2} \right) = -V \cdot \nabla \left( \frac{u^2}{2} \right) - \frac{\partial}{\partial \phi} \left( \frac{u^2}{2} \right) + uu' \tan \phi \frac{1}{a} + \frac{g}{a \cos \phi} \frac{\partial z}{\partial \phi} = uX, \tag{7}
\]

and

\[
\frac{\partial}{\partial t} \left( \frac{v^2}{2} \right) = -V \cdot \nabla \left( \frac{v^2}{2} \right) - \frac{\partial}{\partial \phi} \left( \frac{v^2}{2} \right) - uv' \tan \phi \frac{1}{a} - \frac{g}{a \cos \phi} \frac{\partial z}{\partial \phi} = -vY. \tag{8}
\]

It may be seen that the effect represented by the term \( uu' \tan \phi / a \) is to transfer kinetic energy between the zonal and meridional components of the wind. By adding (7) and (8), this term is eliminated, yielding the following relation for the rate of change of the total kinetic energy of the "horizontal" wind, \( k = \frac{1}{2}(u^2 + v^2) \):

\[
\frac{\partial k}{\partial t} = -V \cdot \nabla k - \omega \frac{\partial k}{\partial \phi} - gV \cdot \nabla z - V \cdot F, \tag{9}
\]

where \( F = Xi + Yj \).

Equations (7), (8) and (9) may be averaged with respect to longitude to give the following relations:

\[
\frac{\partial}{\partial t} \left( \overline{\frac{u^2}{2}} \right) = -V \cdot \nabla \left( \overline{\frac{u^2}{2}} \right) - \frac{\partial}{\partial \phi} \left( \overline{\frac{u^2}{2}} \right) + \frac{u^2}{a} \tan \phi \frac{1}{a} + f \overline{uu' \tan \phi} \frac{1}{a} + f \overline{uu' \tan \phi} \frac{1}{a} - \frac{g}{a \cos \phi} \frac{\partial z}{\partial \phi} - u \overline{X}, \tag{10}
\]

and

\[
\frac{\partial}{\partial t} \left( \overline{\frac{v^2}{2}} \right) = -V \cdot \nabla \left( \overline{\frac{v^2}{2}} \right) - \frac{\partial}{\partial \phi} \left( \overline{\frac{v^2}{2}} \right) - \frac{v^2}{a \cos \phi} \frac{\partial z}{\partial \phi} - v \overline{Y}, \tag{11}
\]

and

\[
\frac{\partial k}{\partial t} = -V \cdot \nabla k - \omega \frac{\partial k}{\partial \phi} - g \overline{V \cdot \nabla z} - \overline{V \cdot F}, \tag{12}
\]

where the bar, defined by \( \overline{()} = \int_{0}^{2\pi} f \delta^{*}(\cdot) \, d\lambda \), denotes the zonal average.

The kinetic energy averaged around a latitude circle may be further resolved into components representing the kinetic energy of the zonally averaged (mean) wind and the mean eddy kinetic energy, according to the equations

\[
\overline{u'^2} = \overline{u'^2} + \overline{u''^2}, \tag{13}
\]

\[
\overline{v'^2} = \overline{v'^2} + \overline{v''^2}, \tag{14}
\]

\[
k = \frac{1}{2} (\overline{u'^2} + \overline{v'^2}) + \frac{1}{2} (\overline{u''^2} + \overline{v''^2}) = \frac{1}{2} (\overline{u'^2} + \overline{v'^2}). \tag{15}
\]

In these relations, the primes denote deviations from the zonal mean so that, for example, \( u = \overline{u} + u' \). We shall now present equations for the time rate of change of these separate components of \( k \).

If we multiply (1) and (2) by \( \overline{u} \) and \( \overline{v} \), respectively, applying the continuity equation (3), and average the resulting equations along a latitude circle, we obtain the relations

\[
\frac{\partial}{\partial t} \left( \overline{\frac{u^2}{2}} \right) = -\frac{\overline{u}}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \overline{uw} \cos \phi \right) - \frac{\partial}{\partial \phi} \left( \overline{u' \tan \phi} \frac{1}{a} \right) u \overline{X}, \tag{16}
\]

and

\[
\frac{\partial}{\partial t} \left( \overline{\frac{v^2}{2}} \right) = -\frac{\overline{v}}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \overline{vw} \cos \phi \right) - \frac{\partial}{\partial \phi} \left( \overline{v' \tan \phi} \frac{1}{a} \right) - \frac{\partial}{\partial \phi} \left( \overline{v' \tan \phi} \frac{1}{a} \right) - v \overline{Y}. \tag{17}
\]

These equations may be expanded further to read

\[
\frac{\partial}{\partial t} \left( \overline{\frac{u^2}{2}} \right) = \quad \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \overline{u'^2} + \overline{u' \tan \phi} \frac{1}{a} \right) \cos \phi
\]

\[
\frac{\partial}{\partial t} \left( \overline{\frac{v^2}{2}} \right) = \quad \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \overline{v'^2} + \overline{v' \tan \phi} \frac{1}{a} \right) \cos \phi
\]

\[
\frac{\partial k}{\partial t} = \quad \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \overline{u'^2} + \overline{v'^2} \right) \cos \phi
\]

\[
\frac{\partial k}{\partial t} = \quad \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \overline{u'^2} + \overline{v'^2} \right) \cos \phi
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\[
\frac{\partial k}{\partial t} = \quad \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \overline{u'^2} + \overline{v'^2} \right) \cos \phi
\]

Adding (18) and (19), we obtain for the rate of change of the kinetic energy of the zonally-averaged flow,
\[
\frac{\partial}{\partial t} \left( \frac{\vec{V}^2}{2} \right) = \left[ \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \frac{\vec{V}^2}{2} - \vec{u} \cdot \vec{u} \cdot \vec{v} \cdot \vec{v} \right) \right] \cos \phi \\
+ \frac{\partial}{\partial \phi} \left( \frac{\vec{V}^2}{2} - \vec{u} \cdot \vec{u} \cdot \vec{v} \cdot \vec{v} \right) \right] \\
- \vec{u} \cdot \vec{u} \cdot \frac{\tan \phi}{a} + \vec{V} \cdot \frac{1}{a} \frac{\partial \vec{V}}{\partial \phi} \\
+ \vec{u} \cdot \vec{u} \cdot \frac{\cos \phi}{a} \frac{\partial}{\partial \phi} \left( \frac{\vec{u} \cdot \vec{u}}{\cos \phi} \right) + \vec{u} \cdot \vec{u} \cdot \frac{\partial \vec{V}}{\partial \phi} \\
+ \vec{V} \cdot \frac{\partial \vec{V}}{\partial \phi} - \frac{g}{a} \frac{\partial \vec{V}}{\partial \phi} - \dot{a},
\]
where \( \dot{a} = \vec{V} \cdot \vec{F} = (\vec{u} \cdot \vec{X} + \vec{V} \cdot \vec{Y}) \) is the rate of frictional dissipation of the mean flow.

Equations for the rate of change of the mean eddy kinetic energy may be obtained by subtracting (18) and (19) from (10) and (11), respectively, in accordance with the relations (13), (14) and (15). The resulting equations may be expressed in the form

\[
\frac{\partial}{\partial t} \left( \frac{\vec{u}^2}{2} \right) = - \left[ \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \frac{\vec{u}^2}{2} \cos \phi + \frac{\partial \omega}{\partial \phi} \frac{\vec{u}^2}{2} \right) \right] \\
- \vec{u} \cdot \vec{u} \cdot \frac{\cos \phi}{a} \frac{\partial}{\partial \phi} \left( \frac{\vec{u} \cdot \vec{u}}{\cos \phi} \right) - \vec{u} \cdot \vec{u} \cdot \frac{\partial \vec{u}}{\partial \phi} \\
+ \vec{u} \cdot \vec{u} \cdot \left( f + \vec{u} \cdot \tan \phi \right) \\
+ \frac{\tan \phi}{a} \vec{u} \cdot \vec{u} \cdot \vec{v} + \frac{\tan \phi}{a} \vec{u} \cdot \vec{u} \cdot \vec{v} \\
- \frac{g}{a \cos \phi} \vec{u} \cdot \vec{u} \cdot \frac{\partial \vec{z}}{\partial \phi} - \vec{u} \cdot \vec{X},
\]

\[
\frac{\partial}{\partial t} \left( \frac{\vec{V}^2}{2} \right) = - \left[ \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \frac{\vec{V}^2}{2} \cos \phi + \frac{\partial \omega}{\partial \phi} \frac{\vec{V}^2}{2} \right) \right] \\
- \vec{V} \cdot \vec{V} \cdot \frac{1}{a} \frac{\partial \vec{V}}{\partial \phi} - \vec{V} \cdot \vec{V} \cdot \frac{\partial \omega}{\partial \phi} \\
- \vec{u} \cdot \vec{V} \cdot \left( f + \vec{u} \cdot \tan \phi \right) - \frac{\tan \phi}{a} \vec{u} \cdot \vec{V} \cdot \vec{v} \\
- \frac{\vec{V} \cdot \vec{V}}{a} \frac{\partial \vec{z}}{\partial \phi} - \vec{V} \cdot \vec{Y},
\]

and

\[
\frac{\partial}{\partial t} \left( \frac{\vec{V}^2}{2} \right) = - \left[ \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \frac{\vec{V}^2}{2} \cos \phi + \frac{\partial \omega}{\partial \phi} \frac{\vec{V}^2}{2} \right) \right] \\
- \vec{V} \cdot \vec{V} \cdot \frac{\cos \phi}{a} \frac{\partial}{\partial \phi} \left( \frac{\vec{u}}{\cos \phi} \right) - \vec{V} \cdot \vec{V} \cdot \frac{1}{a} \frac{\partial \vec{V}}{\partial \phi} \\
- \vec{u} \cdot \vec{V} \cdot \vec{u} \cdot \vec{v} - \vec{V} \cdot \vec{V} \cdot \frac{\partial \omega}{\partial \phi} \\
+ \vec{u} \cdot \vec{V} \cdot \vec{u} \cdot \frac{\tan \phi}{a} \\
- \vec{V} \cdot \vec{V} \cdot \frac{\partial \vec{z}}{\partial \phi} - \vec{V} \cdot \vec{Y},
\]

where \( \vec{d} = \vec{V} \cdot \vec{F} \) is the rate of frictional dissipation of the eddy flow.

If we integrate (20) and (23) over a closed mass of fluid (e.g., the entire atmosphere), neglecting the generally small effects of variations of surface pressure, we obtain

\[
\frac{\partial}{\partial t} \int \frac{\vec{V}^2}{2} \frac{dm}{dm} = \int \frac{\vec{u} \cdot \vec{u}}{a} \frac{\partial \vec{u}}{\partial \phi} \frac{\partial \vec{V}}{\partial \phi} \frac{\vec{u} \cdot \vec{u}}{a} \frac{\partial \vec{V}}{\partial \phi} \\
+ \vec{V} \cdot \vec{V} \cdot \frac{1}{a} \frac{\partial \vec{V}}{\partial \phi} - \vec{u} \cdot \vec{u} \cdot \frac{\tan \phi}{a} + \vec{u} \cdot \vec{V} \cdot \vec{u} \cdot \frac{\tan \phi}{a} \\
+ \vec{V} \cdot \vec{V} \cdot \frac{\partial \vec{z}}{\partial \phi} - \vec{V} \cdot \vec{Y},
\]

and

\[
\frac{\partial}{\partial t} \int \frac{\vec{V}^2}{2} \frac{dm}{dm} = - \int \vec{V} \cdot \vec{V} \cdot \vec{z} \frac{dm}{dm} + \int \vec{V} \cdot \vec{F} \frac{dm}{dm},
\]

where \( dm = g^{-1}a^3 \cos \phi \frac{d\lambda}{d\phi} \frac{d\phi}{d\rho} \frac{d\rho}{d\phi} \), and \( M \) indicates that the integration is over the whole atmosphere.

Adding (24) and (25), we obtain

\[
\frac{\partial}{\partial t} \int k \frac{dm}{dm} = - \int \vec{V} \cdot \vec{V} \cdot \vec{z} \frac{dm}{dm} - \int \vec{V} \cdot \vec{F} \frac{dm}{dm},
\]

which could have been written directly from (12) with the use of the continuity equation.

The appearance of the first integral in (24) with opposite sign in (25) indicates that this term measures the transfer of energy between the mean flow and eddy flow. It depends on the product of the so-called Reynolds eddy stresses and the shear of the mean velocity throughout the fluid.

The second integrals of (24) and (25) represent conversions between potential energy and the kinetic energy of the mean and eddy components of the flow, respectively. This may be demonstrated by considering the thermodynamical energy equation (5). Using the continuity equation (4) and the hydrostatic equation (3), and integrating over \( M \), we obtain [see White and Saltzman (1956)]

\[
\frac{\partial}{\partial t} \int \vec{C}_p T \frac{dm}{dm} = \int \vec{w} \cdot \vec{z} \frac{dm}{dm} + \int \vec{h} \frac{dm}{dm} \\
= \int \vec{g} \cdot \vec{V} \cdot \vec{z} \frac{dm}{dm} + \int \vec{h} \frac{dm}{dm},
\]
where \( \int_M C_p T \, dm \) is the total potential and internal energy of the atmosphere. (Under hydrostatic conditions, the potential energy bears a fixed ratio to the internal energy, so that we may henceforth speak of "Potential" energy alone.) If, now, we apply the bar operator to (27), we obtain

\[
\frac{\partial}{\partial t} \int_M C_r T \, dm = \int_M \nabla \cdot \nabla z \, dm + \int_M h \, dm \\
= \int_M \frac{\partial}{\partial t} \left( \frac{\partial z}{\partial \phi} \right) \, dm \\
+ \int_M \nabla \cdot \nabla z^2 \, dm + \int_M h \, dm. \tag{28}
\]

From this equation it may be seen that the second integrals in (24) and (25), taken together, represent the conversion between total kinetic energy and total potential energy; taken separately, they represent the conversions between "zonal available potential energy" and the kinetic energy of the mean flow, and between "eddy available potential energy" and the kinetic energy of the eddy flow, respectively, according to the terminology introduced by Lorenz (1955).

The last terms in (24) and (25) measure the rate of frictional dissipation of the kinetic energy of the mean and eddy flows, respectively.

The equations discussed in this section are well known and essentially date back to the classical paper of Reynolds (1894) [see, also, Van Mieghem (1952) and Arakawa (1953)]. In accordance with the introductory remarks, in the following sections we shall use the methods of Fourier analysis to define scales of eddies, and we shall derive energy equations corresponding to (24) and (25) which govern the behavior of these individual scales of motion.

4. Basic concepts from the theory of Fourier analysis

Any real, single-valued function \( f(\lambda) \), which is piecewise differentiable in the interval \((0, 2\pi)\), may be written in terms of a Fourier representation,

\[
f(\lambda) = \sum_{n=-\infty}^{\infty} F(n) e^{in\lambda}, \tag{29}
\]

where the complex coefficients, \( F(n) \), are given by

\[
F(n) = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda) e^{-in\lambda} \, d\lambda. \tag{30}
\]

For the purposes of this discussion, we shall consider the Fourier representation of meteorological quantities specified along a given latitude circle. Thus, in (29) and (30), \( \lambda \) is taken as longitude, and \( n \) is the number of waves around the latitude circle. The functions \( f(\lambda) \) and \( F(n) \) to be considered here are listed in Table 1.

<table>
<thead>
<tr>
<th>Table 1. Fourier transform pairs considered in this study.</th>
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<tr>
<td>( f(\lambda) )</td>
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<td>( F(n) )</td>
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The quantity \( F(n) \) is the representation of \( f(\lambda) \) in the domain of wave number and is called the spectral function of \( f \). The set of equations, (29) and (30), is often referred to as a Fourier transform pair.

Using (29) and (30), we may write the Fourier transform pairs for the derivatives of \( f(\lambda, \phi, p, t) \). Specifically, we have

\[
\frac{\partial f}{\partial \lambda} = \sum_{n=-\infty}^{\infty} \hbar F(n) e^{in\lambda}, \tag{31}
\]

and

\[
\int \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial f}{\partial \lambda} \, d\lambda; \tag{32}
\]

and

\[
\frac{\partial f}{\partial \xi} = \sum_{n=-\infty}^{\infty} F(\xi) e^{in\lambda}, \tag{33}
\]

and

\[
F(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial f}{\partial \xi} \, d\lambda, \tag{34}
\]

where \( \xi \) may be \( \phi, p \), or \( t \), and the use of subscript denotes a partial differentiation (i.e., \( F(\xi) = \partial F(n)/\partial \xi \)).

We now consider the product of two functions, \( f(\lambda) \) and \( g(\lambda) \), whose spectral functions defined by (30) are \( F(n) \) and \( G(n) \), respectively. For these functions, we may write

\[
\frac{1}{2\pi} \int_0^{2\pi} \left[ f(\lambda) g(\lambda) \right] e^{-in\lambda} \, d\lambda = \frac{1}{2\pi} \int_0^{2\pi} f(\lambda) \left[ \sum_{n=-\infty}^{\infty} G(m) e^{im\lambda} \right] e^{-in\lambda} \, d\lambda. \tag{35}
\]

If we assume that \( g(\lambda) \) is uniformly convergent, so that the order of summation and integration may be interchanged, (35) can be written in the form

\[
\frac{1}{2\pi} \int_0^{2\pi} \left[ f(\lambda) g(\lambda) \right] e^{-in\lambda} \, d\lambda = \sum_{m=-\infty}^{\infty} G(m) \frac{1}{2\pi} \int_0^{2\pi} f(\lambda) e^{-i(n-m)\lambda} \, d\lambda, \tag{36}
\]
or, finally,
\[
\frac{1}{2\pi} \int_0^{2\pi} f(\lambda) g(\lambda) e^{-im\lambda} \, d\lambda = \sum_{m=-\infty}^{\infty} G(m) F(n - m). \tag{37}
\]

This expression, which gives the spectral function for the product of two variables, is often called the multiplication theorem. As a special case, we may obtain Parseval's theorem by setting \( n = 0 \) in (37):
\[
\frac{1}{2\pi} \int_0^{2\pi} f(\lambda) g(\lambda) \, d\lambda = \sum_{m=-\infty}^{\infty} G(m) F(-m). \tag{38}
\]

If, further, \( f = g \), we have
\[
\frac{1}{2\pi} \int_0^{2\pi} f^2(\lambda) \, d\lambda = \sum_{m=-\infty}^{\infty} |F(m)|^2. \tag{39}
\]

In (39), use has been made of the fact that \( F(-m) \) is the complex conjugate of \( F(m) \), which implies that \( F(m) F(-m) = |F(m)|^2 \).

It may be noted, also, from (30) that \( F(0) = \mathcal{F} \).

5. Equations in the domain of wave number

**Transformation of fundamental equations.**—With use of the relations presented above, we may now transform the basic equations (1) to (5) from the space domain to the domain of wave number. These transformations are effected explicitly by multiplying the basic equations by \((2\pi)^{-1} e^{-im\lambda}\), integrating around a latitude circle, and applying (29) to (34) and (37).

Thus, the equations of motions for the horizontal components of the wind take the form (see table 1)
\[
\frac{\partial}{\partial t} U(n) = - \sum_{m=-\infty}^{\infty} \left[ \frac{im}{a \cos \phi} U(m) U(n - m) + \frac{1}{a} U_\phi(m) V(n - m) + U_m(m) \Omega(n - m) \right. \\
- \tan \phi U_{\phi}(m) V(n - m) \\
\left. - \frac{g}{a} \frac{\partial}{\partial \lambda} n + j V(n) - P(n) \right], \tag{40}
\]

and
\[
\frac{\partial}{\partial t} V(n) = - \sum_{m=-\infty}^{\infty} \left[ \frac{im}{a \cos \phi} V(m) U(n - m) + \frac{1}{a} V_{\phi}(m) V(n - m) \right. \\
+ \frac{\tan \phi}{a} U(m) U(n - m) \right. \\
\left. - \frac{g}{a} \frac{\partial}{\partial \lambda} \Omega(n) - j V(n) - Q(n) \right]. \tag{41}
\]

The hydrostatic equation (3) takes the form
\[
A_p(n) = - (R/\rho \theta) B(n). \tag{42}
\]

The continuity equation (4) takes the form
\[
\Omega_{\phi}(n) = - \left[ \frac{im}{a \cos \phi} U(n) + \frac{1}{a} V_{\phi}(n) - \tan \phi \frac{\partial}{\partial \lambda} V(n) \right]. \tag{43}
\]

Finally, the thermodynamical energy equation (5) takes the form
\[
\frac{\partial}{\partial t} B(n) = - \sum_{m=-\infty}^{\infty} \left[ \frac{im}{a \cos \phi} B(m) U(n - m) + \frac{1}{a} B_{\phi}(m) V(n - m) + B_m(m) \Omega(n - m) \right. \\
\left. - \frac{R}{C_p} \frac{\partial}{\partial \lambda} B(m) \Omega(n - m) \right] + \frac{1}{C_p} H(n). \tag{44}
\]

Equations (40) to (44) represent a closed system of equations governing the five dependent variables \( U \), \( V \), \( \Omega \), \( A \) and \( B \) as functions of \( n \), \( \phi \), \( p \) and \( t \), provided that the heating distribution \( H(n) \) is specified. Although this aspect will not be treated here, it is worth noting that such a system of equations can serve as the basis for a hemispherical numerical prediction scheme.

**Mechanical energy equation.**—We shall now derive the equations for the rate of change of kinetic energy of disturbances of given scale. We may conceive of a disturbance of a scale proportional to \( 1/n \) as the harmonic component of the complete flow whose wave number is \( n \), wave number zero corresponding to the mean flow.

From the Parseval theorem (39), we may write
\[
\tilde{\kappa} = \frac{1}{2\pi} \int_0^{2\pi} \frac{V^2}{2} \, d\lambda = \frac{\mathcal{V}^2}{2} + \sum_{n=1}^{\infty} K(n), \tag{45}
\]

where
\[
K(n) = |U(n)|^2 + |V(n)|^2 \tag{46}
\]

is the spectral function for the zonally-averaged eddy kinetic energy per unit mass.

By multiplying (40) and (41) by \( U(-n) \) and \( V(-n) \), respectively, and applying (43), we obtain
the following expressions for the rate of change of the separate components of $K(n)$:

$$
\frac{\partial}{\partial t} |U(n)|^2 = - \left[ U(-n) V(n) + U(n) V(-n) \right] \frac{\cos \phi}{a} \frac{\partial}{\partial \phi} \left( \frac{\bar{u}}{\cos \phi} \right) - \left[ U(-n) \Omega(n) + U(n) \Omega(-n) \right] \frac{\partial \bar{u}}{\partial \rho}
$$

$$
+ U(-n) U(n) U(-n - m) [U(m) V(-n - m) \cos \phi] - \frac{1}{a \cos \phi} \int \sum_{m=-\infty}^{\infty} \frac{i n}{a} U(m) [V(-n) U(n - m) - U(-n) V(n - m)]
$$

$$
+ \frac{1}{a \cos \phi} \int \sum_{m=-\infty}^{\infty} \frac{i n}{a} U(m) [V(-n) U(n - m) + U(-n) V(n - m)]
$$

and

$$
\frac{\partial}{\partial t} |V(n)|^2 = - \left[ V(-n) V(n) + V(n) V(-n) \right] \frac{1}{a} \frac{\partial}{\partial \phi} \left( \frac{\bar{u}}{\cos \phi} \right) - \left[ U(-n) \Omega(n) + V(n) \Omega(-n) \right] \frac{\partial \bar{u}}{\partial \rho}
$$

$$
+ \frac{1}{a \cos \phi} \int \sum_{m=-\infty}^{\infty} \frac{i n}{a} V(m) [V(-n) U(n - m) - V(n) U(-n - m)]
$$

$$
+ \frac{1}{a \cos \phi} \int \sum_{m=-\infty}^{\infty} \frac{i n}{a} V(m) [V(-n) \Omega(n - m) + V(n) \Omega(-n - m)]
$$

The desired equation for the rate of change of the total kinetic energy of a given wave number may now be obtained by using (46) and integrating over the entire mass of the atmosphere. In this integration, we neglect the terms which arise as a result of variations of surface pressure, as was done in the derivation of (24). Thus, we finally obtain

$$
\frac{\partial}{\partial t} \int_M K(n) \, dm = - \int_M \left[ \Phi_{uu}(n) \frac{\cos \phi}{a} \frac{\partial}{\partial \phi} \left( \frac{\bar{u}}{\cos \phi} \right) + \Phi_{uv}(n) \frac{1}{a} \frac{\partial \bar{v}}{\partial \phi} + \Phi_{uv}(n) \frac{\partial \bar{u}}{\partial \rho} + \Phi_{uu}(n) \frac{\cos \phi}{a} \frac{\partial \bar{v}}{\partial \phi} - \Phi_{uu}(n) \frac{\tan \phi}{a} \right] \, dm
$$

$$
+ \int_M \sum_{m=-\infty}^{\infty} \left[ U(m) \left[ \frac{1}{a \cos \phi} \Psi_{uu}(m, n) + \frac{1}{a} \Psi_{v}(m, n) + \Psi_{uv}(m, n) - \frac{\tan \phi}{a} \Psi_{uv}(m, n) \right] + \frac{1}{a} \Psi_{uv}(m, n) \right] \, dm
$$

$$
+ V(m) \left[ \frac{1}{a \cos \phi} \Psi_{uu}(m, n) + \frac{1}{a} \Psi_{v}(m, n) + \Psi_{uv}(m, n) + \frac{\tan \phi}{a} \Psi_{uv}(m, n) \right] \, dm
$$

$$
- \int_M \left[ \frac{1}{a \cos \phi} \Phi_{uu}(n) + \frac{1}{a} \Phi_{v}(n) \right] \, dm - \int_M \left[ \Phi_{ux}(n) + \Phi_{xy}(n) \right] \, dm. \quad (49)
$$
where

\[ \Phi_f(n) = \left[ F(n) G(-n) + F(-n) G(n) \right], \quad (50) \]

and

\[ \Psi_f(m, n) = \left[ F(n - m) G(-n) + F(-n - m) G(n) \right]. \quad (51) \]

As a special case, we may obtain the expression for the time rate of change of the kinetic energy of the mean flow by noting that \( \mathbf{V}^2 = K(0) \). Thus, setting \( n = 0 \) in (49) and using the continuity equation (43), we obtain

\[
\frac{\partial}{\partial t} \int_M \frac{\mathbf{V}^2}{2} \, dm = \int_M \sum_{n=1}^\infty \left[ \Phi_{v\omega}(n) \frac{\cos \phi}{a} \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \phi} \right) 
+ \Phi_{v\psi}(n) \frac{\partial}{\partial \phi} + \Phi_{\omega\omega}(n) \frac{\partial \theta}{\partial \phi} + \Phi_{\omega\psi}(n) \frac{\partial \theta}{\partial \phi} \right] \, dm
- \int_M g \frac{\partial \psi}{\partial \phi} \, dm - \int_M \frac{\partial}{\partial t} \, dm. \quad (52) \]

Equations (49) and (52) represent the transforms of (25) and (24) in the wave-number "space." The terms of these equations may be interpreted as follows.

The terms in the first integral on the right-hand side of (49) depend on the products of the shear of the mean flow and the transport of momentum by the separate scales of eddies. It seems plausible to regard this transport of momentum by an individual eddy scale as a separate physical process, which is distinct from the transport of momentum by other eddy scales and distinct from the physical processes represented by quadratic functions of the spectra appearing in the second integral. These latter functions all involve more than one scale of disturbance and can best be described as representing an "interaction" process.

On this basis, we may regard the first integral as a "transformation function" measuring the transfer of energy between the individual scales of disturbances and the mean flow: and we may regard the second integral, which vanishes when summed over all wave numbers, as a measure of the transfer of energy, due to non-linear interaction, between a disturbance of a given wave number and disturbances of all other wave numbers. Some insight into the behavior of these terms has recently been given by Fjörtoft (1953), who demonstrated that, for the two-dimensional non-

The divergent case, a change in the kinetic energy of one scale of motion is accompanied by changes in the kinetic energy of disturbances of both smaller and larger scale.

With the use of the hydrostatic equation (42) and the continuity equation (43), we may write the third integral in the successive forms

\[
\int_M \frac{1}{a \cos \phi} \Phi_{v\omega}(n) \, dm \quad \int_M \frac{1}{a \cos \phi} \Phi_{v\psi}(n) \, dm
= - \int_M \frac{1}{a \cos \phi} \Phi_{v\omega}(n) \, dm
+ \int_M \frac{R}{\rho} \Phi_{v\psi}(n) \, dm. \quad (53) \]

If we regard the process represented by (53) as physically distinct for each wave number, we may, in accordance with the discussion at the end of section 3, regard this integral as a measure of the conversion between eddy available potential energy and the eddy kinetic energy of the individual wave components which comprise the flow. In the following part of this section, the available potential energy equation applying in the wave-number domain is presented, and it may be verified that (53) appears with opposite sign in this equation. Qualitatively, the last integral of (53) demonstrates that the baroclinic growth of a disturbance of a given wave number depends on the degree to which the variations in the vertical motion of that wave number are in phase with the variations of temperature.

The last integral in each of (49) and (52) represents the frictional dissipation of the different scales of disturbances and of the mean flow, respectively.

*Available potential energy equation.*—Following the procedures introduced by Lorenz (1955) and Phillips (1956), we may use the first law of thermodynamics to derive equations for the rate of change of the total, zonal and eddy potential energy available for conversion into kinetic energy. These equations take the following forms, respectively, differing only in minor respects from those derived by Lorenz (1955):

\[
\frac{\partial}{\partial t} \int_M \frac{C_p \gamma}{T} T' \, dm
= \int_M \frac{R}{\rho} \omega T \, dm + \int_M \gamma \frac{T'^2}{k^n} \, dm, \quad (54) \]

\[
\frac{\partial}{\partial t} \int_M \frac{C_p \gamma}{T} \frac{T''}{\rho} \, dm
= \int_M \frac{R}{\rho} \omega T \, dm + \int_M \gamma \frac{T''^2}{k^n} \, dm

+ \int_M \left[ \frac{C_p \gamma}{T} \frac{\partial T}{\partial \phi} \right] \, dm, \quad (55) \]

---

\textsuperscript{2} It may be verified from (38) that \( \sum_{n=1}^{\infty} \Phi_f(n) = \overline{f} f' \). Accordingly, the general expression for the spectrum of the meridional transport of any physical quantity \( f \) (e.g., momentum, sensible heat, water vapor) is given by \( \Phi_f(n) \). The evaluation of such spectra from hemispheric data will demonstrate the relative importance of the various scales of eddies in effecting the horizontal transport processes in the general circulation. Computations of this type have recently been performed by Van Isacker and Van Mieghem (1956) and Kubota and Iida (1955).
and
\[
\frac{\partial}{\partial t} \int_{\mathcal{M}} \frac{C_\gamma}{2} \frac{T^4}{d} dm = \int_{\mathcal{M}} \frac{R}{p} \omega \frac{T^3}{a} dm + \int_{\mathcal{M}} \gamma \frac{T^4}{h} dm
\]
\[-\int_{\mathcal{M}} \left[ \frac{C_\gamma}{a} \frac{T^4}{a} \frac{\partial T}{\partial \phi} + \frac{\gamma}{\mu} \left( \frac{\omega T^4}{a} \frac{\partial \theta}{\partial \rho} \right) \right] dm, \tag{56}
\]
where the brackets denote a cosine-weighted average with respect to \(\phi\), \(\{\} = \{\} + \{\}''\); the wavy bar is defined by \(\{\} = \{\} + \{\}^*\); the asterisk denotes a deviation from this average, \(\{\} = \{\} + \{\}^*\); \(\mu = R/C_p\); \(\theta\) = potential temperature; and
\[\gamma = -\mu R/p^{1+v} (\partial / \partial \rho)^{-1}.\]

Applying the same procedure used to obtain (49) from the equations of motion to the thermodynamical energy equation (44), we may obtain an equation describing the variations of the separate scales of eddy available potential energy. This equation, which is the Fourier transform of (56), may be written in the following form, with use of the notation defined by (50) and (51):
\[
\frac{\partial}{\partial t} \int_{\mathcal{M}} \frac{C_\gamma}{2} \{ T'^2 \} dm
\]
\[-\int_{\mathcal{M}} \sum_{n=1}^\infty \left[ \frac{C_\gamma}{a} \Phi_{n+}(n) \frac{\partial T}{\partial \phi} + \frac{\gamma}{\mu} \left( \Phi_{n+}(n)'' \frac{\partial \theta}{\partial \rho} \right) \right] dm
\]
\[+ \int_{\mathcal{M}} \frac{R}{p} \omega T^4 dm + \int_{\mathcal{M}} \gamma \{ T'^{\prime\prime} \} dm. \tag{58}\]

**Discussion.**—The equations presented in this section govern the behavior of the finite disturbances which are observed in the atmosphere. Through the measurement of the terms of (49) and (57) from observational data, many aspects of the cycle of energy transformation involving these disturbances can be examined quantitatively. At this point it is of interest to speculate, on the basis of our present meteorological intelligence, about the nature of such a cycle.

In accordance with the ideas expressed by Lorenz (1955), the radiational heating tends to create a "sonal available potential energy" which is transformed into "eddy available potential energy" largely through the horizontal transport of heat by the existing eddies. By means of the baroclinic processes represented by (53), it appears that cyclone-scale disturbances of intermediate wave numbers (e.g., wave numbers seven to ten) grow at the expense of this available potential energy. It is known from studies of the angular momentum transport in the atmosphere [see, for examples, Kuo (1951), Starr (1953), and Starr and White (1954)] that the eddies tend to transfer their energy to the mean flow; and, in particular, synoptic evidence indicates that it is the largest eddies located at about 30 deg lat which are responsible for this transfer of energy to the mean flow. This has been supported quantitatively by a case study by Kao (1954) and a more recent study by Van Isacker and Van Mieghem (1956). Thus, one might expect that energy is being fed from the intermediate wavelengths not only to smaller eddies but also to the longer wavelengths which correspond in part to the so-called "semi-permanent" centers. (It is recognized that the...
6. Simplifications

The computations required to evaluate the integrals in (49) and (57) from observational data are of an extremely laborious nature, and it is only with the advent of high-speed computing facilities that such computations have become practicable. Even with these high-speed machines, it is desirable to make certain simplifications to make the computational problem more tractable.

Use of a discrete number of harmonics.—First, it is, of course, necessary to replace the summations over infinity by finite sums. In view of the limitations in the data coverage over the hemisphere, it seems desirable at present to use 36 points at 10 deg long intervals around a latitude circle as the basis for the Fourier expansions. In this case, one could not compute more than about twelve wave numbers with accuracy. It would appear, however, that enough of the variance of the hemispheric chart would be captured by this limited number of waves to make it feasible to use this approximation for the present problem. As one possibility, one could assume that all wave numbers larger than twelve act according to the Kolmogoroff theory and are included as frictional effects.

Use of geostrophic winds.—Direct wind observations for the hemisphere are not available in sufficient quantities to compute the spectra of the wind field directly. Observations of the angular momentum transport [i.e., Mintz (1951), and Widger (1949)] have already revealed, however, that horizontal variations in the general circulation are gross enough so that geostrophic winds are suitable for computations of the type required to measure the terms in the first two integrals in (49) except those involving \( \bar{e} \). It is not permissible to use geostrophic winds to evaluate the third integral in (49), since this term would vanish identically under this assumption. When geostrophic winds are used, the spectra for \( u \) and \( v \) may be obtained from the spectral function for the contour height by means of the relations

\[
U(n) = -\frac{g}{fu} A_\phi(n),
\]

and

\[
V(n) = \frac{gn_i}{fu \cos \phi} A(n).
\]

Computation of \( \Omega(n) \).—With the use of the adiabatic form of the thermodynamical energy equation, it is possible to obtain an estimate of \( \Omega(n) \) for the midtroposphere which may be used in evaluating the important baroclinic term (53).

The adiabatic energy equation may be written in the form

\[
\omega = -\frac{1}{\kappa \left( \frac{\partial T}{\partial \rho} - \frac{\alpha}{C_p} \right)} \left[ \frac{\partial T}{\partial t} + V \cdot \nabla T \right].
\]

Assuming that the stability factor,

\[
[\left( \frac{\partial T}{\partial \rho} - \frac{\alpha}{C_p} \right) = \kappa,
\]

can be assigned some horizontally uniform value, we may apply equations (30) to (34) and (37) to obtain the transform

\[
\Omega(n) = -\frac{1}{\kappa} \int \left[ \int B(n) \right. \\
\left. + \sum_{m=-\infty}^{\infty} \left[ \frac{im}{a \cos \phi} B(m) U(n - m) \\
+ \frac{1}{a} B_{\alpha}(n - m) V(m) \right] \right],
\]

or, in terms of “thickness,”

\[
\Omega(n) = \frac{g}{\left( \frac{\alpha}{\partial \theta} \right) \left( \frac{\partial T}{\partial \rho} \right)} \left[ \int \frac{\partial A_\phi(n)}{\partial t} \right. \\
\left. + \sum_{m=-\infty}^{\infty} \left[ \frac{im}{a \cos \phi} A_\phi(m) U(n - m) \\
+ \frac{1}{a} A_{\alpha\phi}(n - m) V(m) \right] \right].
\]

It may be seen that, to compute \( \Omega(n) \) at a given instant in time, it is necessary to evaluate \( \partial B(n)/\partial t \), or alternately \( \partial A_\phi(n)/\partial t \), at that time. This may be accomplished by solving the so-called “thermal vorticity” equation used in simple baroclinic models of the atmosphere.

Remarks concerning frictional effects.—Most of the frictional dissipation takes place near the ground surface, below the level of the atmosphere where the kinetic energy is primarily located. In this connection, it is pertinent to note that the effects measured by the terms involving \( \Omega \) in the first integral on the right side of (49) probably act in the sense of true eddy viscosity (see Kuo, 1951), and, as such, it may be possible to include them in the friction term along with the effects
of the small-scale disturbances. A direct calculation of these combined dissipative effects from observational data is extremely difficult, and, therefore, it is probably best to estimate them as a residual of computing the other terms in the energy equation.

Acknowledgments.—The writer wishes to express his gratitude to Profs. V. P. Starr and E. N. Lorenz, and to Dr. H.-L. Kuo, for their helpful suggestions and discussions.

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